Random Walk Covering of Some Special Trees

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Abstract

For simple random walk on a finite tree, the cover time is the time taken to visit every vertex. For the balanced $b$-ary tree of height $m$, the cover time is shown to be asymptotic to $2m^2b^{m+1}(\log b)/(b - 1)$ as $m \to \infty$. On the uniform random labeled tree on $n$ vertices, we give a convincing heuristic argument that the mean time to cover and return to the root is asymptotic to $6(2\pi)^{1/2}n^{3/2}$, and prove a weak $O(n^{3/2})$ upper bound. The argument rests upon a recursive formula for cover time of trees generated by a simple branching process.

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1 Introduction

Let $G$ be a finite connected graph on $n$ vertices. Random walk on $G$ is the discrete-time Markov chain $(X_j; j \geq 0)$ with transition matrix $P$ of the form

$$P(v, w) = \begin{cases} 1/r_v & \text{if } (v, w) \text{ is an edge} \\ 0 & \text{if not} \end{cases}$$

where $r_v$ is the degree of $v$.

For each vertex $v$ let $T_v$ be the first hitting time:

$$T_v = \min\{j \geq 0 : X_j = v\}.$$

It is elementary that the cover time

$$C = \max_v T_v$$

is a.s. finite. Work of several authors has indicated in broad terms how $C$ relates to structural properties of the graph $G$. See [1] for a survey giving general bounds and explicit examples, and [6, 7, 8, 13] for subsequent work. The purpose of this paper is to carry through the analysis of two more explicit examples, the balanced $b$-ary tree and the uniform random labeled tree, to get the results stated in the abstract above.

Usually, if a problem is quite well understood on general graphs, then the special case of trees is trivial. This is true for mean hitting times $E_v T_w$: on a tree there is a simple explicit formula (Lemma 2), whereas on a general graph there is no such intuitively comprehensible formula. However, cover times are a little unusual in this regard. Consider the parameter $\bar{t} = \max_{v,w} E_v T_w$.

An obvious iteration argument gives

$$EC = O(\bar{t} \log n). \quad (1)$$

Now a large class of “not-too-little-connected” graphs satisfy

$$\bar{t} = \Theta(n)$$

and for such graphs we have $EC = \Theta(n \log n)$, the upper bound from (1) and the lower bound from [2]. Thus for well-connected graphs the cover time is often immediately available (up to constant factors), and theoretical interest lies in less well-connected graphs. This explains this interest in trees, which are by nature not well-connected.
An opposite class of graphs satisfy

\[ \bar{t} = \Omega(n^{1+\varepsilon}). \]  

(2)

For the “natural” graphs in this class which have been studied, we find that \( EC = \Theta(\bar{t}) \), though it is easy to exhibit artificial counterexamples. But these natural graphs have strong 1-dimensional structure; the uniform random labeled tree (which in a sense [3] has Hausdorf dimension 2) is qualitatively different. It is known that \( \bar{t} = \Theta(n^{3/2}) \), and this implies, using (1), that \( \Omega(n^{3/2}) = EC = O(n^{3/2} \log n) \). Our result, removing the log term, may seem merely a pedantic refinement, but actually has qualitative significance. Random walk on these trees rescales [4] to diffusion on a certain compact random fractal, and absence of the log term is equivalent to the fact that the diffusion covers its state space in a.s. finite time.

Between these classes is a class of “critical” graphs with

\[ \bar{t} = \Theta(n \log n). \]

There are two natural examples: the two-dimensional integer torus, and the balanced \( b \)-ary tree. For each, (1) gives \( \Omega(n \log n) = EC = O(n \log^2 n) \). It has been shown [12, 9] that on the \( b \)-ary tree it is now the upper bound which is correct, i.e. \( EC = \Theta(n \log^2 n) \), and our contribution is to exhibit the correct constant factor. (It turns out that the simple upper and lower bounds in [9] are asymptotically each off by a factor of 2.) It has recently been shown [13] that \( EC = \Theta(n \log^2 n) \) also on the 2-dimensional torus, but computing the correct constant seems difficult in this example.

