

Random Information Spread in Networks

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Abstract

Let $G = (V, E)$ be an undirected loopless graph with possible parallel edges and $s, t \in V$. Assume that s is labelled at the initial time step and that every labelled vertex copies its labelling to neighbouring vertices along edges with one labelled endpoint independently with probability p in one time step. In this paper, we establish the equivalence between the expected s - t first arrival time of the above spread process and the notion of the stochastic shortest s - t path. Moreover, we give a short discussion of analytical results on special graphs including the complete graph and s - t series-parallel graphs. Finally we propose some lower bounds for the expected s - t first arrival time.

Keywords. random processes on graphs, stochastic shortest s - t path, s - t reliability polynomial, virus propagation in networks

1 Introduction

The spreading of information in networks is a random process. Consider an undirected loopless graph that possibly has parallel edges $G = (V, E)$. Let $s \in V$ be a chosen vertex that is labelled at time step 0. In addition we assume that that every edge of G that has a labelled endpoint independently copies the label to the possibly unlabelled endpoint with the given infection probability p in one time step.

Some applications are network based models for virus spread in epidemiology or the analysis of gossiping across social networks. The analysis of malware propagation in computer networks, such as spread of computer viruses or worms, represents another important application in technical networks.

The epidemic threshold is an interesting measure that characterises the transition from local to global spread. Random graphs and small-world networks gained most attention for investigations in this field (see for instance [1] or [8]). Certain regularity conditions are necessary in order to make the analysis of large or infinite graphs possible. An application for the analysis of the outbreak of Severe Acute Respiratory Syndrome (SARS) is presented in [6].

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We deal with the following questions in this paper. First we present the problem of finding the expected s - t first arrival time using the generating function approach (see Section 2). Second we establish in Section 3 the relationship between the spread process and the stochastic shortest s - t path problem stated in [9, Chapter 9]). In Section 4 we introduce the s - t spreading resistance of a graph. We show that this notion is related to Kulkarni's concept [4] of the expected length of a shortest s - t path, whenever the edge lengths are exponentially distributed random variables with intensity p . Subsequently, we establish two reduction techniques for calculating the expected first arrival time in s - t series-parallel graphs (see Section 6). Due to the computational complexity of the spread process, we propose in Section 7 methods for yielding lower and upper bounds for the expected s - t first arrival time in terms of the s - t reliability polynomial and the distance between s and t .

2 The spreading process

All graphs considered in this paper are finite and loopless, but possibly with parallel edges. In this section, we present how the spread process propagates in arbitrary graphs.

2.1 Assumptions

Let $G = (V, E)$ be a graph and $X_k \subseteq V$ be the set of labelled vertices in G at the discrete non-negative integral time step k . Assume that $u \in X_k$ copies the message to an adjacent unlabelled vertex v with probability $p_{uv} = 1 - q_{uv}$ along the edges $\{u, v\}$ in G . We refer to the numbers p_{uv} and q_{uv} as the *infection* and *noninfection probability from u to v* .

The spread is performed simultaneously along all edges that link labelled vertices to unlabelled vertices within one unit of time. The copying of labels along the edges is assumed to be stochastically independent. In addition the message spread is symmetric, that is for all $u, v \in V$, $p_{uv} = p_{vu}$. We presuppose that $p_{uv} > 0$ for any $\{u, v\} \in E$, whereas $\{u, v\} \notin E$ implies $p_{uv} = 0$.

Therefore we can describe the spread process by a homogeneous Markov chain $\{X_k\}_{k \in \mathbb{N}}$, where the state space is a subset of 2^V . If G is a connected graph then the labelled vertices spread until all vertices of G are labelled; that is, $\lim_{k \rightarrow \infty} X_k = V$.

Let $G = (V, E)$ be a graph and $v \in V$. The *open neighbourhood of v* , denoted by $N(v)$, is the set of vertices that are adjacent to v in G . For each $A \subseteq V$, the *open neighbourhood of A* is defined by

$$N(A) := \bigcup_{v \in A} N(v) \setminus A.$$

Furthermore for $A, B \subseteq V$, the *A - B edge set* in G , denoted by (A, B) , is the set of edges of G with one endpoint in A and the other endpoint in B . That is,

$$(A, B) := \{\{u, v\} \in E : u \in A, v \in B\}.$$

Definition 1. Let $G = (V, E)$ be a graph and $A, B \subseteq V$ with $A \subseteq B$. The *transition probability* $P_G(A, B)$ is the probability that the vertex subset B becomes infected in one time step if the vertex subset A is infected beforehand.

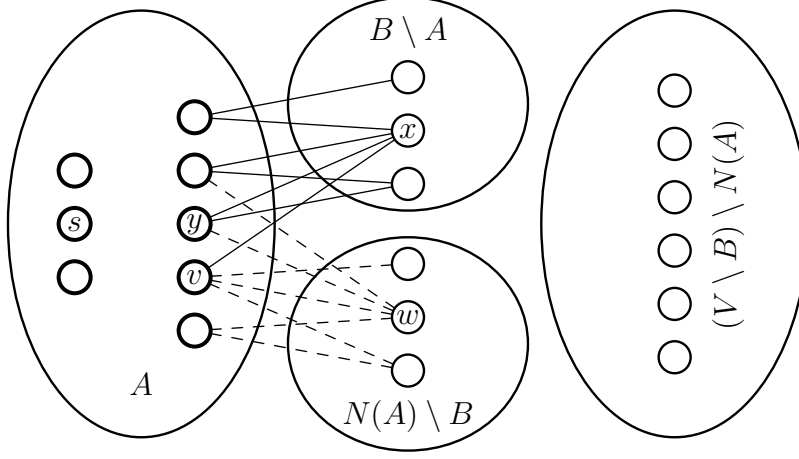


Figure 1: Transition probability of the spread process from A to B in G

Theorem 2. Let $G = (V, E)$ be a graph and $A, B \subseteq V$ with $A \subseteq B$. The transition probability $P_G(A, B)$ is then given by

$$P_G(A, B) = \prod_{v \in A} \prod_{e \in (\{v\}, N(A) \setminus B)} q_e \prod_{x \in B \setminus A} \left[1 - \prod_{f \in (\{x\}, A)} q_f \right].$$

Moreover if G is a simple graph, then

$$P_G(A, B) = \prod_{v \in A} \prod_{w \in N(A) \setminus B} q_{vw} \prod_{x \in B \setminus A} \left[1 - \prod_{y \in N(x) \cap A} q_{xy} \right].$$

PROOF. Figure 1 explains the derivation of $P_G(A, B)$ in a simple graph G . There must be no label propagation along edges from A to $N(A) \setminus B$ that are represented as broken lines. This gives the first double product. The remaining products correspond to transmissions along solid drawn edges. For each vertex $x \in B \setminus A$, there must be at least one edges that transports the label which gives the term in brackets. ■

At this point, we denote the transition probability from A to B in the graph G as $P(A, B)$, whenever G is known from the context.

