EXPONENTIAL EXTINCTION TIME OF THE CONTACT PROCESS ON FINITE GRAPHS

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ABSTRACT. We study the extinction time τ of the contact process started with full occupancy, on finite trees of bounded degree. We show that, if the infection rate is larger than the critical rate for the contact process on $\mathbb{Z}$, then, uniformly over all trees of degree bounded by a given number, the expectation of τ grows exponentially with the number of vertices. Additionally, for any sequence of growing trees of bounded degree, τ divided by its expectation converges in distribution to the unitary exponential distribution. These also hold if one considers a sequence of graphs having spanning trees with uniformly bounded degree, and provide the basis for powerful coarse-graining arguments. To demonstrate this, we consider the contact process on a random graph with vertex degrees following a power law. Improving a result of Chatterjee and Durrett [CD09], we show that, for any non-zero infection rate, the extinction time for the contact process on this graph grows exponentially with the number of vertices.

MSC 2010: 82C22, 05C80.

KEYWORDS: contact process, interacting particle systems, metastability.

1. INTRODUCTION

We study the contact process on finite graphs. Our main goal is to present robust results and techniques which justify that in the supercritical regime, the contact process survives for a time which is exponential in the number of vertices of the underlying graph. We start by introducing notations and recalling important facts.

Let $G = (V, E)$ be a graph with undirected edges. The contact process on G with parameter $\lambda > 0$ is a continuous-time Markov process $(\xi_t)_{t\geq 0}$ on the space of subsets of V whose transitions are given by:

$$\begin{array}{llll} \text{for every } x \in \xi_t, & \xi_t \to \xi_t \setminus \{x\} & \text{with rate } 1, \\ \text{for every } x \notin \xi_t, & \xi_t \to \xi_t \cup \{x\} & \text{with rate } \lambda\,|\{y \in \xi_t : \{x,y\} \in E\}|, \end{array} \tag{1.1}$$

where for a set A, we write $|A|$ to denote its cardinality.

Given $A \subseteq V$, we write $(\xi_t^A)_{t\geq 0}$ to denote the contact process started from the initial configuration equal to A. When we write $(\xi_t)_{t\geq 0}$, with no superscript, the initial configuration will either be clear from the context or unimportant. We often abuse notation and associate configurations $\xi \subseteq V$ with the corresponding indicator functions (that is, elements of $\{0,1\}^V$).

The contact process can be thought of as a model for the spread of an infection in a population. Vertices of the graph (sometimes referred to as *sites*) represent individuals. In a configuration $\xi \in \{0,1\}^V$, individuals in state 1 are said to

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be *infected*, and individuals in state 0 are *healthy*. Pairs of individuals that are connected by edges in the graph are in proximity to each other in the population. With this terminology, the dynamics can be described as follows. First, infected individuals *recover* with rate 1. Second, an individual that is infected transmits its infection to a neighbouring site with rate λ (assuming no multiple edges). Note that there is no recovery period in the model: every healthy individual is susceptible for the infection.

We begin by presenting some of the properties of the contact process for certain choices of the graph G, namely: the lattice $\mathbb{Z}^d$, d-regular infinite trees, and the finite counterparts of these graphs. For proofs of these properties and a detailed treatment of the topic, we refer the reader to [Li1, Li2].

On $\mathbb{Z}^d$ (equipped with its usual nearest-neighbour graph structure), there exists a number $\lambda_c = \lambda_c(\mathbb{Z}^d)$ such that the following holds. If $\lambda \leq \lambda_c$, then the contact process *dies out*, meaning that for any finite initial configuration, the empty configuration $\underline{0}$ is almost surely eventually reached. On the other hand, if $\lambda > \lambda_c$, then the contact process *survives strongly*, that is, for any non-empty initial configuration (and any $x \in V$), $P\left[\xi_t(x) = 1 \text{ for arbitrarily large values of } t\right] > 0$.

The interest in the contact process on trees was prompted when it was discovered that death and strong survival are not the only possibilities in this case [Pe92]. For $d \geq 2$, let T_d denote the infinite $(d+1)$-regular tree with a distinguished vertex o called the root. The different phases of the process are captured by two constants $0 < \lambda_1(T_d) < \lambda_2(T_d) < +\infty$. If $\lambda \leq \lambda_1$, then the contact process dies out, while if $\lambda > \lambda_2$, then it survives strongly. If $\lambda \in (\lambda_1,\ \lambda_2]$, then the process *survives weakly*. That is, if started with a non-empty finite initial configurantion, then the infection has positive probability of always being present on the graph, yet each individual site eventually becomes permanently healthy.

If G is a finite graph, then the contact process on G dies out. This does not however rule out qualitative changes of the behaviour of the contact process as λ varies, as we now describe. To be precise, for $A \subseteq V$, let us write $\tau_G^A = \inf\{t : \xi_t^A = \underline{0}\}$ for the *extinction time* of the process started from occupancy in A. We may omit the subscript G when the context is clear enough, and simply write τ when the contact process is started from full occupancy, that is, $\tau = \tau^{\underline{1}}$. Consider the graph $\{0, \ldots, n\}^d$ (viewed as a subgraph of $\mathbb{Z}^d$) and the distribution of τ for this graph, as n goes to infinity. If $\lambda < \lambda_c(\mathbb{Z}^d)$, then $\tau/\log n$ converges in probability to a constant [DL88], [Ch94]; see also Theorem 3.3 in [Li2]. If $\lambda > \lambda_c$, then $\log E[\tau]/n^d$ converges to a positive constant, and $\tau/E[\tau]$ converges in distribution to the unit exponential distribution [DS88, Mo93, Mo99]. In particular, when $\lambda > \lambda_c$, the order of magnitude of the extinction time is exponential in the number of vertices of the graph; the process is said to exhibit *metastability*, meaning that it persists for a long time in a state that resembles an equilibrium and then quickly moves to its true equilibrium ($\underline{0}$ in this case). Metastability for the contact process in this setting was also studied in [CGOV84] and [Sc85]. Finally, if $d = 1$, is is also known that, if $\lambda = \lambda_c$, then $\tau/n \to \infty$ and $\tau/n^4 \to 0$ in probability [DST89].

For the case of finite trees, the picture is less complete, and the available results concerning the extinction time are contained in [St01]. Fix $d \geq 2$, let T_d^h be the finite subgraph of T_d defined by considering up to h generations from the root and again take the contact process started from full occupancy on this graph, with associated extinction time τ. If $\lambda < \lambda_2$, then there exist constants $c, C > 0$ such that $P(ch \leq \tau \leq Ch) \to 1$ as $h \to \infty$. If $\lambda > \lambda_2$, then for any $\sigma < 1$ there exist $c_1, c_2 > 0$ such that $P\left[\tau > c_1 e^{c_2(\sigma d)^h}\right] \to 1$ as $h \to \infty$. This implies that τ is at least as large as a stretched exponential function of the number of vertices, $(d+1)^h$.

As far as we know, no rigorous results are available concerning finite graphs which are not regular.

For $n \in \mathbb{N}$ and $d > 0$, let $\Lambda(n, d)$ be the set of all trees with n vertices and degree bounded by d, and let $\mathcal{G}(n, d)$ be the set of graphs having a spanning tree in $\Lambda(n, d)$. In this paper, we prove the following theorems.

Theorem 1.1. *For any $d \geq 2$ and $\lambda > \lambda_c(\mathbb{Z})$, there exists $c > 0$ such that, for any n large enough,*

$$\inf_{T \in \Lambda(n,d)} \frac{\log E[\tau_T]}{n} \geq c.$$

Theorem 1.2. *Let $d \geq 2$, $\lambda > \lambda_c(\mathbb{Z})$, and $G_n \in \mathcal{G}(n, d)$. The distribution of $\tau_{G_n}/E[\tau_{G_n}]$ converges to the unitary exponential distribution as n tends to infinity.*

Theorem 1.3. *Let $d \geq 2$ and $\lambda > \lambda_c(\mathbb{Z})$. There exists $c > 0$ such that*

$$\inf_{T \in \Lambda(n,d)} P\left[\tau_T \geq e^{cn}\right] \to 1 \text{ as } n \to \infty.$$

Theorem 1.2 is a (weak) way of exposing the metastability of the contact process (see part (3) of Proposition 3.2 for a finer statement ; note also that from this statement, it is easy to extend the above theorems to more general initial configurations than full occupancy, with appropriate modifications). In Theorems 1.1 and 1.3, one can replace $\Lambda(n, d)$ by the set of all graphs having a subgraph in $\Lambda(n, d)$, and in particular, one can replace $\Lambda(n, d)$ by $\mathcal{G}(n, d)$. For instance, the above results cover the case of any sequence of increasingly large connected subsets of $\mathbb{Z}^d$. At the cost of requiring $\lambda > \lambda_c(\mathbb{Z})$, we thus recover and extend previously mentionned results, without any strong assumption on the regularity of the graph. For such values of λ, this shows in particular that on regular trees with finite depth, the extinction time is actually exponentially large in the number of vertices.

This is however not quite the way in which we think our results are most useful. Rather, they are the basic ingredient of a general strategy to prove that the extinction time of the contact process is exponentially large in the number of vertices as soon as the infection parameter is above the natural critical value of the particular graphs we consider (instead of $\lambda_c(\mathbb{Z})$). We now expose this strategy on certain random graphs whose degree distribution follows a power law. The case of finite homogeneous trees will be discussed in a subsequent work.

We consider Newman-Strogatz-Watts (NSW) random graphs, that are defined as follows [NSW01]. For any $n \in \mathbb{N}$, we construct a graph G^n on n vertices. The vertex set is simply $\{1, \ldots, n\}$. The random set of edges will be constructed from a probability p on $\{3, 4, \ldots\}$ with the property that, for some $a > 2$, $c_0 = \lim_{m\to\infty} p(m)/m^a$ exists and is in $(0, \infty)$. We let $d_1, \cdots, d_n$ be independent random variables distributed according to p, and conditioned on the event that $d_1 + \cdots + d_n$ is even. Next, from each vertex $i \in \{1, \ldots, n\}$ we place d_i *half-edges*; when two half-edges are connected, an edge is formed. We pair up the $d_1 + \cdots + d_n$ half-edges in a random way that is uniformly chosen among all possibilities.

Let us write $\mathbb{P}$ to denote a probability measure under which both the random graph and the contact process on this graph are defined. In [CD09], it is shown that, for any $\lambda > 0$ and any $\delta > 0$, we have $\mathbb{P}[\tau_{G^n} \geq e^{n^{1-\delta}}] \to 1$ as $n \to \infty$ (see also [BBCS05] for earlier results). In particular, the critical infection parameter for these graphs is 0. We show:

Theorem 1.4. *For any $\lambda > 0$, there exists $c > 0$ such that*

$$\mathbb{P}\left[\tau_{G^n} \geq e^{cn}\right] \to 1 \text{ as } n \to \infty.$$

Although it would be simple to deduce Theorem 1.4 from Theorem 1.3 assuming $\lambda > \lambda_c(\mathbb{Z})$, we stress again that Theorem 1.4 covers any non-zero infection parameter. We think that Theorem 1.4 is true for all $a > 1$, but we only give the proof for $a > 2$, which is the harder case (when we increase a, the degrees of the vertices become stochastically smaller, so the graph becomes less connected).

For finite boxes of $\mathbb{Z}^d$, the proof that the extinction time is exponential in the number of vertices relies on a coarse-graining argument. This coarse-graining enables to map the initial contact process into a coarse-grained (discrete-time) contact process with an increasingly large infection parameter. The remarkable feature of $\mathbb{Z}^d$ is its scale invariance, which ensures that the coarse-grained graph is still $\mathbb{Z}^d$ (or rather, a finite box of $\mathbb{Z}^d$). Now, simple percolation arguments show that on finite boxes of $\mathbb{Z}^d$, the time of survival is exponential in the number of vertices if the infection parameter is sufficiently large, say larger than λ_{target}. To sum up, the proof for finite boxes of $\mathbb{Z}^d$ consists in defining a coarse-graining scheme, and then in fixing a finite length of coarse-graining such that the coarse-grained system has infection parameter larger than λ_{target}.

For NSW graphs, coarse-grained blocks will consist of certain *stars*, that is, vertices with a given large number of neighbours. The difficulty in trying to adapt the strategy to these graphs (or to finite homogeneous trees) is that there is no easy scale invariance as on $\mathbb{Z}^d$. It then becomes a very delicate matter to control the geometry of the coarse-grained graph, and thus to define a suitable equivalent of λ_{target}. However, Theorems 1.1 and 1.3 roughly tell us that *we do not need to control this geometry*, and that we can choose $\lambda_{\text{target}} = \lambda_c(\mathbb{Z})$.

The proof in [CD09] is also based on a coarse-graining approach. There, the question of controlling the coarse-grained geometry was bypassed by choosing the coarse-grained scale so large as to ensure that the coarse-grained graph was a complete graph. Since the diameter of the NSW graphs goes to infinity, this cannot be ensured unless the length scale of the coarse-graining diverges. In other words, in this approach, stars should have more and more vertices as n increases, and the number of points in the coarse-grained scale must thus be $o(n)$. With our approach, we can choose instead a large but *fixed* size for the relevant stars in the graph, so that the coarse-grained graph still contains of order n sites, and Theorem 1.4 follows from this construction.

Organization of the paper. Section 2 is a brief reminder on some properties of the contact process that will be useful for our purposes. In Section 3, we show a weaker version of Theorem 1.1, which states that the expectation of the extinction time is larger than e^{cn^α} for some $\alpha > 0$. In order to do this, we consider two cases: either the tree contains a large segment, or it contains a large number of disjoint smaller segments. In the first case, the result follows from the known behavior of the extinction time on finite intervals of $\mathbb{Z}$. In the second case, we adapt an argument of [CD09] and show that, even if the segments are not too large, the time scale of extinction in individual segments is large enough for the infection to spread to other, possibly inactive, segments, so that the segments can jointly sustain activity for the desired amount of time. At this point, using a general metastability argument from [Mo93], we prove Theorem 1.2.