\( EC \) behaves differently in the two examples considered in this paper. So it is perhaps surprising that they can be studied in the same way, by considering cover times \( C_m \) for the trees \( T_m \) consisting of the first \( m \) generations of a simple branching (Galton-Watson) process. In section 3 we set up a recursion (Corollary 6) for certain quantities closely related to the \( C_m \). This is the obvious approach; but it seems not entirely obvious just how to set up a recursion efficiently. The reader is invited to think about this before reading further. Section 4 applies the recursion to the \( b \)-ary tree. Section 5 analyses the recursion in the critical setting, leading to the results for the uniform random labeled tree in section 6.

2 Preliminaries

We first quote two elementary facts about random walk on finite trees: see e.g. [11]. The notation \( E_v \) means “expectation, for the walk started at \( v \)”. 


Lemma 1 For a leaf \( v \), the mean return time \( E_v T_v^+ = 2(n - 1) \), where \( n \) is the number of vertices.

Lemma 2 For an edge \((v, w)\), the mean hitting time \( E_v T_w = 2n_v - 1 \), where 
\[ n_v = |\{ x : \text{ the path from } x \text{ to } v \text{ avoids } w \}|. \]

General mean hitting times \( E_v T_z \) can be deduced from Lemma 2 by summing along the path from \( v \) to \( z \).

For a non-negative integer-valued r.v. \( N \) let \( \Gamma(N) \) denote a continuous r.v. whose distribution, conditional on \( N = m \), is \( \text{Gamma}(m, 1) \), i.e. has density \( f(t) = t^{m-1}e^{-t}/(m-1)! \) when \( m \geq 1 \). (\( \Gamma(0) = 0 \).) For a family \( (N_i) \) of r.v.’s, let \( (\Gamma(N_i)) \) denote a family which, given \( (N_i = m_i \text{ for all } i) \), are independent \( \text{Gamma}(m_i, 1) \). For non-negative real-valued \( Y \) let \( \mathcal{P}(Y) \) denote a discrete r.v. whose distribution, conditional on \( Y = y \), is \( \text{Poisson}(y) \).

Lemma 3 Let \( (m_1, \ldots, m_j) \) be non-negative. Throw a fair \((j+1)\)-sided die until “1” has appeared at least \( m_1 \) times, “2” has appeared at least \( m_2 \) times, \( \ldots \), and “j” has appeared at least \( m_j \) times. Let \( N \) be the number of times that “j+1” has appeared. Then
\[ N \overset{d}{=} \mathcal{P}(\max_{1 \leq i \leq j} \Gamma(m_i)) \]
where \( \overset{d}{=} \) denotes equality in distribution.

This follows from the obvious Poissonization argument, i.e. by considering the die throws as occurring at the times of a Poisson (rate \( j \)) process.

Consider a rooted tree with \( n \) vertices. For random walk on the tree started at the root, let \( C \) be the cover time, and let \( C^+ \) be the time to cover and return to the root. We now make an artificial construction whose use will become clear in the proof of Lemma 5 below. Call the root “0”. Make an extended tree by adding a new vertex “00”, connected only to 0, and make 00 the root of the extended tree. For random walk on the extended tree started at 00, let \( R \) be the number of traversals of the directed edge \((00, 0)\) until the cover time. Call \( R \) the \textit{excursion} r.v. associated with the original tree. We can relate \( EC \) to \( ER \) as follows.

Lemma 4
\[ 2(n - 1)(ER - 1) - \max_i E_i T_0 \leq EC \leq EC^+ = 2(n - 1)(ER - 1). \]
Proof. On the extended tree, let $C^*$ be the time required to cover and return to the root 00. The mean return time to 00 is $2n$ by Lemma 2, and so by Wald’s identity

$$EC^* = 2nER.$$ 

We can write

$$C^* = C + A + B + (2R - 1)$$

where $C$ is the time spent by the extended walk in the original tree until that tree is covered; $A$ is the time subsequently required to hit 0 (which occurs at time $T$ say); and $B$ is the time subsequently required to hit 00. It remains to count the time spent in the first step and in the steps $0 \rightarrow 00 \rightarrow 0$ before $T$, and this uses time $2R - 1$.

Now $C$ is the cover time for the original tree, and $C + A$ is the “cover and return to root” time ($C^*$). Also, $EA \leq \max_i E_i T_0$, and $B + 1$ is distributed as the return time of 00, so $EB + 1 = 2n$. Putting these estimates together gives the lemma.