Corollary 3. Let $G = (V, E)$ be a graph and $A, B \subseteq V$ with $A \subseteq B$. Furthermore, assume that all infection probabilities are identical; that is, $p_{uv} = p = 1 - q$ for all $\{u, v\} \in E$. The transition probability from A to B in G is then given by

$$P(A, B) = q^{|(A, N(A) \setminus B)|} \prod_{x \in B \setminus A} \left[1 - q^{|\{x\}, A|} \right].$$

Note that $P(A, B)$ can be only nonzero if $B \subseteq N(A) \cup A$ holds.

2.2 First arrival times

We first describe some general properties of the spread process. In order to simplify the presentation, we now assume that all infection probabilities are identical.

Definition 4. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a nonempty vertex subset. The graph G_X is obtained by *merging* the vertex subset X in G . The *merged vertex* in G_X is denoted by X' , where G_X contains an edge $\{X', y\}$ for each edge $\{x, y\} \in E$ in G with $x \in X$ and $y \in V \setminus X$.

Lemma 5. Let $G = (V, E)$ be a graph and $A, X \subseteq V$. If $A' = (A \setminus X) \cup \{X'\}$ is set, then $P_G(X, A) = P_{G_X}(\{X'\}, A')$ holds.

PROOF. Let $N_G(X)$ and $N_{G_X}(\{X'\})$ be the corresponding open neighbourhood of X in G and $\{X'\}$ in G_X , then $|(X, N_G(X) \setminus A)| = |(\{X'\}, N_{G_X}(\{X'\}) \setminus A)|$ holds. The merging operation provides $|(X, \{a\})|_G = |(\{X'\}, \{a\})|_{G_X}$ for all $a \in A \setminus X$. Consequently, the exponents of q from Corollary 3 remain unchanged while transforming G into G_X . ■

Thus it is advantageous to work with G_X instead of G since the number of vertices is reduced, but the graph G_X may not be simple even in the case that G is a simple graph. However, G_X can be transformed into a simple graph by replacing all parallel edges joining a pair of vertices $u, v \in V$ by just one edge $\{u, v\}$. That is, two parallel edges namely, $e = \{u, v\}$ and $f = \{u, v\}$ with infection probabilities p_e and p_f are replaced by a new edge g with infection probability $p_g = p_e + p_f - p_e p_f$. In this case we obtain the more general model including edges with different probabilities.

Definition 6. Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Denote by Z_{AB} the random variable of the first time that all vertices in the set B are infected if the vertex set A is infected at the beginning of the spread process.

The following lemma describes the transition from one state to another in terms of a Markov chain. This result will be utilised to establish Theorem 9. In the following \subsetneq denotes the proper subset inclusion.

Lemma 7. Let $G = (V, E)$ be a graph and $A, B \subseteq V$ such that $A \cap B \subsetneq B$, then

$$\Pr(\{Z_{AB} = n\}) = \sum_{C \supseteq A} P(A, C) \Pr(\{Z_{CB} = n - 1\}) \quad (1)$$

holds for all $n \geq 1$.

PROOF. The condition $A \cap B \subsetneq B$ implies that there is at least one vertex $t \in B$ that is not yet infected, when the spread process reaches the vertex subset A . The probability that all vertices in B are infected for the first time after n time steps can be divided into disjoint events by considering one further time step. In one time step a superset C of A can be infected, where the transition probabilities are given by the $P(A, C)$. In order to infect B for the first time after n time steps, it is necessary to reach B from C in exactly $n - 1$ time steps for the first time to account for the already elapsed extra time step. ■

Definition 8. Let $G = (V, E)$ be a graph and $A, B \subseteq V$. The *ordinary generating function of the A - B first arrival times* is defined as

$$\Phi_{AB}(G, z) = \sum_{n \geq 0} \Pr(\{Z_{AB} = n\}) z^n.$$

It clearly follows from the definition of Z_{AB} that $\Phi_{AB}(G, z) = 1$ holds for the case of $A \supseteq B$.

Theorem 9. *The ordinary generating function of the A - B first arrival times satisfies the recurrence relation*

$$\Phi_{AB}(G, z) = \sum_{C \supseteq A} z P(A, C) \Phi_{CB}(G, z)$$

for all $A, B \subseteq V$ with $A \cap B \subsetneq B$, where the initial conditions $\Phi_{CB}(G, z) = 1$ hold for all $C \supseteq B$.

PROOF. The statement follows directly from Lemma 7. Note that the assumption of $A \cap B \subsetneq B$ implies $\Pr(\{Z_{AB} = 0\}) = 0$, as B cannot be reached from A in zero time steps. ■

Observe that the nice triangular structure of the system of linear equations in Theorem 9, caused by $P(A, B) = 0$ for all pairs (A, B) , whenever A is not a subset of B , permits the practical solution of systems with more than 10^5 equations. It is now possible to deduce a recurrence relation for the expectation of the random variable Z_{AB} by using the generating function for the A - B first arrival time probabilities. A similar approach for the later discussed exponential model is presented in [4].

Corollary 10. *Let $G = (V, E)$ be a graph and $A, B \subseteq V$ with $A \cap B \subsetneq B$. The expectation of the random variable Z_{AB} , denoted by $T_{AB}(G)$, obeys the recurrence equation*

$$T_{AB}(G) = \mathbb{E}[Z_{AB}] = \sum_{C \supseteq A} P(A, C) [T_{CB}(G) + 1] \quad (2)$$

with the initial conditions $T_{AB}(G) = 1$, for any $A \supseteq B$. ■

In the case that the sets A and B are singletons say $A = \{s\}$, $B = \{t\}$, we will write $T_{st}(G)$ instead of $T_{\{s\}\{t\}}(G)$. This notation is also used when only one of the two sets is a singleton, e.g. $T_{At}(G)$ means $T_{A\{t\}}(G)$. The same notation also applies to the random variable $Z_{AB}(G)$ and the later defined spreading resistance $\rho_{At}(G)$.

