Asymptotic Behavior of Aldous' Gossip Process

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Abstract

Aldous (2007) defined a gossip process in which space is a discrete $N \times N$ torus, and the state of the process at time t is the set of individuals who know the information. Information spreads from a site to its nearest neighbors at rate $1/4$ each and at rate $N^{-\alpha}$ to a site chosen at random from the torus. We will be interested in the case in which $\alpha < 3$, where the long range transmission significantly accelerates the time at which everyone knows the information. We prove three results that precisely describe the spread of information in a slightly simplified model on the real torus. The time until everyone knows the information is asymptotically $T = (2 - 2\alpha/3)N^{\alpha/3} \log N$. If ρ_s is the fraction of the population who know the information at time s and ε is small then, for large N, the time until ρ_s reaches ε is $T(\varepsilon) \approx T + N^{\alpha/3} \log(3\varepsilon/M)$, where M is a random variable determined by the early spread of the information. The value of ρ_s at time $s = T(1/3) + tN^{\alpha/3}$ is almost a deterministic function $h(t)$ which satisfies an odd looking integro-differential equation. The last result confirms a heuristic calculation of Aldous.

1 Introduction

We study a model introduced by Aldous (2007) for the spread of gossip and other more economically useful information. His paper considers various game theoretic aspects of random percolation of information through networks. Here we concentrate on one small part, a first passage percolation model with nearest neighbor and long-range jumps introduced in his Section 6.2. The work presented here is also related to

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work of Filipe and Maule (2004) and Cannas, Marco, and Montemurro (2006), who considered the impact of long-range dispersal on the spread of epidemics and invading species.

Space is the discrete torus $\Lambda(N) = (\mathbb{Z} \bmod N)^2$. The state of the process at time t is $\xi_t \subset \Lambda(N)$, the set of individuals who know the information at time t. Information spreads from i to j at rate

$$\nu_{ij} = \begin{cases} 1/4 & \text{if } j \text{ is a (nearest) neighbor of } i \\ \lambda_N/N^2 & \text{if not.} \end{cases}$$

If $\lambda_N = 0$, this is ordinary first passage percolation. If we start with $\xi_0 = \{(0,0)\}$, then the shape theorem for nearest-neighbor first passage percolation, see Cox and Durrett (1981) or Kesten (1986), implies that until the process exits $(-N/2, N/2)^2$, the radius of the set ξ_t grows linearly and ξ_t has an asymptotic shape. From this we see that if $\lambda_N = 0$, then there is a constant c_0 so that the time T_N, until everyone knows the information, satisfies

$$\frac{T_N}{N} \xrightarrow{P} c_0,$$

where $\xrightarrow{P}$ denotes convergence in probability.

To simplify things, we will remove the randomness from the nearest neighbor part of the process, and formulate it on the (real) torus $\Gamma(N) = (\mathbb{R} \bmod N)^2$. The state of the process at time t is $\mathcal{C}_t \subset \Gamma(N)$. The "balloon process" $\mathcal{C}_t$ starts with one "center" chosen uniformly from the torus at time 0. When a center is born at x, a disk with radius 0 is put there, and its radius grows as $r(s) = s/\sqrt{2\pi}$, so that the area of the disk at time s after its birth is $s^2/2$. If the area covered at time t is C_t, then births of new centers occur at rate $\lambda_N C_t$. The location of each new center is chosen uniformly from the torus. If the new point lands at $x \in \mathcal{C}_t$, it will never contribute anything to the growth of the set, but we will count it in $\tilde{X}_t$, the total number of centers.

Here we will be concerned with $\lambda_N = N^{-\alpha}$. To begin we will get rid of trivial cases. If the diameter of $\mathcal{C}_t$ grows linearly, then $\int_0^{c_0 N} C_t\, dt = O(N^3)$. So if $\alpha > 3$, with probability tending to 1 as N goes to ∞, there is no long range jump before the initial disk covers the entire torus, and the time T_N until the entire torus is covered satisfies

$$\frac{T_N}{N} \xrightarrow{P} c_1, \quad \text{where} \quad c_1 = \sqrt{\pi}.$$

If $\alpha = 3$, then with probabilities bounded away from 0, (i) there is no long range jump and $T_N \approx c_1 N$, and (ii) there is one that lands close enough to $(N/2, N/2)$ to make $T_N \le (1-\delta)Nc_1$. Using $\Rightarrow$ for weak convergence, this suggests that

Theorem 0. *When $\alpha = 3$, $T_N/N \Rightarrow$ a random limit concentrated on $[0, c_1]$ and with an atom at c_1.*

This is easily proved by observing that the set-valued process $\{\mathcal{C}_{Nt}/N, t \ge 0\}$ converges to a limit. Further details are left to the reader.

For the remainder of the paper we suppose $\lambda_N = N^{-\alpha}$ with $\alpha < 3$. The overlaps between disks in $\mathcal{C}_t$ poses a difficulty in analyzing the process, so we begin by studying a simpler "balloon branching process" $\mathcal{A}_t$, in which A_t is the sum of the areas of all of the disks at time t, births of new centers occur at rate $\lambda_N A_t$, and the location of each new center is chosen uniformly from the torus. Let X_t be the number of centers at time t in $\mathcal{A}_t$.