Given a tree $T \in \Lambda(n, d)$, we decompose it into two subtrees T_1, T_2 by removing an edge; we argue that this can be done so that T_1 and T_2 both contain a non-vanishing proportion of the vertices of T. In Section 4, we compare the contact process $(\xi_t)_{t \geq 0}$ on T to a pair of processes $(\zeta_{T_1,t})_{t \geq 0}$ on T_1 and $(\zeta_{T_2,t})_{t \geq 0}$ on T_2. The process ζ_{T_1} evolves as a contact process on T_1 until extinction. Once extinct, the process stays extinct for some time, and then rises from the ashes (we call it a *Phoenix contact process*). This rebirth of the process reflects the fact that, as long as the true process ξ has not died out, the tree T_1 constantly receives new infections that can restore its activity. The process ζ_{T_2} evolves independently, following the same rules. We show that the true process ξ dominates $\zeta_{T_1} \cup \zeta_{T_2}$ up to the extinction of ξ, with probability close to 1. With this comparison at hand, we argue that, modulo a factor that is polynomial in the number of vertices, the expected extinction time for T is larger than the product of the expected extinction times for T_1 and T_2. This, together with the lower bound e^{cn^α} mentioned in the previous paragraph, is then used to prove Theorem 1.1, from which Theorem 1.3 follows.

In Section 5, we re-state some of the results explained above for a discrete-time version of the contact process. Finally, we prove Theorem 1.4 in Section 6.

Notations. For $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the integer part of x. When talking about the size of a graph, we always mean its number of vertices.

2. A reminder on the contact process

We start this section by presenting the graphical construction of the contact process and its self-duality property. Fix a graph $G = (V, E)$ and $\lambda > 0$. We take the following family of independent Poisson point processes on $[0, \infty)$:

$$\begin{array}{ll} (R^x) : x \in V & \text{with rate } 1; \\ (N^e) : e \in E & \text{with rate } \lambda. \end{array}$$

Let H denote a realization of all these processes. Given $x, y \in V,\ s \leq t$, we say that x and y are connected by an *infection path in H* (and write $(x, s) \leftrightarrow (y, t)$ in H) if there exist times $t_0 = s < t_1 < \cdots < t_k = t$ and vertices $x_0 = x, x_1, \ldots, x_{k-1} = y$ such that

- $R^{x_i} \cap (t_i,\ t_{i+1}) = \varnothing$ for $i = 0, \ldots, k-1$;
- $\{x_i, x_{i+1}\} \in E$ for $i = 0, \ldots, k-2$;
- $t_i \in N^{x_{i-1}, x_i}$ for $i = 1, \ldots, k-1$.

Points of the processes (R^x) are called *recovery marks* and points of (N^e) are *links*; infection paths are thus paths that traverse links and do not touch recovery marks. H is called a *Harris system*; we often omit dependence on H. For $A, B \subseteq V$, we write $A \times \{s\} \leftrightarrow B \times \{t\}$ if $(x, s) \leftrightarrow (y, t)$ for some $x \in A,\ y \in B$. We also write $A \times \{s\} \leftrightarrow (y, t)$ and $(x, s) \leftrightarrow B \times \{t\}$. Finally, given another set $C \subseteq V$, we write $A \times \{s\} \leftrightarrow B \times \{t\}$ *inside* C if there is an infection path from a point in $A \times \{s\}$ to a point in $B \times \{t\}$ and the vertices of this path are entirely contained in C.

Given $A \subseteq V$, put

$$\xi_t^A(x) = \mathbb{1}_{\{A \times \{0\} \leftrightarrow (x,t)\}} \text{ for } x \in V,\ t \geq 0 \tag{2.1}$$

(here and in the rest of the paper, $\mathbb{1}$ denotes the indicator function). It is well-known that the process $(\xi_t^A)_{t \geq 0} = (\xi_t^A(H))_{t \geq 0}$ thus obtained has the same distribution as that defined by the infinitesimal generator (1.1). The advantage of (2.1) is that it allows us to construct in the same probability space versions of the contact processes with all possible initial distributions. From this joint construction, we also obtain the *attractiveness* property of the contact process: if $A \subseteq B \subseteq V$, then $\xi_t^A(H) \subseteq \xi_t^B(H)$ for all t. From now on, we always assume that the contact process is constructed from a Harris system, and will write $P_{G,\lambda}$ to refer to a probability measure under which such a system (on graph G and with rate λ) is defined; we usually omit G, λ.

Now fix $A \subseteq V,\ t > 0$ and a Harris system H. Let us define the *dual process* $(\hat{\xi}_s^{A,t})_{0 \leq s \leq t}$ by

$$\hat{\xi}_s^{A,t}(y) = \mathbb{1}_{\{(y, t-s) \leftrightarrow A \times \{t\} \text{ in } H\}}.$$

If $A = \{x\}$, we write $(\hat{\xi}_s^{x,t})$. This process satisfies two important properties. First, its distribution (from time 0 to t) is the same as that of a contact process with same initial configuration. Second, it satisfies the *duality equation*

$$\xi_t^A \cap B \neq \varnothing \text{ if and only if } A \cap \hat{\xi}_t^{B,t} \neq \varnothing. \tag{2.2}$$

In particular,

$$\xi_t^{\underline{1}}(x) = 1 \text{ if and only if } \hat{\xi}_t^{x,t} \neq \varnothing, \tag{2.3}$$

where $(\xi_t^{\underline{1}})$ is the process started from full occupancy.

We now recall classical results about the contact process on an interval.

Proposition 2.1. *For $n \in \mathbb{Z}_+$, $A \subseteq \mathbb{Z}_+$, let*

$$\sigma_n^A = \inf\left\{t \geq 0 : \xi_t^A(n) = 1\right\},$$

where $(\xi_t^A)_{t\geq 0}$ denotes the contact process on $\mathbb{Z}_+$ with initial configuration A. For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $\overline{c}_1 > 0$, n_0 such that the following results hold.

(1) *For any n,*

$$P\left[\sigma_n^{\{0\}} < \frac{n}{\overline{c}_1}\right] > \overline{c}_1.$$

(2) *For any $A \subseteq \{0, \ldots, n\}$ and any $n \geq n_0$,*

$$P\left[\sigma_0^A + \sigma_n^A \geq \frac{n}{\overline{c}_1},\ \xi_{n/\overline{c}_1}^A \neq \underline{0}\right] \leq e^{-n}.$$

(3) *If $(\xi_{n,t}^{\underline{1}})_{t\geq 0}$ denotes the contact process on $\{0, \ldots, n\}$ started with full occupancy, then for any $n \geq n_0$ and any $t \geq 0$, we have*

$$P\left[\xi_{n,t}^{\underline{1}} = \underline{0}\right] \leq te^{-\overline{c}_1 n}.$$

This follows from the classical renormalization argument that compares the contact process with supercritical oriented percolation, see for instance the proof of [Li1, Corollary VI.3.22].

3. Metastability

In this section, we show that for $\lambda > \lambda_c(\mathbb{Z})$, the extinction time of the contact process is at least e^{cn^α} for some $\alpha > 0$, uniformly over $T \in \Lambda(n, d)$. This is clear by Proposition 2.1 if the tree has diameter at least n^α. Else, we rely on a recursive decomposition of the tree into many subtrees each of which having size at least $\sqrt{n}$. We then pick long intervals (of logarithmic size) inside each of these trees, and study how these can sustain the infection. For a suitable choice of the parameters, the time it takes for such an interval, isolated from the rest of the graph, to turn extinct, is much larger than the time it takes for the infection to travel from one interval to another, since the diameter of the whole graph is assumed small. Over time, the number of infected intervals can be compared to a random walk on the integers with a drift to the right. Analyzing this walk gives the desired result. Using this construction, we also show metastability of the contact process, in the sense that after time n^2, either the contact process is extinct, or it is equal to the contact process that was started from full occupancy (and thus the initial configuration is forgotten). Since n^2 is much smaller than the extinction time, this is a quantitative way of presenting the metastability of the contact process, and indeed we will conclude the section by proving Theorem 1.2.

We begin with the following basic graph-theoretic observation, on which our recursive decomposition of the tree is based.

Lemma 3.1. *For a tree $T \in \Lambda(n,d)$, there exists an edge whose removal separates T into two subtrees T_1 and T_2 both of size at least $\lfloor n/d \rfloor$.*

Proof. Associate to each edge the value of the smallest cardinality of the two subtrees resulting from the edge's removal. Let $\{x,y\}$ be an edge having maximal value. We suppose that the subgraph T_y containing vertex y is the smaller and that the value of its subtree is no more than $\lfloor n/d \rfloor - 1$. Let the remaining edges of vertex x be $\{x,x_1\}$, $\{x,x_2\}, \cdots \{x,x_r\}$, where $r \le d-1$. Let T_j be the subtree containing x_j obtained by removing the edge $\{x,x_j\}$, and let n_j be its cardinality. By maximality, all the n_j must be no more than $\lfloor n/d \rfloor - 1$, but we also have

$$|T_y| = |T \setminus (\{x\} \cup T_1 \cup \cdots \cup T_r)| = n - (1 + n_1 + n_2 + \cdots + n_r) \le \lfloor n/d \rfloor - 1.$$

That is, $n \le (d-1)(\lfloor n/d \rfloor - 1) + \lfloor n/d \rfloor \le n - (d-1)$, a contradiction (the case $d = 1$ being trivial). $\square$

Proposition 3.2. *For any $\lambda > \lambda_c(\mathbb{Z})$, there exists $\alpha > 0$ and $\overline{c}_2 > 0$ such that the following holds.*

(1) *For any n large enough, any $T \in \Lambda(n,d)$, any non-empty $A \subseteq T$, one has*

$$P\left[\tau^A \ge e^{\overline{c}_2 n^\alpha}\right] \ge \overline{c}_2.$$

In particular, $E[\tau^A] \ge \overline{c}_2 e^{\overline{c}_2 n^\alpha}$.

(2) *Moreover,*

$$P\left[\tau \ge e^{\overline{c}_2 n^{\alpha/2}}\right] \ge 1 - e^{-\overline{c}_2 n^{\alpha/2}},$$

where we recall that we write τ as a shorthand for $\tau^{\underline{1}}$.

(3) *For n large enough and any $G \in \mathcal{G}(n,d)$, if the contact process on G started with an arbitrary non-empty configuration survives up to time n^2, then the chance that at this time, it is equal to the contact process starting from full occupancy, is at least $1 - e^{-n^{-\alpha/2}}$.*

From now on, d is fixed and we consider a tree T of maximal degree d and size $n \to \infty$. Let $\beta > 0$ to be determined later, not depending on n. Applying Lemma 3.1 repeatedly $\beta \log n$ times, we obtain $L_n = 2^{\beta \log n}$ disjoint subtrees each of size at least $\frac{n}{(2d)^{\beta \log n}} \ge \sqrt{n}$, provided $\beta \le 1/(2\log(2d))$ (for clarity, we simply assume that L_n is an integer, without writing that the integer part should be taken). We write $T_1, \ldots, T_{L_n}$ for the trees thus obtained.

Since the tree T has maximal degree bounded by d, so do the subtrees (T_j). Now, the size of a tree with maximal degree d is at most

$$1 + d + \ldots + d^{\mathsf{diam}} = \frac{d^{\mathsf{diam}+1} - 1}{d-1},$$

where diam denotes its diameter. As a consequence, for n large enough, each T_j must have a diameter at least $\frac{\log n}{4 \log d}$, and thus contain a path of $\frac{\log n}{4 \log d}$ distinct vertices. We write I_j to denote such a path, which we identify with an interval of length $\frac{\log n}{4 \log d}$.

In what follows, we will distinguish between the two possibilities:

(A) the diameter of T is at least n^α,

(B) the diameter of T is less than n^α,

where $\alpha > 0$ is a fixed number whose value will be specified in the course of the proof. It is worth keeping in mind that α will be chosen much smaller than β, itself chosen as small as necessary.

Proof of parts (1-2) of Proposition 3.2. Assume that the tree T satisfies (A). For part (1), by attractiveness, it suffices to consider initial configurations with a single occupied site z. Condition (A) ensures that one can find an interval of length at least n^α. We write $[x,y]$ to denote such an interval, with x and y its endpoints. Consider the event that within time $2n/\overline{c}_1$, the contact process has infected site x, and thereafter the contact process begun at this time restricted to $[x,y]$ and with only x occupied has infected y. This event has probability at least $\overline{c}_1^2$ by part (1) of Proposition 2.1. If this event occurs, then at time $2n/\overline{c}_1$, the contact process on T dominates the contact process on $[x,y]$ begun with full occupancy. The desired bound now follows from bounds on survival times for supercritical contact processes on an interval, see part (3) of Proposition 2.1. Part (2) also follows using the interval $[x,y]$ and part (3) of Proposition 2.1.

We now consider that the graph satisfies (B), and adapt an approach due to [CD09]. For any $A \subseteq I_i$, we write $(\xi^A_{i,t})_{t\ge 0}$ for the contact process on I_i with initial configuration A, and define

$$p_i(A) = P\left[\xi^A_{i,Kn^\alpha} = \xi^{\underline{1}}_{i,Kn^\alpha} \neq \underline{0}\right],$$

where $K = 2/\overline{c}_1$. For any $i \le L_n$, we say that the interval I_i is *good at time* t if $p_i(\xi_t) \ge 1 - n^{-2\beta}$, where for simplicity we write $p_i(\xi_t)$ instead of $p_i(\xi_t \cap I_i)$.