3 Covering Galton-Watson trees

Let $q = (q_0, q_1, \ldots)$ be a probability distribution with $q_0 < 1$. Consider a simple Galton-Watson branching process, with 1 founder individual in generation 0, with offspring distribution $q$. Let $T_m$ be the family tree of the first $m$ generations of this process. Let $R_m$ be the excursion r.v. associated with $T_m$, as above. Note $R_0 = 1$. Here is the key fact.

Lemma 5

$$R_{m+1} \overset{d}{=} 1 + \mathcal{P}(\max_{1 \leq j \leq Q} \Gamma(R^j_m))$$

where $(R^j_m; j \geq 1)$ are independent copies of $R_m$, where $Q$ has distribution $q$ and is independent of the $R^j_m$, and where the max of an empty set is zero.

Proof. Fix $m$. Write $v_1, \ldots, v_Q$ for the individuals in the first generation, 0 for the founder individual, and $T_m$ for the family tree of the first $m$ generations. Let 00 be an artificial root attached to 0, and call the extended tree $T^*_m$. Write $T'_j$ for the tree consisting of 0, $v_j$ and the descendants of $v_j$ through generation $m + 1$. Then $T'_j$ is distributed as $T^*_m$, independently as $j$ varies.

There is a natural construction of random walk on $T^*_m$ in terms of random walks $(X^j(i); i \geq 0)$ on the $T'_j$ started at 0. Let $U^j_0, U^j_1, \ldots$ be
the successive return times of $X^j$ to 0, and let $E^*_r = (X^j(U^j_{r-1}, X^j(U^j_{r-1} + 1), \ldots, X^j(U^j_r)))$ be the $r$'th excursion of the random walk $X^j$. The walk $X$ on $T_{m+1}^*$ is constructed as follows, in terms of its excursions from 0. Each time the walk hits 0, toss a fair $Q+1$-sided die. If it lands $j$ $(1 \leq j \leq Q)$ the use the next excursion $E^*_j$ of $X^j$ as the next excursion of $X$. If the die lands $Q+1$, take the next excursion of $X$ to be $0 \to 00 \to 0$. It is clear that the process $X$ constructed thus is indeed simple random walk on $T_{m+1}^*$.

Now $T'_j$ is covered after (say) $R^i_m$ traversals of the directed edge $(0, v_j)$. Thus $T_{m+1}^*$ is covered after there have been at least $R^i_1$ rolls of “$v_1$”, $\ldots$, and $R^Q_m$ rolls of “$v_Q$”. Then Lemma 3 tells us how many rolls of 00 there have been: adding 1 for the first step gives the lemma.

We can reformulate this recursion more cleanly. Define $B_m = \Gamma(R_m)$. For real $t \geq 0$ let $H(t) = \Gamma(1 + P(t))$. Explicitly, $H(t)$ is the distribution with density $h(t, \cdot)$ defined at (4) below. For a r.v. $B$, write $H(B)$ for a r.v. whose distribution, conditional on $B = t$, is $H(t)$. Then applying $\Gamma(\cdot)$ to both sides of Lemma 5 gives

**Corollary 6 (The recursion formula)**

$$B_{m+1} \overset{d}{=} H(\max_{1 \leq i \leq Q} B^i_m).$$

Note that $EB_m = ER_m$, and so Lemma 4 relates $EC_m$ to $EB_m$. Thus for trees generated by branching processes, we have reduced the study of mean cover times to the study of $EB_m$, which can be accomplished via Corollary 6.

Writing $N(\mu, \sigma^2)$ for the Normal distribution, we can approximate $P(t) \approx N(t, t)$ for large $t$, and $\Gamma(m) \approx N(m, m)$ for large $m$, so

$$H(t) \approx N(t + 1, 2t) \text{ as } t \to \infty. \quad (3)$$

Here we are being imprecise with our approximation symbol $\approx$. But $H(t)$ has explicit density $h(t, \cdot)$ defined by

$$h(t, x) = e^{-(t+x)} \sum_{j=0}^{\infty} \frac{(tx)^j}{(j!)^2}, \quad 0 \leq x < \infty. \quad (4)$$

Our uses of the Normal approximation (3) can be justified from (4) and straightforward analysis. The series in (4) can be expressed in terms of Bessel functions, but this doesn’t seem helpful.
4 b-ary trees

Fix \( b \geq 2 \). Let \( B_m \) be the balanced \( b \)-ary tree of height \( m \). The number \( n_m \) of vertices satisfies

\[
n_m = 1 + b + b^2 + \ldots + b^m = \frac{b^{m+1} - 1}{b - 1}.
\]

Let \( C_m \) be the cover time, starting from the root.