3 Stochastic shortest paths

In this section, we recast the problem of the spread process in a stochastic shortest path problem. A nice introduction to solve stochastic shortest path problems in an algebraic way in the case of discrete arc lengths assuming finitely many values is found in [9, Chapter 9].

Definition 11. Let $G = (V, E)$ be a graph and $s, t \in V$. Furthermore let D_{st} be the random variable of the length of a shortest s - t path in G , if the lengths $L(e)$ of the edges $e \in E$ in G are independent geometric random variables with parameter p , where p equals the success (infection) probability of the spread process.

Theorem 12. *Let $G = (V, E)$ be a graph and $s, t \in V$, then $D_{st} = Z_{st}$.*

PROOF. For all $e \in E$, we let $L(e)$ be independent random variables representing the edge lengths, that are drawn from a geometric distribution with success probability $p = 1 - q$. That is for all $n \geq 1$,

$$\Pr(\{L(e) = n\}) = (1 - q)q^{n-1}.$$

Now for all $A \subseteq V$ the random variable D_{At} denotes the shortest length among all possible lengths of shortest u - t paths in G with $u \in A$, that is, $D_{At} = \min\{D_{ut} : u \in A\}$. Note that whenever the target vertex t satisfies $t \in A$, we have $\Pr(\{D_{At} = 0\}) = 1$.

Now for every $A \subseteq V$ and $v \in N(A)$, let the random variable $L(A, v)$ be the smallest edge length among all lengths of edges contained in $(A, \{v\})$. That is,

$$L(A, v) = \min\{L(e) : e \in (A, \{v\})\}.$$

We observe that

$$\Pr(\{L(A, v) = 1\}) = 1 - q^{|(A, \{v\})|}.$$

With the notions of the length $L(e)$ of an edge e and $L(A, v)$, we can restate the random variable D_{At} as

$$\begin{aligned} D_{At} &= \min\{D_{vt} + L(\{u, v\}) : \{u, v\} \in (A, N(A))\} \\ &= \min\{D_{vt} + L(A, v) : v \in N(A)\}, \end{aligned}$$

whenever $A \subseteq V \setminus \{t\}$ holds. By defining the event

$$\Lambda(A, B) := \{L(A, v) = 1 : v \in B\} \cap \{L(A, v) > 1 : v \in N(A) \setminus B\}$$

for each $A \subseteq V$ and $B \subseteq N(A)$, we obtain

$$\Pr(\Lambda(A, B)) = \prod_{v \in B} \Pr(\{L(A, v) = 1\}) \prod_{v \in N(A) \setminus B} \Pr(\{L(A, v) > 1\}) = P(A, A \cup B),$$

which coincides to the transition probability of Section 2. The application of the law of total probability on the event $\{D_{At} = n\}$ yields

$$\Pr(\{D_{At} = n\}) = \sum_{B \subseteq N(A)} P(A, A \cup B) \Pr(\{D_{At} = n\} \mid \Lambda(A, B)).$$

For each $A \subseteq V \setminus \{t\}$ and $B \subseteq N(A)$, we have

$$\Pr(\{D_{At} = n\} \mid \Lambda(A, B)) = \Pr\left(\left\{\min_{v \in N(A)} D_{vt} + L(A, v) = n\right\} \mid \Lambda(A, B)\right).$$

Now the conditional probability on the right hand side can be restated by reducing for all $v \in N(A)$ the realisation of the random variables $L(A, v)$ by one and asking for the probability of a shortest A - t path of length $n - 1$. That is,

$$\Pr(\{D_{At} = n\} \mid \Lambda(A, B)) = \Pr\left(\left\{\min_{v \in N(A)} D_{vt} + L(A, v) = n - 1\right\} \mid \Lambda'(A, B)\right)$$

with

$$\Lambda'(A, B) = \{L(A, v) = 0 : v \in B\} \cap \{L(A, v) \geq 1 : v \in N(A) \setminus B\}.$$

Now all $v \in B$ with $L(A, v) = 0$ can be included in the set A , as edge lengths of size zero do not contribute to the length of a shortest A - t path, whereas for each $v \in N(A) \setminus B$ the condition $L(A, v) \geq 1$ is always satisfied. Thus the condition $\Lambda'(A, B)$ can be omitted and

$$\Pr(\{D_{At} = n\} \mid \Lambda(A, B)) = \Pr(\{D_{A \cup B, t} = n - 1\})$$

follows. Hence, the recurrence relation

$$\begin{aligned}\Pr(\{D_{At} = n\}) &= \sum_{B \subseteq N(A)} P(A, A \cup B) \Pr(\{D_{A \cup B, t} = n - 1\}) \\ &= \sum_{C \supseteq A} P(A, C) \Pr(\{D_{Ct} = n - 1\}),\end{aligned}$$

is valid for all $A \subseteq V \setminus \{t\}$ with initial conditions $\Pr(D_{Ct} = 0) = 1$ for all $C \subseteq V$ with $t \in C$. This recurrence relation coincides with (1), whenever Z_{At} is replaced by D_{At} . Hence, $D_{At} = Z_{At}$ is valid for all $A \subseteq V$ and so $D_{st} = Z_{st}$ follows. ■

An important consequence of the above correspondence is the symmetry of the s - t first arrival time probabilities, which is only obvious in the view of the stochastic shortest path formulation of the spread process.

Corollary 13. *Let $G = (V, E)$ be a graph and $s, t \in V$ then $Z_{st} = Z_{ts}$.* ■

4 Spreading resistance

The spread process is connected to a continuous time Markov chain, which was already examined by Kulkarni and Corea in [4, 2].

Definition 14. Let $G = (V, E)$ be a connected graph with $A \subseteq V$ and $t \in V$. Let $T_{At}(G)$ be the expected A - t first arrival time in G . The number $\rho_{At}(G)$ defined by

$$\rho_{At}(G) = \lim_{q \rightarrow 1} (1 - q) T_{At}(G),$$

whenever this limit exists, is called the A - t spreading resistance of G .