For $k \in \mathbb{N}$, we let $X_k \in \{0,\ldots L_n\}$ be the number of good intervals at time kKn^α. For $i \le L_n$ and $k \ge 0$, let us write $\mathcal{E}_{i,k}$ for the event that the interval I_i is good at time kKn^α. By definition,

$$P[\mathcal{E}_{i,k+1} \mid \mathcal{E}_{i,k}] = P[p_i(\xi_{(k+1)Kn^\alpha}) \ge 1 - n^{-2\beta} \mid \mathcal{E}_{i,k}].$$

By attractiveness, the latter is larger than

$$\begin{aligned}
&P\left[p_i\left(\xi^{\xi_{kKn^\alpha}}_{i,Kn^\alpha}\right) \ge 1 - n^{-2\beta} \mid \mathcal{E}_{i,k}\right] \\
&\quad \ge P\left[p_i\left(\xi^{\underline{1}}_{i,Kn^\alpha}\right) \ge 1 - n^{-2\beta}, \xi^{\xi_{kKn^\alpha}}_{i,Kn^\alpha} = \xi^{\underline{1}}_{i,Kn^\alpha} \mid \mathcal{E}_{i,k}\right] \\
&\quad \ge 1 - P\left[p_i\left(\xi^{\underline{1}}_{i,Kn^\alpha}\right) < 1 - n^{-2\beta}\right] - \underbrace{P\left[\xi^{\xi_{kKn^\alpha}}_{i,Kn^\alpha} \neq \xi^{\underline{1}}_{i,Kn^\alpha} \mid \mathcal{E}_{i,k}\right]}_{\le n^{-2\beta}}.
\end{aligned}$$

We now argue that for n large enough,

$$P\left[p_i\left(\xi^{\underline{1}}_{i,Kn^\alpha}\right) < 1 - n^{-2\beta}\right] \le n^{-2\beta}. \tag{3.1}$$

Letting $(\xi^{A,t}_s)_{s\ge t}$ be the contact process started at time t with A occupied, one can rewrite the probability on the l.h.s. of (3.1) as

$$\begin{aligned}
P\left[P\left[\xi^{\xi^{\underline{1}}_{i,Kn^\alpha}, Kn^\alpha}_{i,2Kn^\alpha} \neq \xi^{\underline{1},Kn^\alpha}_{i,2Kn^\alpha} \text{ or } \xi^{\underline{1},Kn^\alpha}_{i,2Kn^\alpha} = \underline{0} \mid \xi^{\underline{1}}_{i,Kn^\alpha}\right] > n^{-2\beta}\right] \\
\le n^{2\beta} P\left[\xi^{\underline{1}}_{i,2Kn^\alpha} \neq \xi^{\underline{1},Kn^\alpha}_{i,2Kn^\alpha} \text{ or } \xi^{\underline{1}}_{i,2Kn^\alpha} = \underline{0}\right].
\end{aligned}$$

By part (3) of Proposition 2.1, the contact process on I_i started with full occupancy survives up to time $2Kn^\alpha$ with probability larger than

$$1 - 2Kn^\alpha \exp\left(-\overline{c}_1 |I_i|\right) = 1 - 2Kn^{\alpha - \overline{c}_1/4\log d}.$$

On this event, the probability that it gets coupled with the contact process started from full occupancy at time Kn^α within time Kn^α is larger than $1 - e^{-|I_i|} = 1 - n^{-\overline{c}_1/4\log d}$ by part (2) of Proposition 2.1. Hence, the l.h.s. of (3.1) is bounded by

$$n^{2\beta}\left(2Kn^{\alpha - \overline{c}_1/4\log d} + n^{-\overline{c}_1/4\log d}\right),$$

which can be made smaller than $n^{-2\beta}$ if $0 < \alpha \ll \beta \ll 1$ are suitably chosen. To sum up, we have shown that for all n large enough,

$$P[\mathcal{E}_{i,k+1} \mid \mathcal{E}_{i,k}] \geq 1 - 2n^{-2\beta}.$$

Moreover, an examination of the above proof shows that this estimate still holds if we condition also on the state of the intervals $(I_j)_{j\neq i}$. In other words, we have shown that for any $x \geq 0$,

$$P\left[X_{k+1} \leq X_k - x \mid X_k\right] \leq P\left[\mathsf{Bin}(L_n, 2n^{-2\beta}) \geq x\right], \tag{3.2}$$

where $\mathsf{Bin}(n,p)$ denotes a binomial random variable of parameters n and p. Note also that with probability tending to 1, all the intervals that are good at time kKn^α remain so at time $(k+1)Kn^\alpha$.

We now show that if $l < L_n$, then

$$P\left[X_{k+1} - X_k \geq 1 \mid \xi_{kKn^\alpha} \neq \underline{0}, X_k = l\right] \geq \frac{\overline{c}_1^2}{2}. \tag{3.3}$$

(Obviously, if $X_k = l \neq 0$, then it must be that $\xi_{kKn^\alpha} \neq \underline{0}$.) By the Markov property, it suffices to show (3.3) for $k = 0$. We thus consider a non-empty initial configuration A with $l < L_n$ good intervals. Let $I_i = [x,y]$ be an interval that is not good at time 0. With probability tending to 1, all good intervals remain good at time Kn^α, so we only need to study the probability that I_i becomes good. The probability of the complementary event is

$$P\left[p_i\left(\xi^A_{Kn^\alpha}\right) < 1 - n^{-2\beta}\right] \leq P\left[\xi^A_{Kn^\alpha} < \xi^{\underline{1}}_{i,Kn^\alpha}\right] + P\left[p_i\left(\xi^{\underline{1}}_{i,Kn^\alpha}\right) < 1 - n^{-2\beta}\right].$$

Inequality (3.1) ensures that the last probability becomes arbitrarily small for n large enough. It thus suffices to show that

$$P\left[\xi^A_{Kn^\alpha} < \xi^{\underline{1}}_{i,Kn^\alpha}\right] \leq 1 - \overline{c}_1^2. \tag{3.4}$$

Let $z \in A$. We consider the event $\mathcal{E}_1$ that within time $Kn^\alpha = 2n^\alpha/\overline{c}_1$, the contact process has infected x, and thereafter the contact process restricted to $[x,y]$ and with only x occupied has reached y. Note that the diameter of T is less than n^α (so that there exists a path of length less than n^α linking z to x), while the length of I_i is $\frac{\log n}{4 \log d} \leq n^\alpha$. As a consequence, part (1) of Proposition 2.1 ensures that the event $\mathcal{E}_1$ has probability at least $\overline{c}_1^2$. Since on the event $\mathcal{E}_1$, we have $\xi^A_{Kn^\alpha} \geq \xi^{\underline{1}}_{i,Kn^\alpha}$, this justifies (3.4), and thus also (3.3).

The conclusion will now follow from (3.2) and (3.3) by a comparison with a random walk on $\mathbb{Z} \cap (-\infty, L_n]$ with a drift to the right. The necessary information on this drifted walk is contained in the following lemma.

Lemma 3.3. *Let $(Z_l)_{l\in\mathbb{N}}$ be the random walk on $\mathbb{Z} \cap (-\infty, L_n]$ with transition probabilities*

$$P[Z_{l+1} = x + k \mid Z_l = x < L_n] = \left| \begin{array}{ll} 0 & \text{if } k > 1, \\ \overline{c}_1^2/2 & \text{if } k = 1, \\ e^{-n^{-\beta}}\, n^{-|k|\beta}/|k|! & \text{if } k \leq -1. \end{array} \right.$$

Let also H_0 be the hitting time of $\mathbb{Z}_- = \mathbb{Z} \cap (-\infty, 0]$, and H_L be the hitting time of L_n. For any n large enough and any $x \leq L_n$, we have

$$P\left[H_0 < H_L \mid Z_0 = x\right] \leq n^{-x\beta/2}.$$

Let us postpone the proof of this lemma, and see how it enables us to conclude. From (3.3), we learn that whatever the initial non-empty configuration, we have $X_1 \geq 1$ with probability bounded away from 0. On this event, we want to couple (X_k) with the random walk of the lemma, so that $X_{k-1} \geq Z_k$ for every $k \geq 0$. In the r.h.s. of (3.2), a binomial random variable appears, while jumps to the left in the lemma follow a Poisson random variable. Since a Bernoulli random variable of parameter p is stochastically dominated by a Poisson random variable of parameter $-\log(1-p)$, it follows that $\mathsf{Bin}(L_n, 2n^{-2\beta})$ is stochastically dominated by a Poisson random variable of parameter

$$-L_n \log(1 - 2n^{-2\beta}) = -n^{\beta \log 2} \log(1 - 2n^{-2\beta}) \leq n^{-\beta}.$$

This and (3.3) guarantee the existence of the coupling. With probability at least $1 - n^{-\beta/2} \geq 1/2$, the random walk hits L_n before entering $\mathbb{Z}_-$. The proof of part (1) will be complete if we can argue that starting from L_n, with probability close to 1, the walk needs to hit and exit L_n at least e^{n^α} times before reaching $\mathbb{Z}_-$. Let us consider a sequence of e^{n^α} excursions from L_n, and show that with high probability, none of them visits $\mathbb{Z}_-$. The first jump out of L_n is distributed according to a Poisson random variable of parameter $n^{-\beta}$, which (for convenience) may be dominated by an exponential random variable of parameter 1. With probability tending to 1, the maximum over e^{n^α} such random variables does not exceed $n^{2\alpha} \leq L_n/4$. In view of the lemma, given an excursion whose first step has size smaller than $L_n/4$, the excursion will visit $\mathbb{Z}_-$ with probability smaller than $n^{-3L_n\beta/4} \leq e^{-2n^\alpha}$, and this finishes the proof of part (1).

As for part (2), the argument is similar, except that in this case $X_0 = L_n$. Consider $e^{n^{\alpha/2}}$ excursions from L_n. With probability at least $1 - e^{-n^{\alpha/2}}$, none of these excursions has size larger than $n^{2\alpha} \leq L_n/4$. As noted above, given an excursion from L_n whose first step has size smaller than $L_n/4$, the excursion will visit $\mathbb{Z}_-$ with probability smaller than $n^{-3L_n\beta/4} \leq e^{-2n^\alpha}$, thus finishing the proof of part (2). □

Proof of Lemma 3.3. Let $h(x) = P\left[H_0 < H_L \mid Z_0 = x\right]$, $\tilde{h}(x) = n^{-x\beta/2}$, and let $\mathcal{L}$ be the generator of the random walk:

$$\mathcal{L}f(x) = \frac{\overline{c}_1^2}{2}(f(x+1) - f(x)) + e^{-n^{-\beta}} \sum_{k=1}^{+\infty} \frac{n^{-k\beta}}{k!}(f(x-k) - f(x)) \qquad (x < L_n).$$

For $x \in \mathbb{Z} \cap (0, L_n)$, we have $\mathcal{L}h(x) = 0$. On the other hand, for such x, we have

$$\begin{aligned} \mathcal{L}\tilde{h}(x) &= \frac{\overline{c}_1^2}{2}\left(n^{-\beta/2} - 1\right)\tilde{h}(x) + e^{-n^{-\beta}} \sum_{k=1}^{+\infty} \frac{n^{-k\beta}}{k!}(n^{k\beta/2} - 1)\tilde{h}(x) \\ &\leq \frac{\overline{c}_1^2}{2}\left(n^{-\beta/2} - 1\right)\tilde{h}(x) + \sum_{k=1}^{+\infty} \frac{n^{-k\beta}}{k!} n^{k\beta/2}\tilde{h}(x) \\ &\leq \left[\frac{\overline{c}_1^2}{2}\left(n^{-\beta/2} - 1\right) + e^{n^{-\beta/2}} - 1\right]\tilde{h}(x), \end{aligned}$$

so $\mathcal{L}\tilde{h}(x) \leq 0$ as soon as n is large enough. As a consequence, $\mathcal{L}(h - \tilde{h}) \geq 0$ on $\mathbb{Z} \cap (0, L_n)$. By the maximum principle,

$$\max_{\mathbb{Z}\cap(0,L_n)} (h - \tilde{h}) \leq \max_{\mathbb{Z}_-\cup\{L_n\}} (h - \tilde{h}) = 0,$$

and the lemma is proved. □

The following observation will be useful in the proof of part (3) of Proposition 3.2.

Remark 3.4. Let a Harris system for the contact process on some graph $G = (V, E)$ be given (and fixed). Assume that $\xi_t^A = \xi_t^1$ for some $A \subseteq V$ and $t > 0$. This implies that in the Harris system, any infection path from $V \times \{0\}$ to $V \times \{t\}$ intersects the offspring of elements of A. Let $(\hat{\xi}_s^{B,t})_{0 \leq s \leq t}$ be the dual contact process for time t, started with configuration B. If furthermore, $\hat{\xi}^{B,t}$ survives up to time t, then there must exist an infection path from $A \times \{0\}$ to $B \times \{t\}$.

Proof of part (3) of Proposition 3.2. We continue with case (B), but considering that T is the spanning tree of some graph $G = (V, E)$. For an arbitrary $z \in V$, we wish to bound

$$P\left[\xi^z_{n^2} \neq \xi^{\underline{1}}_{n^2},\ \xi^z_{n^2} \neq \underline{0}\right].$$

The probability above is equal to $P[\exists y : \xi^z_{n^2}(y) \neq \xi^{\underline{1}}_{n^2}(y),\ \xi^z_{n^2} \neq \underline{0}]$. For any fixed y, we will thus bound

$$P\left[\xi^z_{n^2}(y) \neq \xi^{\underline{1}}_{n^2}(y),\ \xi^z_{n^2} \neq \underline{0}\right]. \tag{3.5}$$

Letting $(\hat{\xi}^{y,n^2}_t)_{0 \le t \le n^2}$ be the dual contact process for time n^2 started with configuration $\{y\}$, we can rewrite this probability as

$$P\left[\xi^z_{n^2}(y) = 0,\ \hat{\xi}^{y,n^2}_{n^2} \neq \underline{0},\ \xi^z_{n^2} \neq \underline{0}\right].$$

As in the proof of part (1), we consider X_k the number of good intervals at time kKn^α. By attractiveness, if an interval is good for the contact process on T, then it must be good for the contact process on G. Note that, for H_L as in Lemma 3.3, a classical large deviation estimate on sums of i.i.d. random variables with an exponential moment gives us that

$$P\left[H_L > n\right] \le e^{-\sqrt{n}},$$

and as a consequence,

$$P\left[L_n \notin \{X_k,\ k \le n\},\ \xi^z_{nKn^\alpha} \neq \underline{0}\right] \le e^{-\sqrt{n}}. \tag{3.6}$$

Let $\mathcal{E}_{3/4}$ be the event that starting from z occupied, at least 3/4 of all the intervals $(I_i)_{i \le L_n}$ are good at time $n^2/2$ (which, for simplicity, is assumed to be a multiple of Kn^α). As the proof of part (2) reveals, once X_k has reached L_n, the probability that it makes an excursion below $3L_n/4$ before time n^2 is smaller than e^{-n^α}. Combining this with (3.6), we obtain

$$P\left[\xi^z_{n^2} \neq \underline{0},\ \mathcal{E}^c_{3/4}\right] \le 2e^{-n^\alpha},$$

where $\mathcal{E}^c_{3/4}$ denotes the complement of $\mathcal{E}_{3/4}$. Similarly, if we let $\hat{\mathcal{E}}_{3/4}$ denote the event that for the dual process $\hat{\xi}^{y,n^2}$, at least 3/4 of the intervals are good at time $n^2/2 - Kn^\alpha$, then

$$P\left[\hat{\xi}^{y,n^2}_{n^2} \neq \underline{0},\ \hat{\mathcal{E}}^c_{3/4}\right] \le 2e^{-n^\alpha}.$$

Consider the event $\tilde{\mathcal{E}}_i$ defined by:

> during the time interval $[n^2/2, n^2/2 + Kn^\alpha]$, the direct contact process restricted to I_i becomes identical with the contact process started with full occupancy (on I_i), while the dual contact process restricted to I_i survives.

