The Metric Coalescent Process

joint with David Aldous

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Two Related Processes

Two stochastic processes:

1. The Compulsive Gambler
   - Finite agent based model,
   - Finite Markov Information Exchange (FMIE) framework.

2. Metric Coalescent
   - Measure-valued Markov process,
   - Defined for any metric space \((S, d)\).
General Setup: interacting particle systems reinterpreted as stochastic social dynamics.

1. $n$ agents; each in some state $X_i(t) \in \mathcal{S}$ for each time $t \geq 0$

2. Each pair of agents $(i, j)$ meet at the times of a Poisson process of rate $\nu_{ij}$

3. At meeting times $t$ between pairs of agents $(i, j)$, the states transition

$$(X_i(t-), X_j(t-)) \rightarrow (X_i(t), X_j(t))$$

according to some deterministic or random rule

$$F : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S} \times \mathcal{S}. $$
Some familiar (and less familiar) examples:

1. Stochastic epidemic models; SIR model, etc.
2. Density dependent Markov chains (For ex. Kurtz 1978)
3. Averaging process, take $\mathcal{G} = \mathbb{R}$ as money. Upon meeting two agents average their money. (Aldous-Lanoue 2012).

\[
F(a, b) = \left( \frac{a + b}{2}, \frac{a + b}{2} \right)
\]

4. The iPod Model, an FMIE variant of the Voter Model (Aldous-Lanoue 2013)

The goal is to study how the (non-asymptotic) behaviour depends on the finite meeting rates $\nu_{ij}$. Analogous to the study of mixing time for finite Markov chains.
Compulsive Gambler Process

Simple FMIE process with agents’ state space $\mathcal{S} = \mathbb{R}_{\geq 0}$, interpreted as money. When agents $i$ and $j$ meet they play a fair, winner take all game. I.e. the transition function is

$$F(a, b) = \begin{cases} 
(a + b, 0) & \text{with prob. } \frac{a}{a+b} \\
(0, a + b) & \text{with prob. } \frac{b}{a+b}
\end{cases}$$

In the finite agent setting, we assume the total initial wealth is normalized

$$\sum_{i \in \text{Agents}} X_i(0) = 1.$$ 

Importantly this allows us to view the state of the process as a probability measure of the set of agents.
This model first studied in the setting of $d$-regular graphs and Galton-Watson Trees by Aldous-Salez. Some results on proportion of agents “still alive” at a time $t > 0$, in particular $t = \infty$ [ALS14].

The rest of today’s talk will focus on a very particular variant of the CG, one with dependent rates $v_{ij}$. 
Extending the CG Process

We can reformulate the CG as a measure-valued Markov process in terms of:

1. A metric space \((S, d)\),
2. A rate function \(\phi(x) : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}\).

The Metric Coalescent (MC) is then a continuous time \(P_{fs}(S)\)-valued Markov process, generalizing the CG as follows. For any \(\mu \in P_{fs}(S)\):

1. The atoms \(s_i, 1 \leq i \leq \#\mu\) of \(\mu\) are identified as the agents,
2. The masses \(\mu(s_i)\) as their respective current wealth,
3. The meeting rates between agents \(i\) and \(j\) given by \(\phi\) and the metric as

\[
\nu_{ij} = \phi(d(s_i, s_j))
\]
A simulation of the Metric Coalescent process on $S = [0, 1]^2$ started from finitely supported approximations of the uniform measure:

Developed by Weijian Han.
Main Theorem

Let \((S, d)\) be a locally compact, separable metric space and let the rate function \(\phi(x)\) satisfy

\[
\lim_{x \downarrow 0} \phi(x) = \infty.
\]

Our main result for the Metric Coalescent is as follows [Lan14]:

Main Theorem

There exists a unique, cadlag, Feller continuous \(P(S)\)-valued Markov process \(\mu_t, t \geq 0\) defined from any initial measure \(\mu_0 \in P(S)\) s.t. if \(\mu_0\) is compactly supported:

1. \(\mu_t \in P_{fs}(S)\) for all \(t > 0\), almost surely;
2. For each \(t_0 > 0\), the process \((\mu_t, t \geq t_0)\) is distributed as the Metric Coalescent started at \(\mu_{t_0}\).
Proof Idea: Naive Approach

The "naive" proof idea for constructing $\mu_t$, $t \geq 0$ for a generic $\mu \in P(S)$ is to approximate $\mu$ with a sequence of finitely supported $\mu^i \in P_{fs}(S)$ for $i \geq 1$. Then for $t \geq 0$ define (the random measure) $\mu_t$ as the weak limit

$$\mu_t = \lim_{i} \mu^i_t.$$

Feller continuity in the Main Theorem retroactively implies that this sequence of random measures does converge, however – even ignoring the coupling issues here – this approach isn’t so fruitful in proving the Main Theorem. Some progress is made in [Lan14] following this idea using moment methods.
**Proof Idea: Exchangeable Coalescents**

**Key Idea:** replace the symmetric “random winners at meeting times” dynamics between agents with “deterministic winners according to a size-biased initial ranking”. This allows us to view the MC as an exchangeable partition process and enables a wide variety of tools. Among these used:

1. A comparison to Kingman’s Coalescent,
2. Two separate applications of de Finetti’s theorem,
3. An explicit formula for moments of

\[
\int f \, d\mu_t
\]

for \( f : S \rightarrow \mathbb{R} \).
Further Directions

Two directions for further research:

1. **Coming Down From Infinity:** We know that for compactly supported $\mu_0$ initial measures, $\mu_t$ is finitely supported for all positive times $t > 0$. It is easy to construct non-compactly supported $\mu_0$ for which this isn’t true. What more can be said?

2. **Time Reversal:** A classical result on Kingman’s Coalescent is its duality under a time reversal to a conditioned Yule process. Viewing the MC as a “geometrization” of KC, can something similar be said?
Thanks for listening!

For further information on these two processes and a complete reference list.
