Triple Collisions of Competing Brownian Particles

Andrey Sarantsev

University of Washington, Seattle

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Consider a system of $N$ Brownian particles $X_1(t), \ldots, X_N(t)$. Rank them from bottom to top: $Y_1(t) \leq \ldots \leq Y_N(t)$.

**Rule for their dynamics:** the particle which is currently the $k$th smallest one moves as a Brownian motion with drift $g_k$ and diffusion $\sigma_k^2$.

$$dX_i(t) = \sum_{k=1}^{N} 1(X_i(t) = Y_k(t)) (g_k \, dt + \sigma_k \, dW_k(t)).$$
A triple collision occurs if \( X_i(t) = X_j(t) = X_k(t) \) for some distinct \( i, j, k \) and some \( t > 0 \).

A simultaneous collision occurs if \( X_i(t) = X_j(t) \) and \( X_k(t) = X_l(t) \) for \( i \neq j \) and \( k \neq l \).

For example:
- \( X_1(t) = X_3(t) = X_6(t) \) is a triple collision;
- \( X_3(t) = X_7(t) \) and \( X_1(t) = X_6(t) \) is a simultaneous collision.

A triple collision is a particular case of a simultaneous collision.

A triple collision is undesirable: a strong solution exists up to the first time of a triple collision.
Theorem (S, 2014)

There are a.s. no triple and no simultaneous collisions if and only if the sequence \((\sigma_k^2)\) is concave, that is,

\[
\sigma_{k+1}^2 - \sigma_k^2 \leq \sigma_k^2 - \sigma_{k-1}^2, \quad k = 2, \ldots, N - 1.
\] (1)

Theorem (S, 2014)

If condition (1) is violated for some \(k\), then with positive probability there is, in fact, a triple collision between the ranked particles \(Y_{k-1}, Y_k\) and \(Y_{k+1}\).
Corollary (S, 2014)

For a system of competing Brownian particles:
if there are a.s. no triple collisions,
then there are a.s. no simultaneous collisions.
Thanks!