Let also $\mathcal{I}$ be the set of indices i such that I_i is good both for the contact process and its dual. We have

$$P\left[\bigcap_{i \le L_n} (\tilde{\mathcal{E}}_i)^c, \mathcal{E}_{3/4}, \hat{\mathcal{E}}_{3/4}\right] \le P\left[\bigcap_{i \in \mathcal{I}} (\tilde{\mathcal{E}}_i)^c, \mathcal{E}_{3/4}, \hat{\mathcal{E}}_{3/4}\right].$$

Given that $\mathcal{E}_{3/4}$ and $\hat{\mathcal{E}}_{3/4}$ both happen, at least 1/2 of the intervals are good both for the contact process and its dual, or in other words, $|\mathcal{I}| \ge L_n/2$. Moreover, the events $\mathcal{E}_{3/4}$ and $\hat{\mathcal{E}}_{3/4}$, and the set $\mathcal{I}$, are independent of the state of the Harris system in the time layer $T \times [n^2/2, n^2/2 + Kn^\alpha]$. By the definition of being good, we have $P[(\tilde{\mathcal{E}}_i)^c \mid i \in \mathcal{I}] \le 2n^{-2\beta}$. Note also that the events $(\tilde{\mathcal{E}}_i)$ are independent. Hence

$$P\left[\bigcap_{i\le L_n} (\tilde{\mathcal{E}}_i)^c, \mathcal{E}_{3/4}, \hat{\mathcal{E}}_{3/4}\right] \le (2n^{-2\beta})^{L_n/2}.$$

Finally, note that when one of the $\tilde{\mathcal{E}}_i$ happens, it must be that $\xi^z_{n^2}(y) = 1$, by Remark 3.4. We have thus proved that

$$\begin{aligned} P\left[\xi^z_{n^2}(y) = 0,\ \hat{\xi}^{y,n^2}_{n^2} \neq \underline{0},\ \xi^z_{n^2} \neq \underline{0}\right] &\le P\left[\xi^z_{n^2}(y) = 0,\ \mathcal{E}_{3/4}, \hat{\mathcal{E}}_{3/4}\right] + 4e^{-n^\alpha} \\ &\le \left(2n^{-2\beta}\right)^{L_n/2} + 4e^{-n^\alpha} \\ &\le 5e^{-n^\alpha}. \end{aligned}$$

Recalling that the probability on the l.h.s. above is that appearing in (3.5), we have thus shown that

$$P\left[\xi^z_{n^2} \neq \xi^{\underline{1}}_{n^2},\ \xi^z_{n^2} \neq \underline{0}\right] \le 5ne^{-n^\alpha}.$$

Now for a general $A \subseteq V$, we have

$$P\left[\xi^A_{n^2} \neq \xi^{\underline{1}}_{n^2},\ \xi^A_{n^2} \neq \underline{0}\right] \le \sum_{z\in T} P\left[\xi^z_{n^2} \neq \xi^{\underline{1}}_{n^2},\ \xi^z_{n^2} \neq \underline{0}\right] \le 5n^2e^{-n^\alpha}.$$

In view of part (1) of Proposition 3.2, we thus have, for $A \neq \varnothing$,

$$P\left[\xi^A_{n^2} \neq \xi^{\underline{1}}_{n^2} \mid \xi^A_{n^2} \neq \underline{0}\right] \le \frac{5n^2}{\bar{c}_2} e^{-n^\alpha},$$

which proves the desired result.

For case (A), the reasoning is similar, only simpler. Let I be an interval of length n^α contained in T. For any $A \subseteq I$, we write $(\xi^A_{I,t})_{t\ge 0}$ for the contact process on I with initial configuration A, and define

$$p(A) = P\left[\xi^A_{I,Kn} = \xi^{\underline{1}}_{I,Kn} \neq \underline{0}\right].$$

We say that I is *good at time* t if $p(\xi_t) \ge 1 - e^{-n^{3\alpha/4}}$, and for $k \in \mathbb{N}$, we let X_k be the indicator function that I is good at time kKn.

In view of the proof of part (1) of Proposition 3.2, we have

$$P\left[X_{k+1} = 1 \mid \xi_{kKn} \neq \underline{0}\right] \ge \bar{c}_1^2, \tag{3.7}$$

while the same reasoning as in case (B) leads to

$$P\left[X_{k+1} = 1 \mid X_k = 1\right] \ge 1 - 2e^{-n^{3\alpha/4}}. \tag{3.8}$$

From (3.7) and (3.8), one can see that, for any $z \in V$,

$$P\left[\xi^z_{n^2} \neq \underline{0},\ I \text{ not good at time } n^{3/2} \text{ for } \xi^z\right] \le 2e^{-n^{5\alpha/8}},$$

where for simplicity we assume that $n^{3/2}$ is a multiple of Kn. Similarly, for any $z \in V$, one has

$$P\left[\hat{\xi}^{y,n^2}_{n^2} \neq \underline{0},\ I \text{ not good at time } n^{3/2} - Kn \text{ for } \hat{\xi}^{y,n^2}\right] \le 2e^{-n^{5\alpha/8}},$$

and we conclude as in case (B). □

Proof of Theorem 1.2. The result follows from [Mo93, Proposition 2.1], using parts (2-3) of Proposition 3.2. □

4. Comparison with Phoenix contact processes

The aim of this section is to prove Theorems 1.1 and 1.3. To this end, we manufacture a "Phoenix contact process". This process evolves as a contact process up to extinction, but has then the ability to recover activity. Separating a tree T into T_1 and T_2 as in Lemma 3.1, we show that with high probability, the true contact process ξ dominates the union of two Phoenix contact processes running independently on T_1 and T_2, as long as these two Phoenix contact processes are not simultaneously in the empty configuration. From this, we derive a recursive relation between $\mathbb{E}[\tau_T]$ and the product $\mathbb{E}[\tau_{T_1}]\mathbb{E}[\tau_{T_2}]$, which enables us to conclude.

Let $T \in \Lambda(n,d)$. Given a Harris system for the contact process on T, for any $x \in T$ and $t \geq 0$, we write $(\xi_s^{x,t})_{s\geq t}$ for the contact process starting at time t with x the only occupied site. We say that the Harris system is *trustworthy* on the time interval $[0,n^4]$ if for any $(x,s) \in T \times [0, n^4/2]$, the following two conditions hold:

(C_1) if $\xi^{x,s}$ survives up to time n^4, then $\xi^{x,s}_{n^4} = \xi^{\underline{1}}_{n^4}$,

(C_2) if $\xi^{x,s}$ survives up to time $s + 2n^2$, then it survives up to time n^4.

We say that the Harris system H is trustworthy on the time interval $[t, t+n^4]$ if $\Theta_t H$ is trustworthy on the time interval $[0,n^4]$, where $\Theta_t H$ is the Harris system obtained by a time translation of t.

For a given Harris system and for $(Y_t)_{t\in\mathbb{R}_+}$ a family of independent auxiliary random variables following a Bernoulli distribution of parameter 1/2, independent of the Harris system, we define the *Phoenix contact process* $(\zeta_{T,t})_{t\geq 0} = (\zeta_t)_{t\geq 0}$ on $\{0,1\}^T$ as follows.

Step 0. Set $\zeta_0 = \underline{1}$, and go to Step 1.

Step 1. Evolve as a contact process according to the Harris system, up to reaching the state $\underline{0}$, and go to Step 2.

Step 2. Let t be the time when Step 2 is reached. Stay at $\underline{0}$ up to time $t+n^4$ and

- if the Harris system is trustworthy on $[t, t+n^4]$ and $Y_t = 1$, then set $\zeta_{t+n^4} = \xi^{\underline{1},t}_{t+n^4}$ (where $\xi^{\underline{1},t}$ is the contact process started with full occupancy at time t and governed by the Harris system), and go to Step 1 ;
- else, go to Step 2.

We say that the process is *active* when it is running Step 1 ; is *quiescent* when it is running Step 2. Note that after initialization, the process alternates between active and quiescent phases. If it happens that during Step 2, the Harris system is trustworthy on $[t, t+n^4]$ and $Y_t = 1$, but $\xi^{\underline{1},t}_{t+n^4} = \underline{0}$, we consider that the process is active at time $t + n^4$, and becomes inactive again immediately afterwards.

Remark 4.1. Note that since the time the process spends on state $\underline{0}$ is not exponential, (ζ_t) is not Markovian. It would however be easy to make the process Markovian, by enlarging its state space into $\big(\{0,1\}^T \setminus \{\underline{0}\}\big) \cup \big(\{\underline{0}\} \times [0,n^4)\big)$, so that when arriving in Step 2, the process is in the state $(\underline{0}, 0)$, and subsequently the second coordinate increases at unit speed.

Remark 4.2. The auxiliary randomization of ζ provided by the family (Y_t) is a technical convenience, which guarantees that if ζ_t is quiescent at some time t, then with probability at least 1/2 it remains so at least up to time $t + n^4$.

Remark 4.3. Each time the process becomes active again, its distribution at this time is that of $\xi^{\underline{1}}_{n^4}$ conditioned on the event that the Harris system is trustworthy on the time interval $[0,n^4]$. We write ν to denote this distribution.

Lemma 4.4. *Let $T \in \Lambda(n,d)$. For any n large enough and any t, the probability that the Harris system on T is trustworthy on $[t, t+n^4]$ is larger than 1/2.*

Proof. It suffices to show the lemma for $t = 0$. We first consider condition (C_1). By part (3) of Proposition 3.2, the probability that

$$\forall z \in T,\ \xi_{n^4}^{z,n^4/2} \neq \underline{0} \Rightarrow \xi_{n^4}^{z,n^4/2} = \xi_{n^4}^{\underline{1},n^4/2} \tag{4.1}$$

goes to 1 as n tends to infinity. Let $(x,s) \in T \times [0, n^4/2]$, and assume that $\xi^{x,s}$ survives up to time n^4, that is,

$$(x,s) \leftrightarrow T \times \{n^4\}.$$

Then there must exist $z \in T$ such that

$$(x,s) \leftrightarrow (z, n^4/2) \leftrightarrow T \times \{n^4\}.$$

On the event (4.1), we thus have $\xi_{n^4}^{x,s} \geq \xi_{n^4}^{\underline{1},n^4/2}$. The converse comparison being clearly satisfied, we have in fact $\xi_{n^4}^{x,s} = \xi_{n^4}^{\underline{1},n^4/2}$. In order to show that condition (C_1) is satisfied for any $(x,s) \in T \times [0, n^4/2]$ with probability tending to 1, it thus suffices to show that

$$P\left[\xi_{n^4}^{\underline{1}} = \xi_{n^4}^{\underline{1},n^4/2}\right] \to 1 \text{ as } n \to \infty. \tag{4.2}$$

In view of part (2) of Proposition 3.2, with probability tending to one, we have $\xi_{n^4}^{\underline{1}} \neq \underline{0}$. On this event, by part (3) of Proposition 3.2, we also have $\xi_{n^4}^{\underline{1},n^4/2} = \xi_{n^4}^{\underline{1}}$ with probability tending to 1, and thus (4.2) is proved.

We now turn to condition (C_2). Note that the event $\xi_{s+2n^2}^{x,s} \neq \underline{0}$ can be rewritten as

$$(x,s) \leftrightarrow T \times \{s + 2n^2\},$$

and under such a circumstance, there must exist $z \in T$ such that

$$(x,s) \leftrightarrow (z, \lceil s/n^2 \rceil n^2) \leftrightarrow T \times \{s + 2n^2\}.$$

It is thus sufficient to show that

$$P\left[\exists z \in T, k \in \{0, \ldots, \lceil n^2/4 \rceil\} :\ \xi_{(k+1)n^2}^{z,kn^2} \neq \underline{0} \text{ but } \xi_{n^4}^{z,kn^2} = \underline{0}\right] \to 0 \text{ as } n \to \infty. \tag{4.3}$$

For a fixed $z \in T$ and integer k, we have by part (3) of Proposition 3.2 that

$$P\left[\xi_{(k+1)n^2}^{z,kn^2} \neq \underline{0} \text{ but } \xi_{(k+1)n^2}^{z,kn^2} \neq \xi_{(k+1)n^2}^{\underline{1},kn^2}\right] \leq e^{-n^{\alpha/2}},$$

so the probability of the event

$$\forall z \in T, k \in \{0, \ldots, \lceil n^2/4 \rceil\} :\ \xi_{(k+1)n^2}^{z,kn^2} = \underline{0} \text{ or } \xi_{(k+1)n^2}^{z,kn^2} = \xi_{(k+1)n^2}^{\underline{1},kn^2} \tag{4.4}$$

tends to 1 as n tends to infinity. On the other hand, with probability tending to 1, $\xi^{\underline{1}}$ survives up to time n^4, and is clearly dominated by $\xi_{(k+1)n^2}^{\underline{1},kn^2}$, for any $k \leq \lceil n^2/4 \rceil$. On the conjunction of this event and the one described in (4.4), we thus have

$$\forall z \in T, k \in \{0, \ldots, \lceil n^2/4 \rceil\} :\ \xi_{(k+1)n^2}^{z,kn^2} = \underline{0} \text{ or } \xi_{(k+1)n^2}^{z,kn^2} \geq \xi_{n^4}^{\underline{1}} \neq \underline{0},$$

and this proves (4.3). □

For the next lemma, recall that τ is the extinction time of the contact process started with full occupancy.

Lemma 4.5. *For any $s > 0$, one has*

$$P[\tau \leq s] \leq \frac{s}{E[\tau]},$$

Moreover, there exists a constant C such that for any $T \in \Lambda(n,d)$, $E[\tau] \leq e^{Cn}$.

Proof. Attractiveness of the contact process implies that for any $r \in \mathbb{N}$,

$$P[\tau \geq rs] \leq (P[\tau \geq s])^r . \tag{4.5}$$

Since

$$E[\tau] \leq s \sum_{r=0}^{+\infty} P[\tau \geq rs] \leq \frac{s}{1 - P[\tau \geq s]}, \tag{4.6}$$

it comes that

$$P[\tau \geq s] \geq 1 - \frac{s}{E[\tau]},$$

which proves the first part. For the second part, note that one can find C such that

$$P[\tau \geq 1] \leq 1 - e^{-Cn} \tag{4.7}$$

uniformly over $T \in \Lambda(n,d)$. The conclusion thus follows from (4.6). □

Lemma 4.6. *For any n large enough, any $T \in \Lambda(n,d)$ and any $t \geq 0$, one has*

$$P[\zeta_t = \underline{0}] \leq \frac{6n^6}{E[\tau]}. \tag{4.8}$$

Proof. Using Lemma 4.5 with $s = n^6$, it is clear that (4.8) holds for any n and any $t \leq n^6$. Note moreover that, writing τ^ν for the extinction time of the contact process started from the distribution ν defined in Remark 4.3, we have

$$P\left[\tau^\nu \leq n^6 - n^4\right] = P\left[\tau \leq n^6 \mid \text{Harris sys. trustworthy on } [0,n^4]\right] \leq \frac{2n^6}{E[\tau]}, \tag{4.9}$$

where we used Lemma 4.4 in the last step.