**Theorem 7**

(i) \( EC_m \sim 2m^2b^{m+1}(\log b)/(b - 1) \) as \( m \to \infty \).

(ii) \( C_m/EC_m \to 1 \) in probability as \( m \to \infty \).

Part (ii) is an easy consequence of (i) and a general fact about covering. For each \( m \) let \( \bar{t}(m) = \max_{v,w} E_{v,T_w} \) be the maximum mean first hitting time for random walk on \( B_m \). Then by [5] Theorem 3 it suffices to prove

\[
\bar{t}(m)/EC_m \to 0.
\]

But using Lemma 2 we can compute \( \bar{t}(m) \) exactly, and we find

\[
\bar{t}(m) = O(mn_m)
\]

verifying (5).

*Proof of Theorem 7.* Using Lemma 4 and (6) it suffices to prove

\[
ER_m \sim (\log b) m^2.
\]

Now \( R_m \) has an obvious submultiplicity property

\[
P(R_m > u + v) \leq P(R_m > u)P(R_m > v); \quad \text{integers } u,v \geq 0
\]

and therefore it suffices to prove

\[
R_m \sim (\log b) m^2 \quad \text{in probability.}
\]

Putting \( B_m = \Gamma(R_m) \), it suffices to prove

\[
B_m^{1/2} \sim (\log b)^{1/2}m \quad \text{in probability.}
\]

This suggests doing a square-root space transformation in the recursion formula, Corollary 6. That is, we consider the kernel \( K(t,\cdot) \) with density

\[
k(t,x) = h(t^2,x^2)/(2x), \quad \text{where } h \text{ is the density of } H(t,\cdot).
\]
Using (3),
\[ K(t) \approx N(t, 1/2) \text{ as } t \to \infty. \] (9)

Writing \( \beta_m = \sqrt{B_m} \), the recursion formula gives
\[ \beta_{m+1} \overset{d}{=} K(\max_{1 \leq i \leq b} \beta_i^m) \] (10)

where \( K(T) \) denotes a r.v. which, conditional on \( T = t \), has density \( k(t, \cdot) \).

Consider the natural way of embedding the trees \( B_m \) as \( B_1 \subset B_2 \subset \ldots \subset B \) such that the leaves of \( B_m \) remain leaves of \( B_{m+1} \). Formally, the infinite tree \( B \) contains a path \( r_0, r_1, r_2, \ldots \) such that, for each \( m \), cutting the edge \( (r_m, r_{m+1}) \) and making \( r_m \) the root of its component yields the balanced \( b \)-ary tree \( B_m \). Given a kernel \( K \), we can define a process \((Z_v : v \in B)\) as follows.

(i) \( Z_v = 0 \) for leaves \( v \).

(ii) If \( v \) has descendants \( v_1, \ldots, v_b \) then \( Z_v \overset{d}{=} K(\max_{1 \leq i \leq b} Z_{v_i}) \).

The recurrence formula (10) says \( \beta_m \overset{d}{=} Z_{r_m} \). (11)

**Proposition 8** Let \( K \) be a stochastically monotone kernel on \([0, \infty)\) such that \( K(x, [x+1, \infty)) > 0 \) for all \( x \) and such that, for each \( -\infty < \theta < \infty \),
\[ \int \exp(\theta(y - x))K(x, dy) \to \phi_\xi(\theta) \equiv E \exp(\theta \xi) \text{ as } x \to \infty. \]

Suppose there exists \( c_0 > 0 \) such that \( \inf_\theta \phi_\xi(\theta)e^{-\theta c_0 b} = 1 \). Then, for \((Z_{r_m})\) defined above,
\[ m^{-1} Z_{r_m} \to c_0 \text{ in probability.} \]

Informally, (9) says we can apply the Proposition with \( \xi = N(0, 1/2) \), giving \( \phi_\xi(\theta) = e^{\theta^2/4} \). Precisely, we need to verify
\[ \int_0^\infty e^{\theta(x-t)} \frac{h(t^2, x^2)}{2x} dx \to e^{\theta^2/4} \text{ as } t \to \infty. \]

This is routine from (4). Then the Proposition holds for \( c_0 = \sqrt{\log b} \), and then (11) verifies (8) and establishes the Theorem.

