Triple Collisions of Competing Brownian Particles

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Competing Brownian Particles

Consider a system of N Brownian particles $X_1(t), \ldots, X_N(t)$. Rank them from bottom to top: $Y_1(t) \le \ldots \le Y_N(t)$.

Rule for their dynamics: the particle which is currently the kth smallest one moves as a Brownian motion with drift g_k and diffusion σ_k^2.

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i(t) = Y_k(t))\left(g_k\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_k(t)\right).$$

Triple and Simultaneous Collisions

- A triple collision occurs if $X_i(t) = X_j(t) = X_k(t)$ for some distinct i, j, k and some $t > 0$.
- A simultaneous collision occurs if $X_i(t) = X_j(t)$ and $X_k(t) = X_l(t)$ for $i \neq j$ and $k \neq l$.

For example:

- $X_1(t) = X_3(t) = X_6(t)$ is a triple collision;
- $X_3(t) = X_7(t)$ and $X_1(t) = X_6(t)$ is a simultaneous collision.

A triple collision is a particular case of a simultaneous collision.

A triple collision is undesirable: a strong solution exists up to the first time of a triple collision.

Necessary and Sufficient Condition

Theorem (S, 2014)

There are a.s. no triple and no simultaneous collisions if and only if the sequence (σ_k^2) is concave, that is,

$$\sigma_{k+1}^2 - \sigma_k^2 \le \sigma_k^2 - \sigma_{k-1}^2, \quad k = 2, \ldots, N-1. \tag{1}$$

Theorem (S, 2014)

If condition (1) is violated for some k, then with positive probability there is, in fact, a triple collision between the ranked particles Y_{k-1}, Y_k and Y_{k+1}.

Corollary

Corollary (S, 2014)

For a system of competing Brownian particles:
if there are a.s. no triple collisions,
then there are a.s. no simultaneous collisions.

Thanks!