Suppose now that $t > n^6$, and consider the event $\mathcal{E}$ defined by

$$\exists s \in (t - n^6/2, t - n^6/4] \text{ such that } \zeta_s = \underline{0}.$$

We write $\tilde{\tau}$ for the first $s \geq t - n^6/2$ such that $\zeta_s = \underline{0}$. On the event $\mathcal{E}$, we have $\tilde{\tau} \leq t - n^6/4$. The event $\mathcal{E}'$ defined by

$$\forall k \in \mathbb{N}, k < \lfloor n^2/4 \rfloor,$$
$$\text{Harris sys. not trustworthy on } [\tilde{\tau} + kn^4, \tilde{\tau} + (k+1)n^4] \text{ or } Y_{\tilde{\tau}+kn^4} \neq 1$$

has probability smaller than $(3/4)^{\lfloor n^2/4 \rfloor}$ by Lemma 4.4. When $\mathcal{E}$ and $(\mathcal{E}')^c$ both hold, the process ζ becomes active at some time $t_A \in [t - n^6/2, t]$, and is distributed according to ν at this time. Hence,

$$\begin{aligned} P[\zeta_t = \underline{0}, \mathcal{E}] &\leq P[\zeta_t = \underline{0}, \mathcal{E}, (\mathcal{E}')^c] + P[\mathcal{E}'] \\ &\leq P\left[\tau^\nu \leq n^6/2\right] + P[\mathcal{E}'] . \end{aligned}$$

Since $P[\mathcal{E}'] \ll 1/E[\tau]$ and in view of (4.9), we have indeed

$$P[\zeta_t = \underline{0}, \mathcal{E}] \leq \frac{3n^6}{E[\tau]} \tag{4.10}$$

for any large enough n. It thus remains to bound

$$P[\zeta_t = \underline{0}, \mathcal{E}^c] . \tag{4.11}$$

Let k be the first positive integer such that $Y_{t-n^6/2+kn^4} = 1$ and the Harris system is trustworthy on

$$[a_k, b_k] \overset{\text{(def)}}{=} [t - n^6/2 + kn^4, t - n^6/2 + (k+1)n^4].$$

For the same reason as above, we may assume that $[a_k, b_k] \subseteq [t - n^6/2, t - n^6/4]$. Since on the event $\mathcal{E}^c$, the process ζ remains active on the time interval $[a_k, b_k]$, and considering the definition of trustworthiness and of the Phoenix process, we know that $\zeta_{b_k} = \xi^{1,a_k}_{b_k}$, and moreover, the latter random variable is distributed according to ν. Hence, up to a negligible event, the probability in (4.11) is bounded by

$$P\left[\tau^\nu \le n^6/2\right],$$

and thus, using (4.9) again,

$$P\left[\zeta_t = \underline{0}, \mathcal{E}^c\right] \le \frac{3n^6}{E[\tau]}. \tag{4.12}$$

The conclusion now follows, combining (4.10) and (4.12). □

Lemma 4.7. *Let T be a tree with size at most n and maximal degree at most d, and let $x \in T$. Define recursively $\gamma_0 = 0$ and, for any $i \in \mathbb{N}$,*

$$\gamma_{i+1} = \inf\{t \ge \gamma_i + 2n^2 : \xi_t(x) = 1\} \quad (+\infty \text{ if empty}).$$

For n large enough, we have

$$P\left[\gamma_{n^2/8} > n^4/2 \mid \xi_{n^4/2} \ne \underline{0}\right] \le e^{-n^2}.$$

Proof. In view of part (1) of Proposition 2.1, for any non-empty $A \subseteq T$, we have

$$P\left[\exists s \le \frac{n}{\overline{c}_1} : \xi^A_s(x) = 1\right] \ge \overline{c}_1.$$

Let $\mathcal{F}_i$ be the σ-field generated by $\{\xi_t, t \le \gamma_i\}$. By induction and the Markov property, we can thus show that for any $k \in \mathbb{N}$,

$$P\left[\gamma_{i+1} - (\gamma_i + 2n^2) \ge \frac{kn}{\overline{c}_1},\ \xi_{\gamma_i + 2n^2 + (k-1)n/\overline{c}_1} \ne \underline{0} \mid \mathcal{F}_i\right] \le (1-\overline{c}_1)^k.$$

Hence,

$$\begin{aligned} P\left[\gamma_{n^2/8} > n^4/2,\ \xi_{n^4/2} \ne \underline{0}\right] &= P\left[\sum_{i=0}^{n^2/8-1} \gamma_{i+1} - (\gamma_i + 2n^2) > n^4/4,\ \xi_{n^4/2} \ne \underline{0}\right] \\ &\le P\left[\sum_{i=0}^{n^2/8-1} B_i n/\overline{c}_1 > n^4/4\right], \end{aligned}$$

where (B_i) are independent geometric random variables of parameter $1 - \overline{c}_1$. For $\lambda > 0$ small enough, we have

$$e^{\phi(\lambda)} \overset{\text{(def)}}{=} E[e^{\lambda B_i}] < +\infty,$$

and we thus obtain

$$P\left[\sum_{i=0}^{n^2/8-1} B_i > \overline{c}_1 n^3/4\right] \le \exp\left(\phi(\lambda)n^2/8 - \lambda \overline{c}_1 n^3/4\right),$$

which, together with part (1) of Proposition 3.2, proves the claim. □

Proposition 4.8. *For n large enough, let $T \in \Lambda(n,d)$ be split into two subtrees T_1, T_2 as described by Lemma 3.1. Define the process $(\tilde{\zeta}_t)_{t \ge 0}$ by*

$$\tilde{\zeta}_t = \zeta_{T_1,t} \cup \zeta_{T_2,t} \qquad (t \ge 0),$$

where ζ_{T_1} and ζ_{T_2} are Phoenix processes defined on T_1 and T_2 respectively, using the Harris system on T together with two independent families of auxiliary random variables, independent of the Harris system. One has

$$P\left[\forall t \le \tau,\ \xi_t \ge \tilde{\zeta}_t\right] \ge 1 - e^{-n^{3/2}}.$$

Proof. Let $(\sigma_i)_{i\geq 1}$ be the sequence of (stopping) times when the process ζ_{T_1} becomes quiescent. We start by showing that, for any i,

$$P\left[\xi_{\sigma_i+n^4} < \zeta_{T_1,\sigma_i+n^4},\ \xi_{\sigma_i+n^4} \neq \underline{0}\right] \leq e^{-n^{7/4}}. \tag{4.13}$$

For some arbitrary $x \in T_1$, consider the stopping times introduced in Lemma 4.7, but started with $\gamma_0 = \sigma_i$, and let N be the largest index satisfying $\gamma_N \leq \sigma_i + n^4/2$. By Lemma 4.7, we have

$$P\left[N < n^2/8,\ \xi_{\sigma_i+n^4} \neq \underline{0}\right] \leq e^{-n^2}. \tag{4.14}$$

Moreover, part (1) of Proposition 3.2 ensures that, for any j,

$$P\left[\xi_{T_1}^{x,\gamma_j} \text{ survives up to time } \gamma_j + 2n^2 \mid \gamma_j < +\infty\right] \geq \overline{c}_2, \tag{4.15}$$

where $\xi_{T_1}^{x,\gamma_j}$ denotes the contact process restricted to T_1 started with x occupied at time γ_j. We introduce the stopping times $\tilde{\gamma}_j$ to deal with the fact that γ_j may be infinite. Let $\tilde{\jmath}$ we be the largest index such that $\gamma_{\tilde{\jmath}} \leq \sigma_i + n^4/2$. We let $\tilde{\gamma}_j = \gamma_j$ if $j \leq \tilde{\jmath}$, $\tilde{\gamma}_{\tilde{\jmath}+1} = \sigma_i + n^4/2 + 2n^2$, and then recursively, $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j = 2n^2$. We have

$$\begin{aligned} P&\left[\forall j \leq N,\ \xi_{T_1,\gamma_j+2n^2}^{x,\gamma_j} = \underline{0},\ \xi_{\sigma_i+n^4} \neq \underline{0}\right] \\ &\leq P\left[N < n^2/8,\ \xi_{\sigma_i+n^4} \neq \underline{0}\right] + P\left[\forall j \leq n^2/8,\ \xi_{T_1,\tilde{\gamma}_j+2n^2}^{x,\tilde{\gamma}_j} = \underline{0}\right], \end{aligned} \tag{4.16}$$

Since for any j, we have $\tilde{\gamma}_{j+1} \geq \tilde{\gamma}_j + 2n^2$, the events indexed by j appearing in the second probability on the r.h.s. of (4.16) are independent. Using also (4.14) and (4.15) (with γ_j replaced by $\tilde{\gamma}_j$), we thus arrive at

$$P\left[\forall j \leq N,\ \xi_{T_1,\gamma_j+2n^2}^{x,\gamma_j} = \underline{0},\ ,\xi_{\sigma_i+n^4} \neq \underline{0}\right] \leq e^{-n^2} + (1-\overline{c}_2)^{n^2/8}. \tag{4.17}$$

We now show that

$$\exists j \leq N,\ \xi_{T_1,\gamma_j+2n^2}^{x,\gamma_j} \neq \underline{0} \ \Rightarrow\ \xi_{\sigma_i+n^4} \geq \zeta_{T_1,\sigma_i+n^4}. \tag{4.18}$$

Indeed, in order for ζ_{T_1,σ_i+n^4} to be non $\underline{0}$, it must be that the Harris system restricted to T_1 is trustworthy on $[\sigma_i, \sigma_i + n^4]$. In this case, by the definition of trustworthiness, if there exists some $j \leq N$ such that $\xi_{T_1,\gamma_j+2n^2}^{x,\gamma_j} \neq \underline{0}$, then it must be that

$$\xi_{T_1,\sigma_i+n^4}^{x,\gamma_j} = \xi_{T_1,\sigma_i+n^4}^{\underline{1},\sigma_i} \geq \zeta_{T_1,\sigma_i+n^4}$$

(the last two being equal when $Y_{\sigma_i} = 1$, otherwise $\zeta_{T_1,\sigma_i+n^4} = \underline{0}$). Since $\xi_{\gamma_j}(x) = 1$, it is clear that $\xi_{\sigma_i+n^4} \geq \xi_{T_1,\sigma_i+n^4}^{x,\gamma_j}$, thus justifying (4.18). This and (4.17) prove (4.13).

In order to conclude, we first show that τ cannot be too large. It comes from (4.5) and (4.7) that

$$P\left[\tau \geq n^4 e^{Cn}\right] \leq e^{-n^2}, \tag{4.19}$$

where C can be chosen uniformly over $T \in \Lambda(n,d)$. If ζ_{T_1} is active at time t and ξ dominates ζ_{T_1} at this time, then the domination is preserved during the whole phase of activity, since ζ_{T_1} is driven by a subset of the Harris system driving the evolution of ξ. When ζ_{T_1} becomes quiescent, the domination is obviously preserved. As a consequence, if the domination of ζ_{T_1} by ξ is broken at some time, it must be when ζ_{T_1} turns from quiescent to active. We thus have

$$P\left[\exists t \leq \tau,\ \xi_t < \zeta_{T_1,t}\right] = P\left[\exists i : \xi_{\sigma_i+n^4} < \zeta_{T_1,\sigma_i+n^4} \text{ and } \xi_{\sigma_i+n^4} \neq \underline{0}\right].$$

Since $\sigma_{i+1} - \sigma_i \geq n^4$, on the event $\tau \leq n^4 e^{Cn}$, there are at most e^{Cn} times when ζ_{T_1} turns from quiescent to active. Using (4.13), we thus obtain

$$P\left[\forall t \leq \tau,\ \xi_t \geq \zeta_{T_1,t}\right] \leq P[\tau \geq n^4 e^{Cn}] + e^{Cn} e^{-n^{7/4}}.$$

The proposition is now proved, using (4.19) together with the fact that

$$P\left[\exists t \leq \tau,\ \xi_t < \tilde{\zeta}_t\right] \leq P\left[\exists t \leq \tau,\ \xi_t < \zeta_{T_1,t}\right] + P\left[\exists t \leq \tau,\ \xi_t < \zeta_{T_2,t}\right].$$

□

Corollary 4.9. *For n large enough, let $T \in \Lambda(n,d)$ be split into two subtrees T_1, T_2 as described by Lemma 3.1. We have*

$$E[\tau_T] \geq n^{-9}\ E\left[\tau_{T_1}\right] E\left[\tau_{T_2}\right].$$

Proof. Let $\tilde{\sigma}$ be the first time when ζ_{T_1} and ζ_{T_2} are simultaneously quiescent. By Proposition 4.8, for any $t \geq 0$, we have

$$P[\tau \leq t] \leq P[\tilde{\sigma} \leq t] + e^{-n^{3/2}}. \tag{4.20}$$

In view of Remark 4.2, at time $\tilde{\sigma}$, both ζ_{T_1} and ζ_{T_2} remain quiescent for a time n^4 with probability at least $1/2$ (one of them just becomes quiescent at time $\tilde{\sigma}$, while the other stays quiescent for a time n^4 with probability at least $1/2$). As a consequence, for any $t \geq 0$,

$$P\left[\tilde{\sigma} \leq t\right] \leq \frac{2}{n^4} \int_0^{t+n^4} P\left[\tilde{\zeta}_s = 0\right]\ \mathrm{d}s.$$

Since ζ_{T_1} and ζ_{T_2} are independent, and using Lemma 4.6, we thus obtain

$$P[\tilde{\sigma} \leq t] \leq \frac{2}{n^4}(t+n^4)\frac{(6n^6)^2}{E[\tau_{T_1}]E[\tau_{T_2}]} = \frac{72n^8(t+n^4)}{E[\tau_{T_1}]E[\tau_{T_2}]}. \tag{4.21}$$

Let us now fix

$$\tilde{t} = 2\frac{E[\tau_{T_1}]E[\tau_{T_2}]}{n^9}.$$

Since we know from part (1) of Proposition 3.2 that $\tilde{t}$ grows faster than any power of n, (4.21) gives us that for n large enough,

$$P\left[\tilde{\sigma} \leq \tilde{t}\right] \leq 1/4.$$

In view of (4.20), we thus obtain

$$P\left[\tau \leq \tilde{t}\right] \leq 1/4 + e^{-n^{3/2}} \leq 1/2,$$

which implies that $E[\tau] \geq \tilde{t}/2$, and thus the corollary. □

Proof of Theorem 1.1. Let $\rho = 1 + 1/d$, and consider, for any $r \in \mathbb{N}$, the quantity

$$V_r = \inf_{n \in (\rho^{r-1}/d, \rho^r]}\ \inf_{T \in \Lambda(n,d)} \frac{\log E[\tau(T)]}{|T|}$$

Theorem 1.1 will be proved if we can show that $\liminf_{r\to\infty} V_r > 0$.