*Proof of Proposition 8.* The proof uses variants of well-known techniques, so we shall be brief. First consider \( S_m = \sum_{i=1}^m \xi_i \), where the \( \xi_i \) are i.i.d. copies of \( \xi \). The fundamental large deviation theorem says
\[ b^{-m} P(S_m > cm) \to 0, \quad c > c_0 \] (12)
\[ \quad \to 1, \quad c < c_0. \] (13)
Suppose instead that \((S_m; m \geq 0)\) is the Markov chain with \(S_0 = 0\) and with transition kernel \(K\) satisfying the hypotheses of the Proposition. It is straightforward to show that (12,13) remain true. Now \(Z_{r_m}\) can be represented as the maximum of \(b^m\) dependent r.v.'s \(S_m\), one associated with each of the \(b^m\) paths from \(r_m\) to the leaves, and each distributed as \(S_m\) above. Thus the upper bound

\[
P(Z_{r_m} > cm) \to 0, \quad c > c_0
\]

follows from (12) and Boole’s inequality. The lower bound uses an embedded branching process argument. Consider first the homogeneous case \(K(x, dy) = P(x + \xi \in dy)\). Given \(c < c_0\) we can by (13) choose \(M\) such that

\[
b^M P(S_M > cM) > 1.
\]

Now define a process of “special” vertices as follows.

(a) Some single vertex at some height (= distance from leaves) \(i_0\) is special.

(b) For \(i \geq 1\), a vertex \(v\) at height \(i_0 + iM\) is special iff it has some descendant \(\tilde{v}\) of height \((i_0 + (i - 1)M\) which is special and is such that \(Z_v \geq Z_{\tilde{v}} + cM\).

Then the special vertices form a supercritical branching process. By varying \(i_0\) and the initial vertex in (a), it is easy to see that a.s. there is some initial vertex for which this branching process does not become extinct; therefore a.s.

\[
\lim \inf \frac{1}{m} Z_{r_m} \geq c.
\]

The non-homogeneous case, with which the Proposition deals, is similar.

## 5 Critical trees

Consider now a branching process whose offspring distribution \(Q\) has mean 1 and variance \(0 < \sigma^2 < \infty\). Such a critical branching process becomes extinct a.s., so the entire family tree \(T\) is finite. As in section 2, extend \(T\) by attaching an artificial root 00 to the original root 0. For random walk on this extended tree started at 00, let \(R\) be the number of crossings from 00 to 0 before covering. Letting \(m \to \infty\) in the recursion formula (Corollary 6) gives an identity satisfied by \(B = \Gamma(R)\).
Corollary 9

\[ B \overset{d}{=} H(\max_{1 \leq i \leq Q} B^i) \]

where \((B^i)\) are i.i.d. copies of \(B\).

To write this more explicitly let \(f(t)\) be the density of \(B\), let \(F(t) = P(B > t) = \int_t^\infty f(s)ds\), let \(h(t, \cdot)\) be the density (4) of \(H(t)\), and let \(\phi\) be the probability generating function of \(Q\). Then

\[ f(x) = q_0 h(0, x) + \int_0^\infty h(t, x) f(t) \phi'(1 - F(t)) dt. \tag{14} \]

We will be interested in the tail behavior of the distribution of \(B\), i.e. the asymptotic behavior of \(f(x)\).