Theorem 15. *Let $G = (V, E)$ be a graph then the A - t spreading resistance obeys the recurrence relation*

$$\rho_{At}(G) = \frac{1}{|(A, N(A))|} \left[1 + \sum_{c \in N(A)} |(A, \{c\})| \rho_{A \cup \{c\}, t}(G) \right]$$

for all $A \subseteq V$ and $t \in V$ with $(A, N(A)) \neq \emptyset$ and initial conditions $\rho_{Ct}(G) = 0$ for all $C \subseteq V$ satisfying $t \in C$.

PROOF. Observe that the system of linear equations for the expected A - t first arrival times that is obtained from (2) with the initial condition $T_{Ct}(G) = 0$, for all $C \subseteq V$ and $t \in C$ can be restated as

$$T_{At}(G) = \frac{1}{1 - P(A, A)} \left[1 + \sum_{C \supseteq A} P(A, C) T_{Ct}(G) \right].$$

As a consequence, the limit value of $(1 - q) T_{At}(G)$ yields to

$$\lim_{q \rightarrow 1} (1 - q) T_{At}(G) = \lim_{q \rightarrow 1} \frac{1 - q}{1 - P(A, A)} \left[1 + \sum_{C \supseteq A} P(A, C) T_{Ct}(G) \right] \quad (3)$$

as q tends to 1. Now consider the two limits from the right-hand side of (3)

$$L_1 = \lim_{q \rightarrow 1} \frac{1 - q}{1 - P(A, A)} \quad \text{and} \quad L_2 = \lim_{q \rightarrow 1} \frac{P(A, C)}{1 - P(A, A)}.$$

By applying Corollary 3 and L'Hôpital's rule to the right-hand side of L_1 , one finds

$$L_1 = \lim_{q \rightarrow 1} \frac{1 - q}{1 - q^{|(A, N(A))|}} = \frac{1}{|(A, N(A))|}$$

as $|(A, N(A))| \geq 1$. Similarly,

$$\begin{aligned} L_2 &= \lim_{q \rightarrow 1} \frac{q^{|(A, N(A) \setminus C)|}}{1 - q^{|(A, N(A))|}} \prod_{c \in C \setminus A} \left[1 - q^{|(A, \{c\})|} \right] \\ &= \sum_{c \in C \setminus A} \frac{|(A, \{c\})|}{|(A, N(A))|} \lim_{q \rightarrow 1} \prod_{\substack{v \in C \setminus A \\ v \neq c}} \left[1 - q^{|(A, \{v\})|} \right]. \end{aligned} \quad (4)$$

Now suppose $|C \setminus A| \geq 2$. Then for every $c \in C \setminus A$ there is a $v \neq c$ that contributes the factor $1 - q^{|(A, \{v\})|}$ to the product in (4). Hence the limit vanishes for every $c \in C \setminus A$. In the case of $C = A \cup \{c\}$ with $c \in N(A)$ one readily finds

$$L_2 = \frac{|(A, \{c\})|}{|(A, N(A))|}.$$

Now assume that $\rho_{A \cup \{c\}, t}(G)$ exists for each $c \in N(A)$ then (3) becomes

$$\rho_{At}(G) = \frac{1}{|(A, N(A))|} \left[1 + \sum_{c \in N(A)} |(A, \{c\})| \rho_{A \cup \{c\}, t}(G) \right]. \quad \blacksquare$$

Corollary 16. *Let $G = (V, E)$ be a connected graph with $A \subseteq V$ and $t \in V$, then the limit*

$$\rho_{At}(G) = \lim_{q \rightarrow 1} (1 - q) T_{At}(G)$$

always exists. \blacksquare

Definition 17. Let $G = (V, E)$ be a graph. The *exponential spreading model* in G is a spread process such that every edge $e \in E$ is weighted with $\omega(e)$ and $\{\omega(e) : e \in E\}$ is a collection of independent exponential random variables with intensity p .

The following result, due to Kulkarni, shows the connection between $\rho_{At}(G)$ and the exponential spreading model in G .

Theorem 18 (Kulkarni [4]). *Let $G = (V, E)$ be a graph with $A \subseteq V$, $t \in V$, and $(A, N(A)) \neq \emptyset$. If $\tau_{At}(G)$ is the expected length of a shortest A - t path of the exponential spreading model in G , then $\tau_{At}(G)$ satisfies the recurrence relation*

$$\tau_{At}(G) = \frac{1}{|(A, N(A))|} \left[\frac{1}{p} + \sum_{c \in N(A)} |(A, \{c\})| \tau_{A \cup \{c\}, t}(G) \right]$$

with the initial condition $\tau_{Ct}(G) = 0$ for all $C \subseteq V$ with $t \in C$. \blacksquare

The substitution of $(1 - q)\tau_{At}(G) = \rho_{At}(G)$ transforms Kulkarni's result into the recursive definition of $\rho_{At}(G)$. This yields an interpretation of the A - t spreading resistance $\rho_{At}(G)$.

5 Special graphs

This section illustrates the problem of the spread process discussed in Sections 2 and 4 on graphs with a special structure. The simplest is a path P_n of length n . If s and t are the end vertices of P_n and p is the infection probability assigned to every edges in P_n , then $T_{st}(P_n) = n/p$ and $\rho_{st}(P_n) = n$. This result is useful when deriving some simple upper bounds as seen in Chapter 7. Another trivial example is a tree, which follows from the fact that every two (distinct) vertices in a tree are joined by a unique path of nonzero length.

Theorem 19. *Let $H(m_1, m_2, \dots, m_n) = (V, E)$ be the graph consisting of n parallel paths of lengths m_i with $1 \leq i \leq n$ between the vertices s and t . Then its s - t spreading resistance is given by*

$$\rho_{st}(H(m_1, m_2, \dots, m_n)) = \sum_{j \geq 0} \sum \frac{1}{n^{j+1}} \binom{j}{i_1, i_2, \dots, i_n},$$

where the inner sum runs for all nonnegative integers i_1, i_2, \dots, i_n such that $j = i_1 + i_2 + \dots + i_n$ and $i_k < m_k$ holds for all k with $k = 1, 2, \dots, n$.