Let r be a positive integer, and T be a tree of degree bounded by d and whose size lies in $\left(\rho^r, \rho^{r+1}\right]$.

Since $1 - \rho^{-1} = 1/(d+1) < 1/d$ and in view of Lemma 3.1, for r large enough, we can split up T into two subtrees T_1, T_2 such that

$$|T_1|, |T_2| \geq |T|(1-\rho^{-1}).$$

As a consequence,

$$|T_1|, |T_2| \geq \rho^{r-1}/d,$$

and also

$$|T_1| \leq |T| - |T_2| \leq |T|\left(1 - (1-\rho^{-1})\right) \leq \rho^r,$$

with the same inequality for T_2. Corollary 4.9 tells us that for r large enough,

$$E[\tau(T)] \geq \frac{1}{|T|^9} E[\tau(T_1)]\ E[\tau(T_2)],$$

that is to say,

$$\log E[\tau(T)] \geq \log E[\tau(T_1)] + \log E[\tau(T_2)] - \log |T|^9.$$

Observing that

$$\log E[\tau(T_1)] + \log E[\tau(T_2)] \geq V_r(|T_1| + |T_2|) = V_r|T|,$$

we arrive at

$$\frac{\log E[\tau(T)]}{|T|} \geq V_r - \frac{\log |T|^9}{|T|}. \tag{4.22}$$

Part (1) of Proposition 3.2 ensures that for r large enough, one has

$$V_r \geq \frac{c}{\rho^{r(1-\alpha)}} \tag{4.23}$$

for some constant $c > 0$. Recalling that $|T| \leq \rho^{r+1}$, we thus have

$$\frac{\log |T|^9}{|T|} \leq \frac{V_r}{\rho^{r\alpha/2}},$$

and (4.22) turns into

$$\frac{\log E[\tau(T)]}{|T|} \geq V_r \left(1 - \frac{1}{\rho^{r\alpha/2}}\right),$$

for any large enough r and any tree whose size lies in $(\rho^r, \rho^{r+1}]$. If the size of the tree lies in $(\rho^r/d, \rho^r]$, then the inequality

$$\frac{\log E[\tau(T)]}{|T|} \geq V_r$$

is obvious, so we arrive at

$$V_{r+1} \geq V_r \left(1 - \frac{1}{\rho^{r\alpha/2}}\right).$$

Since $V_r > 0$ for any r large enough by (4.23), and

$$\prod_r \left(1 - \frac{1}{\rho^{r\alpha/2}}\right) > 0,$$

we have shown that $\liminf_{r\to\infty} V_r > 0$, and this finishes the proof. □

Proof of Theorem 1.3. Let $c > 0$ be given by Theorem 1.1, and $T \in \Lambda(n, d)$. We learn from Lemma 4.5 that

$$P\left[\tau \leq e^{cn/2}\right] \leq \frac{e^{cn/2}}{E[\tau]},$$

which, by our choice of c, is smaller than $e^{-cn/4}$ for n large enough, uniformly over $T \in \Lambda(n, d)$. □

5. Discrete time growth process

For comparison purposes, it is useful to consider a discrete-time analogue of the contact process; we will need to consider such a process in the next section. Though many different definitions may be proposed, we have decided on the following.

Fix $p \in (0, 1)$ and let $\{I^r_{x,y} : r \in \{1, 2, \ldots\},\ x, y \in \mathbb{Z},\ |x - y| \leq 1\}$ be a family of independent Bernoulli(p) random variables. Fix $\eta_0 \in \{0, 1\}^{\mathbb{Z}}$ and, for $r \geq 0$, let

$$\eta_{r+1}(x) = \mathbb{1}\{\exists y : |x - y| \leq 1,\ \eta_r(y) = 1,\ I^r_{y,x} = 1\}.$$

The following is standard.

Proposition 5.1. *The above process is attractive and there exists $p_c^{(1)} < 1$ such that for $p > p_c^{(1)}$ the process survives in the sense that, for any $\eta_0 \neq \underline{0}$,*

$$P\left[\eta_r \neq \underline{0} \quad \forall r\right] > 0$$

and, if $\eta_0 = \underline{1}$, then η_r decreases stochastically to a non zero limit.

This process generalizes to locally finite graphs, just as does the contact process. In particular it has the self-duality property, and we can follow through the arguments of the preceding sections to arrive at:

Proposition 5.2. *Let $d \geq 2$ and $p > p_c^{(1)}$. There exists $c > 0$ such that*

$$\inf_{T \in \Lambda(n,d)} P\left[\tau_T \geq e^{cn}\right] \longrightarrow 1 \text{ as } n \to \infty.$$

(Here, τ_T is the extinction time for the discrete-time process on T started from full occupancy).

6. Extinction time on Newman-Strogatz-Watts random graphs

Let us briefly recall the definition of the NSW random graph on n vertices, $G_n = (V_n, E_n)$. We take $V_n = \{1, 2, \ldots, n\}$ and suppose given a probability $p(\cdot)$ on the positive integers greater than or equal to 3 with the property that, for some $a > 2$ and $c_0 > 0$, $p(m) \sim \frac{c_0}{m^a}$. To generate the NSW graph G_n, we first choose the degrees for the n vertices $d_1, d_2, \ldots, d_n$, according to i.i.d. random variables of law $p(\cdot)$ conditioned on $\sum_{x=1}^n d_x$ being even. Given this realization, we choose the edges by first giving each vertex x d_x half-edges and then matching up the half-edges uniformly among all possible matchings, so that, say, a half-edge for vertex x matched with a half-edge of vertex y becomes an edge between x and y. In [vdH12, Theorem 10.14], it is shown that from the assumption that p is supported on integers larger than 2, it follows that G_n is a connected graph with probability tending to 1 as $n \to \infty$.

We consider the contact process with parameter $\lambda > 0$ on NSW random graphs. In order to do so, we need to slightly modify the generator given in (1.1) to accomodate the fact that the random graph obtained from the above distribution may have loops and parallel edges. We put

(6.1)
$$\begin{array}{llll} \text{for every } x \in \xi_t, & \xi_t \to \xi_t \setminus \{x\} & \text{with rate } 1, \\ \text{for every } x \notin \xi_t, & \xi_t \to \xi_t \cup \{x\} & \text{with rate } \lambda \ \sum_{y \sim : y \in \xi_t} |\{e \in E_n : \{x,y\} \in E\}|, \end{array}$$

With this definition, loops have no effect on the dynamics and parallel edges are seen as independent channels for the transmission of the infection. The graphical construction defined in the beginning of Section 2 is compatible with 6.1 and requires no modification.

We will prove Theorem 1.4 under the assumption that $a > 2$, as mentioned in the Introduction. We will also assume that λ is small; this is not problematic to us because clearly it is sufficient to prove Theorem 1.4 for λ small enough.

Our approach is to show that with high probability, G_n has a subgraph $G_n' = (V_n', E_n')$ with a set of distinguished vertices $J_n' \subseteq V_n'$ satisfying certain properties that will guarantee that the extinction time of the contact process on G_n' is exponential in n. From this, by attractiveness, we will conclude that the extinction time for the contact process on G_n is also exponential in n with high probability.

This section is organized as follows. We will first list the properties that we want for G_n', J_n' and state in Proposition 6.1 that with high probability, large enough G_n', J_n' satisfying these properties indeed exist. Next, we will show how this proposition implies Theorem 1.4. Finally, we will prove the proposition, showing how G_n', J_n' can be constructed.

The first property on our list is

Property 1: G'_n is a connected tree.

Before listing the other properties, we need some notation. Let $\deg'$ denote the degree of a vertex in G'_n, that is, $\deg'(x) = |\{y \in V'_n : \{x,y\} \in E'_n\}|$, and dist' denote the graph distance in G'_n, that is, $\mathrm{dist}'(x,y)$ is the length of a minimal path from x to y contained in G'_n. For $x, y \in J'_n$, write $x \stackrel{*}{\sim} y$ if the unique path contained in G'_n from x to y contains no vertices of J'_n other than x and y. Let M be a large universal constant to be chosen later, $S = M\left(\frac{1}{\lambda}\log\left(\frac{1}{\lambda}\right)\right)^2$ and $D = 20\,a\log\left(\frac{1}{\lambda}\right)$.

Property 2: $\deg'(x) \geq \frac{S}{4}$ for all $x \in J'_n$.

Property 3: $\mathrm{dist}'(x,y) \leq D$ for all $x, y \in J'_n$ with $x \stackrel{*}{\sim} y$.

Property 4: The graph $G''_n = (V''_n, E''_n)$ given by

$$V''_n = J'_n; \qquad E''_n = \{\{x,y\} : x, y \in J'_n,\ x \stackrel{*}{\sim} y\}$$

is a connected tree with degree bounded by 4.

Proposition 6.1. *If λ is small enough, then there exists $\delta > 0$ such that, with probability tending to 1 as $n \to \infty$, G_n has a subgraph G'_n with a set of vertices $J'_n \subseteq V'_n$ such that properties 1-4 are satisfied and $|J'_n| > \delta n$.*

Proof of Theorem 1.4. Assume G'_n, J'_n are as in the above proposition and G''_n is as in Property 4. G''_n is thus a tree with more than δn vertices and degree bounded by 4. We will couple the contact process $(\xi'_t)_{t\geq 0}$ on G'_n (starting from full occupancy) and a discrete time growth process $(\eta_r)_{r\geq 0}$ on G''_n (again starting from full occupancy); this comes down to a coupling between the Harris system on G'_n and the Bernoulli random variables used to define the growth process. (η_r) is to be thought of as a coarse-grained version of (ξ'_t). Our choice of parameters and Proposition 5.2 will guarantee that the extinction time for (η_r) is exponential in n, and the corresponding fact for (ξ'_t) will be immediate.

Given a vertex $x \in G'_n$, we denote by $\mathcal{N}(x)$ the set containing x and its neighbours (in G'_n). For $x, y \in J'_n$ with $x \stackrel{*}{\sim} y$, let $b(x,y)$ be the set of vertices of G'_n in the unique path from x to y (notice that this path has length less than D).

We suppose given the Harris system on G'_n. We will consider (ξ'_t) on time intervals of size $\kappa = \left(\frac{1}{\lambda}\right)^{30a}$; this scale is chosen because it is large enough for an infection from a site $x \in J'_n$ to reach $y \in J'_n$ with $x \stackrel{*}{\sim} y$ but smaller than the extinction time for the process restricted to $\mathcal{N}(x)$. The following lemma and proposition will make this precise.

Given a set of vertices U in a graph Γ and $\xi \in \{0,1\}^U$, we will say that U is *infested* in ξ if $|\{x \in U : \xi(x) = 1\}| \geq \frac{\lambda}{20}|U|$. In [MVY11] the following is proved.

Lemma 6.2. *There exist $\bar{c}_6$ and N_0 such that, if λ is small enough, the following holds. Let Γ be a star graph consisting of one vertex x of degree $\frac{N}{\lambda^2}$, where $N \geq N_0$, and all other vertices of degree 1. Then, for the contact process with parameter λ on Γ,*

(i.) $P_{\Gamma,\lambda}\left[\,\Gamma \text{ is infested in } \xi_1 \mid \xi_0 = \{x\}\,\right] > 1/2$*;*

(ii.) $P_{\Gamma,\lambda}\left[\,\Gamma \text{ is infested in } \xi_{e^{\bar{c}_6 N}} \mid \Gamma \text{ is infested in } \xi_0\right] > 1 - e^{-\bar{c}_6 N}$.

In the case of the star graph given by $\mathcal{N}(x)$ for some $x \in J'_n$, the N of the above lemma is equal to $\lambda^2\deg'(x) \geq \frac{M}{4}\log^2\left(\frac{1}{\lambda}\right)$. The extinction time for the contact process restricted to $\mathcal{N}(x)$ and started from full occupancy will then be with high probability larger than $e^{\bar{c}_6\frac{M}{4}\log^2(1/\lambda)} = \left(\frac{1}{\lambda}\right)^{\bar{c}_6\frac{M}{4}\log(1/\lambda)} > \left(\frac{1}{\lambda}\right)^{21a\log(1/\lambda)}$ as long as $M > \frac{84a}{\bar{c}_6}$. Now, if $x, y \in J'_n$ and $x \stackrel{*}{\sim} y$, the probability that an infection in $\mathcal{N}(x)$ is transmitted along $b(x,y)$, reaches y within time $20a\log\left(\frac{1}{\lambda}\right)$ and then infests $\mathcal{N}(y)$ within time 1 is larger than $\frac{1}{2}\left(e^{-1}\cdot\lambda e^{-\lambda}\right)^{20a\log(1/\lambda)}$. If $\mathcal{N}(x)$ holds the infection for $(1/\lambda)^{21a\log(1/\lambda)}$ units of time, there will be $\dfrac{(1/\lambda)^{21a\log(1/\lambda)}}{20a\log(1/\lambda)+1}$ chances for such a transmission to occur. Comparing the number of chances with the probability of a transmission, we see that a transmission will occur with very high probability. These considerations lead to

Proposition 6.3. *For any $\sigma > 0$, M can be chosen large enough so that the following holds. Assume that $x, y \in J'_n$, $x \overset{*}{\sim} y$ and, in ξ_0, $\mathcal{N}(x)$ is infested. Let $(\tilde{\xi}_t)$ denote the process constructed on $\mathcal{N}(x)\cup\mathcal{N}(y)\cup b(x,y)$. Then, with probability larger than $1-\sigma$, both $\mathcal{N}(x)$ and $\mathcal{N}(y)$ are infested in $\tilde{\xi}_\kappa$.*

Let $r \in \mathbb{N}$, $x, y \in J'_n$ with $x \overset{*}{\sim} y$ and (ξ'_t) be the contact process on G'_n started from full occupancy. Put $I^r_{x,y} = 1$ if one of the following holds:

- $\mathcal{N}(x)$ is infested in $\xi'_{\kappa r}$ and
$$\left|\left\{\begin{array}{c} z \in \mathcal{N}(y) : \exists w \in \mathcal{N}(x) : \xi'_{\kappa r}(w) = 1, \\ (w,\kappa r) \leftrightarrow (z, \kappa(r+1)) \text{ inside } \mathcal{N}(x)\cup\mathcal{N}(y)\cup b(x,y) \end{array}\right\}\right| > \frac{\lambda}{20}|\mathcal{N}(y)|;$$
- $\mathcal{N}(x)$ is not infested in $\xi'_{\kappa r}$.