Conjecture 10

\[ f(x) \sim 6\sigma^{-2} x^{-2} \text{ as } x \to \infty. \]

Heuristic argument. Suppose \(f\) is regularly varying, with exponent \(-\alpha\), say. For large \(t\), \(f\) is essentially constant over the range of \(h(t, \cdot)\) and so we can re-write (14) as

\[ 1/\phi'(1 - F(x)) \sim \int_0^\infty h(t, x) f(t) / f(x) dt. \tag{15} \]

As \(x \to \infty\) we have \(1 - \phi'(1 - F(x)) \sim F(x) \phi''(1) \sim F(x)\sigma^2\) and so the left side of (15) is \(\approx 1 + F(x)\sigma^2\). The right side, from the assumed regular variation, behaves like

\[ \int_0^\infty h(t, x)(t/x)^{-\alpha} dt. \]

But \(h(t, \cdot)\) is approximately the Normal\((t + 1, 2t)\) density, and the natural expansion of the integral gives leading terms

\[ 1 + \alpha(\alpha + 1)x^{-1}. \]

Equating the sides and differentiating,

\[ f(x)\sigma^2 \approx \alpha(\alpha + 1)x^{-2} \]

which identifies \(\alpha\) as 2.
Remark. This can be made rigorous if it is known that $f$ is indeed regularly varying. Presumably more careful analysis could establish this; unfortunately this result is not by itself sufficient for our applications, so we have not pursued the analysis.

For our formal results let us restrict to the special case where $Q$ has Poisson(1) distribution (the general case is similar). Here Equation (14) for the distribution of $B$ becomes
\[
f(x) = e^{-(x+1)} + \int_0^\infty h(t, x) f(t) \exp(-F(t)) \, dt. \tag{16}
\]

**Proposition 11** There exists $a < \infty$ such that
\[
P(B > x) \leq ax^{-1} \text{ for all } x > 0.
\]
\[
P(R > x) \leq ax^{-1} \text{ for all } x > 0.
\]

**Proof.** The recursion formula (Corollary 6) with $Q \overset{d}{=} \text{Poisson}(1)$ can be regarded as a map $\mu \mapsto r(\mu)$ on distributions. It is clear this map preserves stochastic ordering: if $\mu_1 \preceq \mu_2$ (i.e. if $\mu_1[0, x] \geq \mu_2[0, x]$ for all $x$) then $r(\mu_1) \preceq r(\mu_2)$. So suppose we can exhibit some $\mu$ such that $r(\mu) \preceq \mu$. Then by induction $\text{dist}(B_m) \preceq \mu$ for all $m$, and hence $\text{dist}(B) \preceq \mu$. So fix $a \geq 1$ and consider $\mu$ with density
\[
f(t) = at^{-2}; \ t \geq a.
\]
To prove the bound for $B$ it will suffice to show $r(\mu) \preceq \mu$, and by (16) it suffices to show
\[
\int_0^\infty h(t, x)t^{-2}\exp(-a/t)dt - x^{-2} \leq -e^{-(x+1)}; \ x \geq a. \tag{17}
\]
Expanding $\exp$, this quantity is bounded by the sum of the following three terms.
\[
J(x) = \int_0^\infty h(t, x)t^{-2}dt - x^{-2}
\]
\[
-aJ_1(x) = -a\int_0^\infty h(t, x)t^{-3}dt
\]
\[
\frac{1}{2}a^2J_2(x) = \frac{1}{2}a^2\int_0^\infty h(t, x)t^{-4}dt.
\]
From the explicit expression (4) for $h(t, x)$ routine analysis gives
\[
J(x) \sim 6x^{-3}; \ J_1(x) \sim x^{-3}; \ J_2(x) \sim x^{-4}.
\]
Thus we can choose $x_0$ such that

$$J(x) \leq 7x^{-3}, \quad J_1(x) \geq 0.9x^{-3}, \quad J_2(x) \leq 1.1x^{-4}; \quad x \geq x_0.$$ 

Then the quantity at (17) is, for $x \geq x_0$, bounded by

$$\frac{7 - 0.35a}{x^3} + \frac{0.55a}{x^3} \left(\frac{a}{x} - 1\right).$$

Taking $a = \max(x_0, 21)$ yields (17). The bound for $R$ (with slightly larger $a$) follows easily from the representation $B = \Gamma(R)$.

## 6 Combinatorial random trees

There are $n^{n-2}$ trees on the $n$ labeled vertices \{0, 1, \ldots, n - 1\} with root 0. Let $T_n$ be the uniform random tree chosen from this set. As far as the random walk is concerned, we can drop the labels and consider $T_n$ as a (non-uniform) random rooted unlabeled tree.