PROOF. Suppose that for all $j = 1, 2, \dots, m_i$, the weights $\omega(e_{ij})$ of the edges e_{ij} in the i -th path of $G = H(m_1, m_2, \dots, m_n)$ are independent exponential random variables with intensity p . Hence the realisation $(\omega(e_{i1}), \omega(e_{i2}), \dots, \omega(e_{im_i}))$ of waiting times for the edges of the i -th path in G corresponds to the sum of the random variables $\omega(e_{ij})$, $j = 1, 2, \dots, m_i$. This implies that $Y_i = \omega(e_{i1}) + \omega(e_{i2}) + \dots + \omega(e_{im_i})$ follows an Erlang- m_i distribution with intensity p ; that is,

$$\Pr(\{Y_i \leq t\}) = 1 - e^{-pt} \sum_{j=0}^{m_i-1} \frac{(pt)^j}{j!},$$

for each $i = 1, 2, \dots, n$. Now define the random variable $Y = \min\{Y_i : i = 1, 2, \dots, n\}$, then

$$\Pr(\{Y \leq t\}) = 1 - \prod_{i=1}^n \Pr(\{Y_i \geq t\}).$$

Because Y is a continuous random variable that takes nonnegative real values, it follows that its expectation obeys

$$\mathbb{E}[Y] = \int_0^\infty \Pr(\{Y \geq t\}) dt.$$

With $m' = \prod_{i=1}^n (m_i - 1)$, we obtain

$$\mathbb{E}[Y] = \int_0^\infty e^{-npt} \sum_{k=0}^{m'} \sum \frac{(pt)^j}{j!} \frac{j!}{i_1! i_2! \dots i_n!} dt = \frac{1}{p} \sum_{j \geq 0} \sum \frac{1}{n^{j+1}} \binom{j}{i_1, i_2, \dots, i_n}.$$

The inner sum in the above equation runs over all nonnegative integers i_k with $k = 1, 2, \dots, n$ satisfying $i_1 + i_2 + \dots + i_n = j$ and $i_k < m_k$ for all k with $1 \leq k \leq n$. The desired claim follows by multiplying $\mathbb{E}[Y]$ by $p = 1 - q$. ■

Davis and Prieditis [3] considered the expected s - t first arrival time for the complete graph with respect to the exponential model. The same analysis can be also accomplished in the case of the spreading process, which is the assertion given by the following result.

Theorem 20. Let $K_n = (V, E)$ be the complete graph with $n \geq 2$ vertices, $A \subseteq V \setminus \{t\}$, $|A| = i$ and $T_i := T_{At}(K_n)$. Then T_i satisfies the recurrence relation

$$T_i = \frac{1}{1 - q^{i(n-i)}} \left[1 + \sum_{j=i+1}^{n-1} \binom{n-1-i}{j-i} q^{i(n-j)} (1 - q^i)^{j-i} T_j \right],$$

with the initial condition $T_{n-1} = 1/(1 - q^{n-1})$.

PROOF. Note that in the case of the complete graph K_n the transition probabilities $P(A, B)$ do not depend on the sets A and B themselves, but on their cardinality. Assume therefore that A is an i element set and that B is an j element superset of A , then we find

$$P(A, B) = q^{i(n-j)} [1 - q^i]^{j-i}. \quad (5)$$

Hence, the A - t first arrival time of K_n is obtained by plugging (5) in the recurrence formula of the A - t first arrival times stated in (2). That is with $A \subseteq V \setminus \{t\}$ and $|A| = j$, we have

$$\begin{aligned} T_i &= 1 + \sum_{A \subseteq C \subseteq V \setminus \{t\}} q^{i(n-|C|)} [1 - q^i]^{|C|-i} T_{|C|} \\ &= 1 + \sum_{j=i}^{n-1} \binom{n-1-i}{j-i} q^{i(n-j)} [1 - q^i]^{j-i} T_j. \end{aligned} \quad (6)$$

The result then follows by factoring out T_i in (6). ■

Consequently, the solution of the recurrence relation in Theorem 20 gives us the expected s - t first arrival time of the K_n by setting $i = 1$. In effect, the s - t spreading resistance of K_n with $n \geq 2$ yields to the ratio of the $(n-1)$ -st harmonic number and $n-1$. This result was deduced in a different fashion in the work of David and Prieditis [3].

6 Reduction techniques on series-parallel graphs

In the case of s - t serial parallel graphs, we present an algebraic approach that utilises the Hadamard product of formal power series. The approach presented here is viewed as a special case of Shier's [9] general algebraic approach for the stochastic shortest path problem under the assumption that all s - t paths in G are edge disjoint.

Theorem 21 (Series reduction technique). Let $G = (V, E)$ be a connected graph with an articulation $a \in V$ such that there are two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ of G satisfying $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = (\{a\}, \emptyset)$. Let $s \in V_1 \setminus \{a\}$ and $t \in V_2 \setminus \{a\}$, then the following equations are true:

$$(a) \quad \Phi_{st}(G, z) = \Phi_{sa}(G_1, z) \Phi_{at}(G_2, z)$$

$$(b) \quad T_{st}(G) = T_{sa}(G_1) + T_{at}(G_2).$$

PROOF. Let $\Pr(\{Z_{st}(G) = n\})$ be the probability of the s - t first arrival time with respect to a spread process in G at $n \geq 0$ time steps. Moreover, denote by $\Pr(\{Z_{sa}(G_1) = n\})$ and $\Pr(\{Z_{at}(G_2) =$

$n)\}$ the s - a and a - t first arrival time probabilities in G_1 and G_2 in $n \geq 0$ time steps, respectively. Then we obtain

$$\Pr(\{Z_{st}(G) = n\}) = \sum_{k=0}^n \Pr(\{Z_{sa}(G_1) = k\}) \Pr(\{Z_{at}(G_2) = n - k\})$$

Correspondingly, let $\Phi_{st}(G, z)$ be the ordinary generating function for the s - t first arrival time probability in G . Similarly, the functions $\Phi_{sa}(G_1, z)$ and $\Phi_{at}(G_2, z)$ denote the ordinary generating functions for the first arrival time that the contact process reaches a in G_1 and t in G_2 , respectively. Hence

$$\Phi_{st}(G, z) = \sum_{n \geq 0} \Pr(\{Z_{st}(G) = n\}) z^n = \Phi_{sa}(G_1, z) \Phi_{at}(G_2, z).$$

follows. By applying the formal differentiation with respect to the indeterminate z and then setting $z = 1$ gives the desired expression $T_{st}(G) = T_{sa}(G_1) + T_{at}(G_2)$. \blacksquare

In order to establish the parallel reduction technique (Theorem 24), we need to introduce the notion of Hadamard multiplication in the ring $\mathbb{C}[[Z]]$ of all formal power series with complex-valued coefficients over the indeterminate z .