Otherwise put $I^r_{x,y} = 0$. The second condition above is just present to guarantee that $I^r_{x,y} = 1$ with high probability regardless of $\xi'_{\kappa r}$. Put $I^r_{x,x} = 1$ if one of the following holds:

- $\mathcal{N}(x)$ is infested in $\xi'_{\kappa r}$ and
$$\left|\left\{\begin{array}{c} z \in \mathcal{N}(x) : \exists w \in \mathcal{N}(x) : \xi'_{\kappa r}(w) = 1, \\ (w,\kappa r) \leftrightarrow (z, \kappa(r+1)) \text{ inside } \mathcal{N}(x) \end{array}\right\}\right| > \frac{\lambda}{20}|\mathcal{N}(x)|;$$
- $\mathcal{N}(x)$ is not infested in $\xi'_{\kappa r}$.

Otherwise put $I^r_{x,x} = 0$.

Let $\eta_0 \equiv 1$ and, for $r \geq 0$,
$$\eta_{r+1}(x) = \mathbb{1}\left\{\begin{array}{c} \eta_r(x) = 1 \text{ and } I^r_{x,x} = 1 \text{ or, for some} \\ y \text{ with } x \overset{*}{\sim} y,\ \eta_r(y) = 1 \text{ and } I^r_{y,x} = 1. \end{array}\right\}.$$

Notice that, if a sequence $x_1, x_2, \ldots, x_R$ in J'_n is such that, for each r, either $x_r = x_{r+1}$ and $I^r_{x_r,x_r} = 1$ or $x_r \overset{*}{\sim} x_{r+1}$ and $I^r_{x_r,x_{r+1}} = 1$, then we will have $\eta_R(x_R) = 1$ and $\mathcal{N}(x_R)$ will be infested in $\xi'_{\kappa R}$.

Now, using a result of Liggett, Schonmann and Stacey [LSS97] (see also Theorem B26 in [Li2]), given $p \in (0,1)$ we can choose M large enough so that the measure of the field $\{\{I^r_{x,x}\}, \{I^r_{x,y}\}\}$ stochastically dominates i.i.d. Bernoulli(p) random variables. We then have

Corollary 6.4. *For any $p > p_c(1)$, if M is large enough, then $\{\eta_r\}$ dominates a growth process on G''_n defined from i.i.d. Bernoulli(p) random variables.*

This, the fact that $|J'_n| \geq \delta n$ and Proposition 5.2 give Theorem 1.4. □

The rest of the section is devoted to the proof of Proposition 6.1. We start with some remarks concerning the random degree sequence $d_1, \ldots, d_n$. Recall that $S = M\left(\frac{1}{\lambda}\log\left(\frac{1}{\lambda}\right)\right)^2$, where M is the constant that was chosen during the proof of Theorem 1.4. We define
$$J_n = \{x \in V_n : \deg(x) \in [S, 2S]\}.$$

Let $\mu = \sum_{m=1}^\infty m \cdot p(m)$. Let us remark that, if the degrees are given by $d_1, \ldots, d_n$ and we choose a half-edge uniformly at random, then the probability that the corresponding vertex has degree m is

$$\frac{m \cdot |x : d_x = m|}{\sum_x d_x} \to \frac{m \cdot p(m)}{\mu} \text{ as } n \to \infty.$$

The probability $q(m) = m \cdot p(m)/\mu$ is called the *size biased distribution.* By our assumption that $p(m) \sim \frac{c_0}{m^a}$, it follows that $q(m) \sim \frac{c_1}{m^{a-1}}$, where $c_1 = \frac{c_0}{\mu}$. If y is large enough, then it can be easily verified by comparison with integrals that

$$\frac{c_0}{2(a-1)} \cdot y^{-(a-1)} < p([y, 2y]) < \frac{2c_0}{a-1} \cdot y^{-(a-1)}, \tag{6.2}$$

$$\frac{c_1}{2(a-2)} \cdot y^{-(a-2)} < q([y, 2y]) < \frac{2c_1}{a-2} \cdot y^{-(a-2)}. \tag{6.3}$$

We will also need the following facts.

Lemma 6.5. *For any small enough $\lambda > 0$, there exists $\epsilon > 0$ such that, with probability tending to 1 as $n \to \infty$, for any $A \subseteq V_n$ with $|A| \leq \epsilon n$ we have*

(i.) $\dfrac{c_0}{4(a-1)S^{a-1}} < \dfrac{|J_n \cap A^c|}{n} < \dfrac{4c_0}{(a-1)S^{a-1}}$;

(ii.) $\dfrac{\sum_{x \in A} d_x}{\sum_{x \in V_n} d_x} < \dfrac{1}{8}$;

(iii.) $\dfrac{c_1}{4(a-2)S^{a-2}} < \dfrac{\sum_{x \in J_n \cap A^c} d_x}{\sum_{x \in V_n} d_x} < \dfrac{4c_1}{(a-2)S^{a-2}}$;

(iv.) $\dfrac{\mu}{2} < \dfrac{\sum_{x \in A^c} d_x}{n} < 2\mu$.

Proof. Assume λ is small enough (and thus S is large enough) so that (6.2) and (6.3) hold with S in the place of y. All four points in the lemma are consequences of the Law of Large Numbers. For $(i.)$, it suffices to take $\epsilon < \frac{c_0}{8(a-1)S^{a-1}}$. For $(ii.), (iii.)$ and $(iv.)$, choose K large enough so that $\mu \sum_{m>K} q(m) < \frac{c_1}{8(a-2)S^{a-2}}$. Note that, as $n \to \infty$, $\frac{1}{n}\sum_{x=1}^n d_x \cdot \mathbb{1}_{\{d_x > K\}}$ converges in probability to

$$\mathbb{E}\left[d_1 \cdot \mathbb{1}_{\{d_1 > K\}}\right] = \sum_{m>K} m \cdot p(m) = \mu \sum_{m>K} q(m). \tag{6.4}$$

Reducing ϵ if necessary, assume $\epsilon < \frac{1}{2}p([K, \infty))$. The event $\{|\{x : d_x > K\}| > \epsilon n\}$ occurs with probability tending to 1 as $n \to \infty$. On this event, we have

$$\sum_{x \in A} d_x < \sum_{x : d_x > K} d_x < \frac{c_1}{4(a-2)S^{a-2}} \cdot n,$$

where the last inequality holds with high probability, by (6.4). The desired results easily follow from this. $\square$

In what follows, λ is fixed and ϵ is taken corresponding to λ as in the lemma. We will often assume that λ is small enough, and also that n is large enough, for other desired properties to hold. Let us say that a degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ is *robust* if it satisfies $(i.), (ii.), (iii.)$ and $(iv.)$. We will henceforth fix a robust sequence $\mathbf{d}$. We will write $\mathbb{P}_\mathbf{d}$ to denote a probability measure under which the random graph G_n is constructed as follows: the degrees of the n vertices are deterministic, given by $\mathbf{d}$, and the half-edges are then matched in a manner that is chosen uniformly at random among all possibilities, as prescribed in the definition of the NSW graph.

We now describe an alternative matching procedure that produces the same random graph. This procedure consists of matching the half-edges sequentially, pair by pair, so that, in each step, we are free to choose one of the half-edges involved in the matching, and the other is chosen at random. To be more precise, let us introduce some terminology. A *semi-graph* $g = (V_n, \mathcal{H}, \mathcal{E})$ is a triple consisting of the set of vertices V_n, a set of half-edges $\mathcal{H}$ and a set of edges $\mathcal{E}$ (of course, if $\mathcal{H} = \varnothing$, then g is a graph). The degree of a vertex in a semi-graph is the number of its half-edges plus the number of edges that are incident to it. Given two half-edges $h, h' \in \mathcal{H}$, we will denote by $h + h'$ a new edge produced by "attaching" h and h'. We will now show how to define a finite sequence of semi-graphs $g_0, g_1, \ldots, g_k$ so that g_k is a graph with the desired distribution. $g_0 = (V_n, \mathcal{H}_0, \mathcal{E}_0)$ is defined with $\mathcal{E}_0 = \varnothing$ and such that each vertex x has d_x half-edges. Assume $g_i = (V_n, \mathcal{H}_i, \mathcal{E}_i)$ is defined and has half-edges. Fix an arbitrary half-edge $h \in \mathcal{H}_i$ (call this an *elected* half-edge) and randomly choose another half-edge h' uniformly in $\mathcal{H}_i - \{h\}$. Then put $g_{i+1} = (V_n, \mathcal{H}_{i+1}, \mathcal{E}_{i+1})$, where $\mathcal{H}_{i+1} = \mathcal{H}_i - \{h, h'\}$ and $\mathcal{E}_{i+1} = \mathcal{E}_i \cup \{h + h'\}$. When no half-edges are left, we are done, and the graph thus obtained is a NSW random graph. Often, instead of updating the sets each time, say from $\mathcal{H}_i, \mathcal{E}_i$ to $\mathcal{H}_{i+1}, \mathcal{E}_{i+1}$ as above, we will hold the notation $g = (V_n, \mathcal{H}, \mathcal{E})$ and say (for example) that h, h' are deleted from $\mathcal{H}$ and $h + h'$ is added to $\mathcal{E}$.

In each step of the above construction, we are free to indicate the elected half-edge. A full description of how to elect a half-edge given all previous steps in the construction (and thus the present state of the semi-graph) is an algorithm to construct the NSW graph (or a subgraph of it, if we stop before exhausting all half-edges – this will be the case for us, since our objective is to construct the subgraph G'_n). We will present an algorithm that will help us find G'_n with high probability. The robustness property will come into play because we will have to deal with the set of half-edges after some matchings have been made.

Other than doing matchings, our algorithm writes labels on edges and vertices. Whenever an edge is constructed, it is labeled either marked or unmarked. Once written, edge labels will not be modified. They will play a role in the definition of G'_n once the algorithm finishes running: unmarked edges will be ignored and some of the marked edges will be included as edges of G'_n. Vertex labels serve to guide the order of the matchings. At any given time, each vertex has one of the four labels: unidentified, preactive, active and read. Before the algorithm starts running, all vertices are set to unidentified; in general, an unidentified vertex is one that has not yet been "seen" by the algorithm, that is, none of its half-edges has been matched yet. Unlike edge labels, vertex labels may be modified by the algorithm. If a vertex has any label different from unidentified, then it is said to be identified. The labels preactive and active can only be associated to vertices in J_n, and at most one vertex will be active at a given time.

The algorithm repeatedly follows a subroutine called a *pass*, which is just a sequence of matchings of half-edges and labelings. At the moment between the completion of a pass and the beginning of the following pass, there will be no active vertices. When a new pass starts, it typically takes a preactive (or sometimes unidentified) vertex $\bar{x} \in J_n$, turns it into active and successively explores the graph around $\bar{x}$ (by performing matchings) until certain conditions are satisfied; then, it labels every vertex that was touched as read except for the vertices of J_n that were found; these are labeled preactive and are activated by future passes.

A labeled semi-graph $g = (V_n, \mathcal{H}, \mathcal{E}, \{\ell_x\}_{x \in V^n}, \{\ell_e\}_{e \in \mathcal{E}}, \prec)$ is a semi-graph with a label ℓ_x attached to each vertex x, a label ℓ_e associated to each edge e and a total order $\prec$ on the set of preactive vertices. It is worth remarking that since a pass only does matchings and relabeling, it does not change the degree of any vertex. In particular, the definition of the set J_n does not change. Let us now define the pass. Obviously, whenever there is an instruction to give a vertex a label, this label replaces the former label of that vertex.

The pass

Input: $g = (V_n, \mathcal{H}, \mathcal{E}, \{\ell_x\}, \{\ell_e\}, \prec)$ with at least one vertex of J_n preactive or unidentified.

(S1) Let $\bar{x}$ be the preactive vertex of highest order; if there are no preactive vertices, let $\bar{x}$ be an arbitrary unidentified site of J_n.
- If $\bar{x}$ has less than $\frac{S}{2}$ half-edges (which can only happen if it is preactive), label it read; the pass is then stopped in status B_1.
- Otherwise, label $\bar{x}$ active and proceed to (S2).

(S2) Define the set $\mathcal{H}^*$ of *relevant half-edges* of the pass as the set of half-edges attached to the active vertex. Endow $\mathcal{H}^*$ with a total order $\prec^*$ chosen arbitrarily. Also let $\bar{C} = 0$; this will be a counting variable whose value will be progressively incremented. Proceed to (S3).

(S3) Let h be the half-edge of highest order in $\mathcal{H}^*$. Choose another half-edge h' uniformly at random in $\mathcal{H} - \{h\}$ and let v' be the vertex of h'. Delete h, h' from all sets that contain them (h from $\mathcal{H}$ and $\mathcal{H}^*$, h' from $\mathcal{H}$ and possibly $\mathcal{H}^*$) and add $h + h'$ to $\mathcal{E}$; its label is given as follows:
- If v' is identified, label $h + h'$ unmarked.
- If v' is unidentified and not in J_n, label $h + h'$ marked. Also label v' read and add its half-edges to $\mathcal{H}^*$ (note that at this point h' is no longer a half-edge of v') so that they have arbitrary order among themselves but lower order than all half-edges previously in $\mathcal{H}^*$.
- If v' is unidentified and in J_n, and if $\bar{C} < 3$, label $h + h'$ marked, label v' preactive, assign it the lowest order in the set of preactive vertices and add 1 to $\bar{C}$.
- If v' is unidentified and in J_n, and if $\bar{C} \geq 3$, label $h + h'$ unmarked and label v' read.

Proceed to (S4).

(S4)
- If $\bar{x}$ still has half-edges, go to (S3).
- If the last half-edge of $\bar{x}$ has been deleted in the previous step and now there are less than $\frac{S}{4}$ marked edges incident to $\bar{x}$, label $\bar{x}$ and all vertices that have been identified in the pass (including the preactive ones) read. The pass is stopped in status B_2.
- Otherwise go to (S5).

(S5)
- If $\bar{C} \geq 3$, label $\bar{x}$ read and end the pass in status G.
- Otherwise go to (S6).

