Pattern avoiding permutations and Brownian excursion

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Definition
A permutation $\sigma$ is said to be $231$-avoiding if there does not exist $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$.

- $\sigma_1 = 3754621$ is NOT $231$-avoiding.

- $\sigma_2 = 2154367$ is $231$-avoiding.
**231-avoiding permutations**

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- $\sigma_1 = 3754621$ is **NOT** **231**-avoiding.
- $\sigma_2 = 2154367$ is **231**-avoiding.
- Knuth ('69): The number of **231**-avoiding permutations of size $n$ is

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$
231-avoiding permutations

- Miner, Pak 2013 – *The shape of random pattern avoiding permutations*.
- Janson, Nakamura, Zeilberger 2013 – *On the asymptotic statistics of the number of occurrences of multiple permutation patterns*.
- Janson 2014 – *Patterns in random permutations avoiding the pattern 132*.
- Madras, Pehlivan 2014 – *Structure of Random 312-avoiding permutations*.
Fixed Points

Theorem (Montmort 1708)

Let $\sigma_n$ be a uniformly random permutation of $\{1, 2, \ldots, n\}$. The number of fixed point of $\sigma_n$ converges in distribution to a Poisson(1) random variable as $n \to \infty$. 

Elizalde '04, '12, Elizalde and Pak '04 give detailed combinatorial results on fixed points of pattern avoiding permutations.

Theorem (Miner-Pak '13)

Let $\sigma_n$ be a uniformly random $231$-avoiding permutation of $\{1, 2, \ldots, n\}$. The expected number of fixed points of $\sigma_n$ is asymptotic to $\frac{\sqrt{\pi}}{2} \left(\frac{1}{4} n\right)^{-1/4}$ as $n \to \infty$. 

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Theorem (Miner-Pak ’13)

Let $\sigma_n$ be a uniformly random 231-avoiding permutation of $\{1, 2, \ldots, n\}$. The expected number of fixed points of $\sigma_n$ is asymptotic to $\frac{\Gamma(1/4)}{2\sqrt{\pi}} n^{1/4}$ as $n \to \infty$. 
Theorem (Hoffman-R-Slivken ’14)

Let $\sigma_n$ be a uniformly random $231$-avoiding permutation of $\{1, 2, \ldots, n\}$.

Let $\text{Fix}_n(t) = \# \{ i \in \{1, 2, \ldots, [t]\} : \sigma_n(i) = i \}$.

Then

$$
\left( \frac{1}{n^{1/4}} \text{Fix}_n(nt) \right)_{t \in [0, 1]} \xrightarrow{d} \left( \frac{1}{2^{7/4} \sqrt{\pi}} \int_0^t \frac{1}{e^{u^{3/2}}} \, du \right)_{t \in [0, 1]},
$$

where $(\exists_t)_{t \in [0, 1]}$ is standard Brownian excursion.
Theorem (Hoffman-R-Slivken ’14)

If $n$ is large and $\sigma_n$ is a uniformly random 231-avoiding permutation of $\{1, 2, \ldots, n\}$ then, appropriately rescaled,

$$(i - \sigma_n(i))_{1 \leq i \leq n}$$

almost looks like a Brownian excursion.
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Figure: $i - \sigma_n(i)$ for “good” values of $i$. 
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If \( n \) is large and \( \sigma_n \) is a uniformly random 231-avoiding permutation of \( \{1, 2, \ldots, n\} \) then, appropriately rescaled,

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Figure: An example for \( n = 10000 \)
A bijection between trees with $n + 1$ vertices and 231-avoiding permutations of $\{1, 2, \ldots, n\}$.
231-avoiding permutations

A bijection between trees with \( n + 1 \) vertices and 231-avoiding permutations of \( \{1, 2, \ldots, n\} \).
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A bijection between trees with $n + 1$ vertices and 231-avoiding permutations of $\{1, 2, \ldots, n\}$.

$$\sigma_t(2) = 2 + 5 - 1 = 6$$

$$\sigma_t(i) = i + |t_v_i| - \text{ht}(v_i)$$
The Good Points

\[ i - \sigma_t(i) = \text{ht}(v_i) - |t_{v_i}| \]