Definition 22. Let $A(z) = \sum_{i \geq 0} a_i z^i$ and $B(z) = \sum_{i \geq 0} b_i z^i$ be arbitrary formal power series in $\mathbb{C}[[Z]]$. The *Hadamard product* of $A(z)$ and $B(z)$ is defined as

$$(A \odot B)(z) := A(z) \odot B(z) = \sum_{i \geq 0} a_i b_i z^i.$$

In other words, the resulting formal power series $A(z) \odot B(z)$ is obtained by termwise multiplication of $A(z)$ and $B(z)$. It is clear that the Hadamard product is well-defined and is closed in $\mathbb{C}[[Z]]$ as presented in [10]. Together with the usual polynomial addition, polynomial multiplication, scalar multiplication and the Hadamard product, $\mathbb{C}[[Z]]$ forms a commutative algebra over the field \mathbb{C} . In addition, the *geometric series*

$$J(z) = \sum_{k \geq 0} z^k = \frac{1}{1 - z}$$

acts as the identity element of $\mathbb{C}[[Z]]$ under Hadamard multiplication.

Let r be a positive integer. We call the formal power series $J^r(z)$ to be the r -*geometric series*, which is defined as

$$J^r(z) = \left(\frac{1}{1 - z} \right)^r = \sum_{k \geq 0} \binom{r + k - 1}{k} z^k.$$

The following lemma presents a closed rational expression after pointwise multiplying m - and n -geometric series.

Lemma 23. Let m and n be positive integers with $m \geq n$, then for any complex numbers a and b ,

$$J^m(az) \odot J^n(bz) = J^{m+n-1}(abz) \sum_{i=0}^{n-1} \binom{m-1}{i} \binom{n-1}{i} (abz)^i.$$

PROOF. Suppose a and b are nonzero complex numbers. Assume further that m and n are positive integers with $m \geq n$. Let $F(z) = \sum_{k \geq 0} f(k)z^k$ be the resulting formal power series obtained by taking the Hadamard product of $J^m(az)$ and $J^n(bz)$. Then for each $k \in \mathbb{N}$, the coefficient of z^k in $F(z)$ yields

$$f(k) = \binom{m+k-1}{k} \binom{n+k-1}{k} (ab)^k.$$

We see that

$$\binom{n+k-1}{k} = \frac{(k+1)^{\overline{n-1}}}{(n-1)!} = \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{k^i}{i!} \quad (7)$$

where the rightmost part of (7) is first obtained by treating $(k+1)^{\overline{n-1}}/(n-1)!$ as a polynomial in k and then applying the Newton's forward difference equation to the said polynomial. Using (7) and setting $w = abz$ in $F(z)$, we now obtain

$$F(w) = \sum_{i=0}^{n-1} \binom{n-1}{i} \binom{m+i-1}{i} w^i J^{m+i}(w). \quad (8)$$

Observe that we can factor $J^{n+m-1}(w)$ from the sum in (8). This gives us

$$\begin{aligned} F(w) &= J^{n+m-1}(w) \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \binom{n-1}{i} \binom{m+i-1}{i} \binom{n-i-1}{j} (-1)^j w^{i+j} \\ &= J^{n+m-1}(w) \sum_{j=0}^{n-1} \sum_{i=0}^j \binom{n-1}{j-i} \binom{m+j-i-1}{j-i} \binom{n-j+i-1}{i} (-1)^i w^j. \end{aligned} \quad (9)$$

The identity

$$\binom{n-1}{j-i} \binom{m+j-i-1}{j-i} \binom{n-j+i-1}{i} = \binom{j}{i} \binom{m+j-i-1}{m-1} \binom{n-1}{j}$$

holds for each i with $0 \leq i \leq j$. In effect, $F(w)$ in (9) becomes

$$F(w) = J^{n+m-1}(w) \sum_{j=0}^{n-1} \binom{n-1}{j} \left[\sum_{i=0}^j \binom{j}{i} \binom{m+j-i-1}{m-1} (-1)^i \right] w^j. \quad (10)$$

We interpret the inner sum of (10) in a combinatorial way. Let Y be an $(m+j-1)$ -element set. Let $I \subseteq Y$ be a j -element subset of marked elements in Y . For each $A \subseteq I$, we define S_A to be the collection of $(m-1)$ -element subsets of Y such that all the elements of A are not included in the said subsets. In view of the principle of inclusion-exclusion, we have

$$\left| \bigcap_{A \subseteq I} \overline{S_A} \right| = \sum_{A \subseteq I} (-1)^{|A|} \left| \bigcap_{a \in A} S_a \right| = \sum_{i=0}^j (-1)^i \binom{j}{i} \binom{m+j-i-1}{m-1}.$$

This gives us the number of ways we can draw an $(m-1)$ -element set from Y such that all the marked elements of I are included. In turn, this corresponds to the number of ways of forming an j -element subset of an arbitrary $(m-1)$ -element set. ■

Theorem 24 (Parallel reduction technique). *Let $G = (V, E)$ be a graph and $\{s, t\}$ be a separating vertex pair of G such that H and K are two connected nontrivial graphs with $H \cup K = G$ and $H \cap K = (\{s, t\}, \emptyset)$, then*

$$\Phi_{st}(G, z) = \Phi_{st}(H, z) + \Phi_{st}(K, z) - (1 - z) \left[\frac{\Phi_{st}(H, z)}{1 - z} \odot \frac{\Phi_{st}(K, z)}{1 - z} \right]$$

PROOF. Let $Z_{st}(G)$ be the random variable of the s - t first arrival time in G , and $Z_{st}(H)$ and $Z_{st}(K)$ be the random variables of the s - t first arrival times in H and K , respectively. As the events $\{Z_{st}(H) > n\}$ and $\{Z_{st}(K) > n\}$ are independent, we have

$$1 - \Pr(\{Z_{st}(G) \leq n\}) = (1 - \Pr(\{Z_{st}(H) \leq n\}))(1 - \Pr(\{Z_{st}(K) \leq n\})).$$

Multiplying both sides of the above equation with z^n and summation over all $n \geq 0$ yields

$$\frac{1 - \Phi_{st}(G, z)}{1 - z} = \frac{1}{1 - z} - \frac{\Phi_{st}(H, z)}{1 - z} - \frac{\Phi_{st}(K, z)}{1 - z} + \frac{\Phi_{st}(H, z)}{1 - z} \odot \frac{\Phi_{st}(K, z)}{1 - z},$$

which leads after rearranging terms to the desired result. \blacksquare

We apply the parallel reduction for the series-parallel graph G with s and t as its start and terminal vertices such that $\{s, t\}$ forms a vertex separating pair. The subsequent result deals with the case that one of the two subgraphs of G is a path of length $r \geq 1$.