(S6)
- If (a) more than $\left(\frac{1}{\lambda}\right)^{2a-3}$ vertices have been identified in the pass, or (b) a path of length $20a \log\left(\frac{1}{\lambda}\right)$ may be formed with marked edges constructed in the pass, or (c) $\mathcal{H}^*$ is empty, then label $\bar{x}$ read and end the pass in status B_3.
- Otherwise go to (S3).

Output: updated labeled semi-graph, status.

Let us explain what happens when a pass ends in status G. It first activates the preactive vertex $\bar{x}$ of highest order, then starts identifying the neighbours of $\bar{x}$; when they are all identified, it starts identifying the vertices at distance 2 from $\bar{x}$, and so on, until it has found three new vertices of J_n, at which point it stops. The "bad" outcomes B_1,B_2 and B_3 are included to guarantee that G_n' has the desired properties mentioned earlier and that the algorithm can successfully continue. B_1 and B_2 are necessary to ensure that the vertices of G_n' that will be the focal points for the comparison growth process all have large degree. B_3 is necessary to ensure that the focal points are not very far from each other and also that the pass does not delete too many half-edges, thus exploring too much of the graph.

We wish the pass to return the status G; the following lemma addresses this.

Lemma 6.6. *Assume that g has less than $\frac{\epsilon}{2}n$ identified vertices before the pass starts and, when the pass defines $\bar{x}$, this vertex has more than $\frac{S}{2}$ half-edges. Then, the pass ends in status G with probability larger than $\frac{9}{10}$.*

Proof. Start noticing that the pass identifies at most $(1/\lambda)^{2a-3}$ vertices and this is much less than $\frac{\epsilon}{2}n$ if n is large. So, at a moment immediately before the pass chooses a half-edge at random, there are less than ϵn identified vertices; let A in the definition of robustness be this set of identified vertices. The chosen half-edge then has probability:

(1) larger than $\frac{7}{8}$ of belonging to an unidentified vertex;

(2) larger than $\frac{c_1}{4(a-2)S^{a-2}}$ of belonging to an unidentified vertex that is in J_n;

(3) larger than $\frac{3}{4}$ of belonging to an unidentified vertex that is not in J_n.

By hypothesis, the pass does not end in status B_1. For it to end in status B_2, at least half of the more than $\frac{S}{2}$ half-edges initially present in $\mathcal{H}^*$ must be matched to half-edges of previously identified vertices. By (1) this has probability less than $P[\mathsf{Bin}(S/2, 1/8) > S/4]$, which is less than $\frac{1}{40}$ if λ is small (and hence S is large). Likewise, we can show using (2) that the probability of the pass ending because of case (a) in (S6) is less than $\frac{1}{40}$.

Let us now show that the same holds for (b) in (S6). For $k \geq 1$, let s_k be the set of vertices at distance k from $\bar{x}$ that are not in J_n and that the pass identifies; also let $s_0 = \{\bar{x}\}$. Since every vertex has degree 3 or more, there will be at least $2|s_k|$ half-edges of vertices of s_k for the pass to match (unless it halts before). Define the event

$$A_k = \left\{ \begin{array}{c} \text{the pass deletes all half-edges of vertices of } s_k\text{; of these,} \\ \text{less than } \frac{5}{8} \text{ are matched to half-edges of vertices not in } J_n \\ \text{that were (at the time of matching) } \mathsf{unidentified} \end{array} \right\}, \quad k \geq 0.$$

We have $\mathbb{P}[A_0] \leq P\left[\mathsf{Bin}(S/2, 3/4) < (S/2)\cdot(5/8)\right].$ Given that $A_1, \ldots, A_k$ have not occurred and the pass reaches distance $k+1$ from $\bar{x}$, the probability of A_{k+1} is less than

$$P\left[\mathsf{Bin}\left(\frac{S}{2}\left(\frac{5}{8}\right)^{k+1} 2^k,\ \frac{3}{4}\right) < \left(\frac{S}{2}\left(\frac{5}{8}\right)^{k+1} 2^k\right)\cdot\frac{5}{8}\right].$$

Letting $K = \frac{2a}{\log(5/4)}\log\left(\frac{1}{\lambda}\right) < 20a\log\left(\frac{1}{\lambda}\right)$, the above estimates show that $\mathbb{P}\left[\cup_{k=0}^{K} A_k\right]$ vanishes as $\lambda \to 0$. Now, assume that $A_0, \ldots, A_K$ have not occurred and the pass reaches level $K+1$. The probability that less than 3 unidentified vertices of J_n are discovered in the matching of half-edges from s_{K+1} is then less than

$$P\left[\mathsf{Bin}\left(\frac{S}{2}\left(\frac{5}{8}\right)^{K+1} 2^K,\ \frac{c_1}{4(a-2)S^{a-2}}\right) < 3\right]. \tag{6.5}$$

Note that

$$\frac{S}{2}\left(\frac{5}{8}\right)^{K+1}2^K \geq \frac{S}{4}\left(\frac{1}{\lambda}\right)^{\frac{2a}{\log(5/4)}\cdot\log(5/4)} = \frac{S}{4}\left(\frac{1}{\lambda}\right)^{2a}$$

and

$$\frac{c_1}{4(a-2)S^{a-2}} = \frac{c_1\lambda^{2(a-2)}}{4(a-2)M^{a-2}\log^{2(a-2)}(1/\lambda)},$$

so the probability in (6.5) is very small if λ is small. Putting these facts together, we get the desired result.

The probability of (c) in (S6) occurring is similarly shown to be less than $\frac{1}{40}$, and this concludes the proof. □

As we mentioned, given $d_1, \dots, d_n$, the starting point of the algorithm is the semi-graph g so that each vertex x is endowed with d_x half-edges and has label unidentified. We will run $\epsilon' n$ successive passes, where $\epsilon' = \frac{\epsilon\lambda^{2a-3}}{2}$. Since each pass identifies at most $\lambda^{-(2a-3)}$ vertices, we see that at the beginning of each pass, less than $\frac{\epsilon}{2}n$ vertices will be identified, so the hypotheses of Lemma 6.6 will hold. Also let $\delta' = \frac{\epsilon'}{2}$

For $1 \leq i < \epsilon' n$, define

$$\begin{aligned} W_i &= \text{Number of } \mathsf{preactive} \text{ vertices before pass } i, \\ X_i &= W_{i+1} - W_i, \\ Y_i &= \mathbb{1}_{\{\text{Pass } i \text{ ends in status } \mathsf{B}_1\}}. \end{aligned}$$

The possible values for X_i are $-1, 0, 1, 2$. If $Y_i = 1$, then $X_i = -1$. By the previous lemma, for any $x_1, \dots, x_{i-1}, y_1, \dots, y_{i-1}$ we have

$$\mathbb{P}_{\mathbf{d}}\left[\, X_i = 2 \mid \{X_j\}_{j=1}^{i-1} = \{x_j\}_{j=1}^{i-1},\ \{Y_j\}_{j=1}^{i-1} = \{y_j\}_{j=1}^{i-1},\ Y_i = 0\, \right] > 9/10. \tag{6.6}$$

Let us now exclude the possibility that many passes end in status B_1.

Lemma 6.7.

$$\mathbb{P}_{\mathbf{d}}\left[\sum_{i=1}^{\lfloor\epsilon' n\rfloor} Y_i > \frac{1}{10}\lfloor\epsilon' n\rfloor\right] \xrightarrow{n\to\infty} 0.$$

Proof. We start remarking that, for $\{Y_i = 1\}$ to occur, there must exist a vertex $x \in J_n$ such that

- x is identified before pass i;
- from the moment x is identified to the beginning of pass i, more than $S/2$ half-edges of x are chosen for matchings;
- x is the preactive vertex of highest order when pass i starts.

Let $h_1, \dots, h_N$ be the sequence of half-edges chosen at random by the algorithm since the beginning of the first pass. As explained above, we have $N \leq \epsilon n$. By (*iii.*) of Lemma 6.5, regardless of what happened before h_j is chosen, the probability that h_j belongs to a vertex of J_n is less than $\frac{4c_1}{(a-2)S^{a-2}}$. On the other hand, for $\{\sum Y_i > (1/10)\lfloor\epsilon' n\rfloor\}$ to occur, more than $\frac{1}{10}\lfloor\epsilon' n\rfloor\frac{S}{2}$ half-edges of vertices of J_n must be chosen. The probability of this is less than

$$P\left[\mathsf{Bin}\left(\lfloor\epsilon n\rfloor,\ \frac{4c_1}{(a-2)S^{a-2}}\right) > \frac{1}{10}\lfloor\epsilon' n\rfloor\frac{S}{2}\right].$$

By Markov's Inequality, this is less than

$$\frac{\lfloor\epsilon n\rfloor\cdot\frac{4c_1}{(a-2)S^{a-2}}}{\frac{1}{10}\lfloor\epsilon' n\rfloor\frac{S}{2}} = C\frac{\epsilon}{\epsilon'}\frac{1}{S^{a-1}} = C\frac{1}{\lambda^{2a-3}}\frac{\lambda^{2(a-1)}}{M^{a-1}\log^{2(a-1)}(1/\lambda)} = C'\frac{\lambda}{\log^{2(a-1)}(1/\lambda)},$$

where C, C' are constants that do not depend on λ or n. The above can be made as small as desired by taking λ small. □

Proposition 6.8. $\mathbb{P}_{\mathbf{d}}\left[W_{\lfloor \epsilon' n\rfloor} > \delta' n\right] \xrightarrow{n\to\infty} 1.$

Proof. We start giving a random mapping representation of the random variables $X_1, \ldots, X_{\lfloor \epsilon' n\rfloor}, Y_1, \ldots, Y_{\lfloor \epsilon' n\rfloor}$. Given sequences $\{x_j\}_{j=1}^{i-1}$, $\{y_j\}_{j=1}^{i}$ and $s \in (0,1)$, let

$$\begin{aligned}
&\phi\left(s, \{x_j\}_{j=1}^{i-1}, \{y_j\}_{j=1}^{i}\right) = m \text{ if}\\
&\mathbb{P}_{\mathbf{d}}\left[X_i \le m-1 \mid \{X_j\}_{j=1}^{i-1} = \{x_j\}_{j=1}^{i-1},\ \{Y_j\}_{j=1}^{i} = \{y_j\}_{j=1}^{i}\right]\\
&\qquad < s \le \mathbb{P}_{\mathbf{d}}\left[X_i \le m \mid \{X_j\}_{j=1}^{i-1} = \{x_j\}_{j=1}^{i-1},\ \{Y_j\}_{j=1}^{i} = \{y_j\}_{j=1}^{i}\right]
\end{aligned}$$

Likewise, let

$$\begin{aligned}
&\psi\left(s, \{x_j\}_{j=1}^{i-1}, \{y_j\}_{j=1}^{i-1}\right)\\
&\qquad = \left| \begin{array}{ll} 0 & \text{if } s \le \mathbb{P}_{\mathbf{d}}\left[Y_i = 0 \mid \{X_j\}_{j=1}^{i-1} = \{x_j\}_{j=1}^{i-1},\ \{Y_j\}_{j=1}^{i-1} = \{y_j\}_{j=1}^{i-1}\right] \\ 1 & \text{otherwise.} \end{array}\right.
\end{aligned}$$

(when we write only $\phi(s)$, $\psi(s)$, we mean the functions above for X_1 and Y_1, with no conditioning in the probabilities that define them). Let $U_1, U_2, \ldots,\ V_1, V_2, \ldots$ be independent random variables with the uniform distribution on $(0,1)$. Set $X_1' = \phi(U_1)$, $Y_1' = \psi(V_1)$ and recursively define, for $1 < i < \epsilon' n$,

$$\begin{aligned}
Y_{i+1}' &= \psi\left(V_{i+1}, \{X_j'\}_{j=1}^{i}, \{Y_j'\}_{j=1}^{i}\right);\\
X_{i+1}' &= \phi\left(U_{i+1}, \{X_j'\}_{j=1}^{i}, \{Y_j'\}_{j=1}^{i+1}\right).
\end{aligned}$$

Now, clearly $\{X_i', Y_i'\}_{i=1}^{\lfloor \epsilon' n\rfloor}$ has the same distribution as $\{X_i, Y_i\}_{i=1}^{\lfloor \epsilon' n\rfloor}$. By (6.6), we have $\{Y_i' = 0,\ X_i' \neq 2\} \subseteq \{U_i \le \frac{1}{10}\}$. We can now estimate

$$\begin{aligned}
\mathbb{P}_{\mathbf{d}}\left[W_{\lfloor \epsilon' n\rfloor} < \frac{\epsilon'}{2} n\right] &\le \mathbb{P}_{\mathbf{d}}\left[\sum_i Y_i > \frac{1}{10}\epsilon' n\right] + P\left[|\{i : Y_i = 0, X_i \neq 2\}| > \frac{1}{5}\epsilon' n\right]\\
&\le \mathbb{P}_{\mathbf{d}}\left[\sum_i Y_i > \frac{1}{10}\epsilon' n\right] + P\left[\left|\left\{i \le \epsilon' n : U_i \le \frac{1}{10}\right\}\right| > \frac{1}{5}\epsilon' n\right].
\end{aligned}$$

The first of these probabilities vanishes by Lemma 6.7, and the second by the Law of Large Numbers. ☐

We will define our subgraph G_n' only on the event $\{W_{\lfloor \epsilon' n\rfloor > \delta' n}\}$. Let

$i_0 = \sup\{i : W_i = 0\}$;
$V_n' =$ vertices that have been identified in passes $i_0, \ldots, \lfloor \epsilon' n\rfloor$;
$E_n' =$ edges that have been constructed in passes $i_0, \ldots, \lfloor \epsilon' n\rfloor$ and are marked;
$G_n' = (V_n', E_n')$;
$\deg'(x) = |\{e \in E_n' : x \in e\}|$;
$J_n' = \{x \in V_n' : x$ has been activated by a pass after i_0 and $\deg'(x) \ge S/4\}$.

All vertices in V_n' are connected by a path of marked edges to the vertex that was activated by pass i_0, and all the edges in this path result from matchings in the passes $i_0, \ldots, \lfloor \epsilon' n\rfloor$. This shows that G_n' is connected. All other facts contained in Properties 1-4 are immediate consequences of the definition of the pass.

Let $\delta = \delta'/2$ and note that $|J_n'| > \delta n$. This follows from the fact that, in the sequence $X_{i_0}, \ldots, X_{\lfloor \epsilon' n\rfloor}$, for every i such that $X_i \neq -1$, the vertex that was activated in pass i must be in J_n'. Since there are at least $\frac{\delta'}{2} n$ such i's, we get $|J_n'| > \delta n$. This completes the proof of Proposition 6.1.

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