**Lemma 12 (Branching process representation of uniform trees)** Let $T$ be the entire family tree of the branching process with Poisson(1) offspring, and let $|T|$ be the total population size. Then $T_n$ has the conditional distribution of $T$ given $|T| = n$.

This is straightforward to verify; see e.g. Kolchin [10] for this approach to combinatorial random trees. We also need a standard result (e.g. [10] Lemma 2.1.4)

**Lemma 13**

$$P(|T| = n) \sim \frac{n^{-3/2}}{\sqrt{2\pi}}.$$

Let $C^+_n$ be the time taken by random walk on $T_n$ to cover and return to the root.

**Conjecture 14**

$$EC^+_n \sim 6\sqrt{2\pi}n^{3/2}.$$ 

By Lemma 4 this is equivalent to

$$ER_n \sim 3\sqrt{2\pi}n^{1/2} \quad \text{(18)}$$
where $R_n$ is the excursion r.v. associated with $T_n$ as in section 3. But $R_n$ is distributed as $R$ conditioned on $|T| = n$ by Lemma 12, where $R$ denotes the excursion r.v. associated with $T$. So

$$P(R > r) = \sum_n P(|T| = n)P(R_n > r). \quad (19)$$

**Heuristic argument.** Suppose $R_n/n^{\alpha} \overset{d}{\to} \xi$ for some $\xi$. Using Lemma 13 and (19),

$$P(R > r) \sim (2\pi)^{-1/2} \sum_n n^{-3/2} P(\xi > r n^{-\alpha})$$

$$\sim (2\pi)^{-1/2} \alpha^{-1} r^{-\frac{1}{2\alpha}} \int x^{\frac{1}{2\alpha}-1} P(\xi > x) dx$$

putting $x = r n^{-\alpha}$. But Conjecture 10 says $P(R > r) \sim 6 r^{-1}$, so this identifies $\alpha = 1/2$ and then $6 = (2\pi)^{-1/2} 2E\xi$, that is $E\xi = 3\sqrt{2\pi}$, suggesting (18).

Returning to the rigorous argument, Moon [11] Corollary 6.3.1 shows that, for a uniform random vertex $j$,

$$ET_j \sim \sqrt{\pi/2} n^{3/2}.$$  

Thus we certainly have $EC_n = \Omega(n^{3/2})$. Our results yield a weak kind of upper bound.

**Proposition 15** $C_n^+ / n^{3/2}$ is tight as $n \to \infty$ through some subsequence.

**Remark.** Although weak, this is enough to imply [4] that the limit process “diffusion on the continuum fractal” covers in finite time.

**Proof of Proposition 15.** Proposition 11, Lemma 13 and (19) imply there exists $A < \infty$ such that

$$\sum_n n^{-3/2} P(R_n > r) \leq Ar^{-1}, \quad r > 0. \quad (20)$$

We shall show

$$R_n/n^{1/2} \text{ is tight as } n \to \infty \text{ through some subsequence.} \quad (21)$$

For if (21) fails, then there exists $\varepsilon > 0$ and $s_n = \omega(n^{1/2})$ such that

$$P(R_n > s_n) \geq \varepsilon \text{ for all } n.$$
But let $n_0(r) = \min\{n : s_n > r\} = o(r^2)$. Then

$$\sum_n n^{-3/2} P(R_n > r) \geq \sum_{n \geq n_0(r)} n^{-3/2} P(R_n > s_n)$$

$$\geq \epsilon \sum_{n \geq o(r^2)} n^{-3/2}$$

$$= \omega(r^{-1}).$$

This contradicts (20) and establishes (21).

Now consider $n \to \infty$ through the subsequence in (21). On a fixed tree, $R$ satisfies the submultiplicivity condition (7), and so on non-random trees we have: $n^{-1/2} R_n$ tight implies $n^{-1/2} E R_n$ bounded. Thus (21), applied conditionally on the trees $T_n$, implies that $n^{-1/2} E(R_n|T_n)$ is tight. Applying Lemma 4 conditionally, we conclude that $n^{-3/2} E(C_n^+|T_n)$ is tight. The Proposition follows.
References