Corollary 25. *Let $G = (V, E)$ be a graph and $\{s, t\}$ a separating vertex pair of G , furthermore assume that H and K are subgraphs of G with $G = H \cup K$ and $H \cap K = (\{s, t\}, \emptyset)$ and that K is a path of length r connecting s and t . Then the following equations hold:*

$$\begin{aligned} (a) \quad \Phi_{st}(G, z) &= \frac{p^r z^r}{(1 - qz)^r} + \left(\frac{p}{q}\right)^{r-1} \frac{1 - z}{(r - 1)!} \frac{d^{r-1}}{dz^{r-1}} \left(\frac{z^{r-1} \Phi_{st}(H, qz)}{1 - qz} \right) \\ (b) \quad T_{st}(G) &= \frac{rq^r}{(1 - q)^{r+1}} - \frac{1}{(r - 1)!} \left(\frac{p}{q}\right)^{r-1} \frac{d^{r-1}}{dz^{r-1}} \left(\frac{z^{r-1} \Phi_{st}(H, qz)}{1 - qz} \right) \Big|_{z=1}. \end{aligned}$$

PROOF. For the first arrival times of the path K of length r in G , one finds

$$\Phi_{st}(K, z) = \sum_{n \geq r} \binom{n-1}{r-1} p^r q^{n-r} z^n = \frac{p^r z^r}{(1 - qz)^r}. \quad (11)$$

By Theorem 24, the ordinary generating function for the first arrival time of G yields

$$\begin{aligned} \Phi_{st}(G, z) &= \Phi_{st}(H, z) + \frac{p^r z^r}{(1 - qz)^r} + (z - 1) \left[\frac{\Phi_{st}(H, z)}{1 - z} \odot \frac{p^r z^r}{(1 - z)(1 - qz)^r} \right] \\ &= \Phi_{st}(H, z) + \frac{p^r z^r}{(1 - qz)^r} + (z - 1) \left[\frac{\Phi_{st}(H, z)}{1 - z} \odot \left(\frac{1}{1 - z} - \left(\frac{p}{q}\right)^{r-1} \frac{1}{(1 - qz)^r} \right) \right] \\ &= \frac{p^r z^r}{(1 - qz)^r} + \left(\frac{p}{q}\right)^{r-1} (1 - z) \left[\frac{\Phi_{st}(H, z)}{1 - z} \odot \frac{1}{(1 - qz)^r} \right] \\ &= \frac{p^r z^r}{(1 - qz)^r} + \left(\frac{p}{q}\right)^{r-1} (1 - z) \left[\frac{1}{(1 - z)^r} \odot \frac{\Phi_{st}(H, qz)}{1 - qz} \right] \\ &= \frac{p^r z^r}{(1 - qz)^r} + \left(\frac{p}{q}\right)^{r-1} \frac{1 - z}{(r - 1)!} \frac{d^{r-1}}{dz^{r-1}} \left(\frac{z^{r-1} \Phi_{st}(H, qz)}{1 - qz} \right). \end{aligned}$$

After evaluating the formal derivative at $z = 1$ we obtain

$$T_{st}(G) = \frac{rq^r}{(1-q)^{r+1}} - \frac{1}{(r-1)!} \left(\frac{p}{q}\right)^{r-1} \frac{d^{r-1}}{dz^{r-1}} \left(\frac{z^{r-1} \Phi_{st}(H, qz)}{1-qz} \right) \Big|_{z=1}. \quad \blacksquare$$

Another instance of the parallel reduction technique is given by two parallel paths of lengths n and m between s and t .

Corollary 26. *Let $G = (V, E)$ be the graph that consists of two parallel paths of lengths $n \geq 1$ and $m \geq 1$ between the vertices $s, t \in V$ where $m \geq n$. Then*

$$\Phi_{st}(G, z) = 1 - \left(\frac{p}{q}\right)^{n+m-2} \frac{1-z}{(1-q^2z)^{m+n-1}} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \binom{m-1}{\ell} (q^2z)^\ell.$$

PROOF. Let $\Phi_{st}(P_\ell, z)$ be the ordinary generating function for the s - t first arrival time in a path of length ℓ . Then the ratio $\Phi_{st}(P_\ell, z)/(1-z)$ becomes

$$\frac{\Phi_{st}(P_\ell, z)}{1-z} = \frac{1}{1-z} - \left(\frac{p}{q}\right)^{\ell-1} \frac{1}{(1-qz)^\ell}$$

by Equation (11) and the application of partial fraction decomposition. With $\ell = n$ and $\ell = m$, it follows from the parallel reduction technique that

$$\Phi_{st}(G, z) = 1 - (1-z) \left(\frac{p}{q}\right)^{n+m-2} \left[\frac{1}{(1-z)^n} \odot \frac{1}{(1-q^2z)^m} \right]. \quad (12)$$

The desired result now follows by applying Lemma 23 to (12). \blacksquare

7 Bounds for the expected first arrival time

As a result of the exponentially increasing number of equations in Corollary 10, the exact computation of the expected s - t first arrival times can be performed in reasonable time only in small networks. Hence, computational feasible bounds for $T_{st}(G)$ are of great interest.

Theorem 27. *Let $G = (V, E)$ be a graph with $s, t \in V$ and $H = (V, F)$ with $F \subseteq E$ being a subgraph of G , then $T_{st}(H) \geq T_{st}(G)$.*

PROOF. Every removal of an edge $e \in E$ of G causes an increase in the expected s - t first arrival time. \blacksquare

By considering a shortest s - t path as a subgraph of G one gets the following corollary giving an upper bound, that is not very tight in general.

Corollary 28. *Let $G = (V, E)$ be a graph and $s, t \in V$. If $d(s, t)$ is the distance of s and t in G , then the inequality*

$$\mathbb{E}[T_{st}(G)] \leq \frac{d(s, t)}{1-q}$$

always holds. \blacksquare

A known lower bound for $\tau_{st}(G)$ was presented by Lyons [5] and is connected to the ideas presented in [7]. That is, if $\text{Res}_{st}(G)$ is the electrical resistance between the vertices s and t in a given connected graph G , where the edges are assumed to have unit resistance, then

$$\frac{\text{Res}_{st}(G)}{1-q} \leq \tau_{st}(G).$$

A lower bound for $T_{st}(G)$ can be found by considering the s - t reliability polynomial of G .

Definition 29. Let $G = (V, E)$ be a connected graph with m edges and $R_{st}(G, q) = c_0 + c_1q + c_2q^2 + \dots + c_mq^m$ the s - t reliability polynomial of G , i.e. the probability that there is at least one intact s - t path in G , if the edges of G are failing independently with probability q . The s - t insertion probability, denoted by $\tilde{R}_{st}(G, q)$, is then defined as

$$\tilde{R}_{st}(G, q) = \sum_{i=1}^m \frac{c_i}{q^i - 1}.$$

Theorem 30. Let $G = (V, E)$ be a graph and $s, t \in V$. If $d(s, t)$ is the distance between s and t in G , then

$$d(s, t) - 1 + \tilde{R}_{st}(G, q) \leq T_{st}(G).$$

PROOF. For each edge $e \in E$, let $L(e)$ be independent geometric random variables, with non infection probabilities q , representing the edge lengths. Denote now by $G_k = (V, E_k)$ with $k = 0, 1, 2, \dots$ a discrete random process that gives an infinite sequence of subgraphs G_k of G , where $e \in E_k$ if and only if the event $\{L(e) \leq k\}$ occurs. Moreover, denote by I_{st} the random variable of the smallest k , so that G_k contains an s - t path.

Now the following inequality holds

$$\Pr(\{I_{st} \leq k\}) \geq \Pr(\{Z_{st} \leq d(s, t) + k - 1\}), \quad (13)$$

for all $k \geq 0$. In order to prove this estimate, we assume an elementary event E in $\{Z_{st} \leq d(s, t) + k - 1\}$, e.g. a realisation of edge lengths, so that the length of a shortest s - t path is smaller or equal than $d(s, t) + k - 1$. In order that E occurs there must be at least one s - t path in G , so that the sum of the edge lengths of the path is smaller or equal than $d(s, t) + k - 1$. Therefore all edge lengths of this s - t path must be smaller than k , as if there would be one edge length in this path that is greater than $k + 1$, then the sum of the path lengths would exceed $(k + 1) + (d(s, t) - 1)$, which contradicts the assumption $Z_{st} \leq d(s, t) + k - 1$. As all edge lengths are smaller than k , the event E is by definition also in $\{I_{st} \leq k\}$, i.e. G_k must also contains this s - t path. Furthermore the probabilities for the event E coincide in both probability spaces, so that (13) follows.

Due to (13), we find that the expectation of the random variable I_{st} is smaller than the expectation of the random variable $Z_{st} - d(s, t) + 1$. Hence,

$$\mathbb{E}[I_{st}] \leq \mathbb{E}[Z_{st} - d(s, t) + 1] = \mathbb{E}[Z_{st}] - d(s, t) + 1$$

and this gives $d(s, t) - 1 + \mathbb{E}[I_{st}]$ as a lower bound for $T_{st}(G)$.

In order to show the relationship between $\mathbb{E}[I_{st}]$ and the s - t reliability polynomial, consider the edge $e = \{u, v\}$ and the event $\{L(e) \leq k\}$. We easily find that $\Pr(\{L(e) \leq k\}) = 1 - q^k$, whenever the non insertion probability is q . Assume now that there are k parallel edges between

u and v , that are failing independently with probability q , then the probability that there is at least one non failing edge is also given by $1 - q^k$. Therefore, the probability for the event $\{I_{st} > k\}$ coincides with the probability of the event that there is no s - t path in the multigraph $G(k)$, where $G(k)$ emerges from G by replacing each edge in G with k parallel edges. Equivalently,

$$\Pr(\{I_{st} > k\}) = 1 - R_{st}(G(k), q) = 1 - R_{st}(G, q^k)$$

and we can write

$$\mathbb{E}[I_{st}] = \sum_{k \geq 1} k \left[R_{st}(G, q^k, \dots, q^k) - R_{st}(G, q^{k-1}, \dots, q^{k-1}) \right].$$

Now suppose that the s - t reliability polynomial is expressed as $R_{st}(G, q) = \sum_{i=0}^m c_i q^i$, then

$$\mathbb{E}[I_{st}] = \sum_{i=0}^m c_i \sum_{k \geq 1} k (q^{ik} - q^{i(k-1)}) = \sum_{i=1}^m \frac{c_i}{q^i - 1} = \tilde{R}_{st}(G, q)$$

and hence, $d(s, t) - 1 + \tilde{R}_{st}(G, q) \leq T_{st}(G)$. ■

8 Conclusion

We have shown that there is a one-to-one correspondence of the spread process and the stochastic shortest path problem. Furthermore, it was shown that this process can be well-described by the exponential model, when the infection probability p tends to zero. Several closed formulae or at least efficient calculation schemes for special graphs, such as complete graphs, parallel paths and s - t series-parallel graphs are presented. Finally some ideas for bounds are proposed that utilise the well-known concept of the s - t reliability polynomial, which can be efficiently computed for graphs with bounded tree-width.

The authors conjecture the following bound for the s - t first arrival time, that behaves good for values of q , that are close to 1, i.e. $\lim_{q \rightarrow 1} T_{st}(G)/\tau_{st}(G) = 1$.

Conjecture 31. Let $G = (V, E)$ be a graph, $s, t \in V$ and $T_{st}(G)$, $\tau_{st}(G)$ be the expected s - t first arrival times of the spread process and the exponential model, respectively. In this case $\tau_{st}(G) \leq T_{st}(G)$ holds for all $0 \leq q < 1$.

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