Suppose we start $\mathcal{C}_0$ and $\mathcal{A}_0$ from the same randomly chosen point. The areas $C_t = A_t$ until the time of the first birth, which can be made to be the same in the two processes. If we couple the location of the new centers at that time, and continue in the obvious way letting $\mathcal{C}_t$ and $\mathcal{A}_t$ give birth at the same time with the maximum rate possible, to the same place when they give birth simultaneously, and letting $\mathcal{A}_t$ give birth by itself otherwise, then we will have

$$\mathcal{C}_t \subset \mathcal{A}_t, \quad C_t \le A_t, \quad \tilde{X}_t \le X_t \quad \text{for all } t \ge 0. \tag{1.1}$$

couple

X_t is a Crump-Mode-Jagers branching process, but saying these words does not magically solve our problems. Define the length process L_t to be $\sqrt{2\pi}$ times the sum of the radii of all the disks at time t.

$$\begin{aligned} L_t &= \int_0^t (t-s)\,dX_s = \int_0^t X_s\,ds, \\ A_t &= \int_0^t \frac{(t-s)^2}{2}\,dX_s = \int_0^t L_s\,ds. \end{aligned} \tag{1.2}$$

LA

Here and later we use $\int_0^t$ for integration over the closed interval $[0,t]$, i.e., we include the contribution from the atom in dX_s at 0. ($X_0 = 1$ while $X_s = 0$ for $s < 0$.) For the second equality on each line integrate by parts or note that $L_t' = X_t$ and $A_t' = L_t$. Since X_t increases by 1 at rate $\lambda_N A_t$, (X_t, L_t, A_t) is a Markov process.

To simplify formulas, we will often drop the subscript N from λ_N. For comparison with C_t, the parameter λ is important, but in the analysis of A_t it is not. If we let

$$X_t^1 = X(t\lambda^{-1/3}), \quad L_t^1 = \lambda^{1/3} L(t\lambda^{-1/3}), \quad A_t^1 = \lambda^{2/3} A(t\lambda^{-1/3}), \tag{1.3}$$

scale

then (X_t^1, L_t^1, A_t^1) is the process with $\lambda = 1$.

To study the growth of A_t, first we will compute the means of X_t, L_t, and A_t. Let $F(t) = \lambda t^3/3!$. Using the independent and identical behavior of all the disks in $\mathcal{A}_t$ it is easy to show that

$$EX_t = 1 + \int_0^t EX_{t-s}\,dF(s).$$

Solving the above renewal equation and using (1.2), we can show

$$
\begin{aligned}
EX_t &= \sum_{k=0}^{\infty} F^{*k}(t) = V(t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!}, \\
EL_t &= \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+1}}{(3k+1)!}, \\
EA_t &= \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+2}}{(3k+2)!}.
\end{aligned} \tag{1.4}
$$

mean

To evaluate $V(t)$ we note that $V'''(t) = \lambda V(t)$ with $V(0) = 1$, $V'(0) = V''(0) = 0$, so

$$
V(t) = \frac{1}{3}\left[\exp(\lambda^{1/3} t) + \exp(\lambda^{1/3}\omega t) + \exp(\lambda^{1/3}\omega^2 t)\right]. \tag{1.5}
$$

Vtdef

Here $\omega = \left(-1 + i\sqrt{3}\right)/2$ is one of the complex cube roots of 1 and $\omega^2 = \left(-1 - i\sqrt{3}\right)/2$ is the other. Note that each of ω and ω^2 has real part $-1/2$. So the second and third terms in (1.5) go to 0 exponentially fast.

If $\mathcal{F}_s = \sigma\{X_r, L_r, A_r : r \le s\}$, then

$$
\frac{d}{dt} E\left[\left.\begin{matrix} X_t \\ L_t \\ A_t \end{matrix}\,\right|\, \mathcal{F}_s\right]\Bigg|_{t=s} = \begin{pmatrix} 0 & 0 & \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} X_s \\ L_s \\ A_s \end{bmatrix}. \tag{1.6}
$$

infgen

Let Q be the matrix in (1.6). By computing the determinant of $Q - \eta I$ it is easy to see that Q has eigenvalues $\eta = \lambda^{1/3}, \omega\lambda^{1/3}, \omega^2\lambda^{1/3}$, and

$$
e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t) \quad \text{is a (complex) martingale.}
$$

Let I_t, J_t, and K_t be $X_t + \eta L_t + \eta^2 A_t$ for the three values of η respectively, and let M_t, $\tilde{J}_t$, and $\tilde{K}_t$ be the corresponding martingales.

th1 **Theorem 1.** *$\{M_t : t \ge 0\}$ is a positive square integrable martingale with respect to the filtration $\{\mathcal{F}_t : t \ge 0\}$. $EM_t = M_0 = 1$.*

$$
\begin{aligned}
EM_t^2 &= \frac{8}{7} - \frac{1}{3}\exp(-\lambda^{1/3} t) + O\left(\exp(-5\lambda^{1/3} t/2)\right), \\
E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 &= \frac{1}{6}\exp(2\lambda^{1/3} t) + O\left(\exp(\lambda^{1/3} t/2)\right).
\end{aligned}
$$

If we let $M = \lim_{t\to\infty} M_t$, then $P(M > 0) = 1$ and

$$
\exp(-\lambda^{1/3} t) X_t,\ \lambda^{1/3}\exp(-\lambda^{1/3} t) L_t,\ \lambda^{2/3}\exp(-\lambda^{1/3} t) A_t \to M/3
$$

a.s. and in L^2. The distribution of M does not depend on λ.

The last result follows from (1.3), which with (1.2) explains why the three quantities converge to the same limit. The key to the proof of the convergence results is to note that $1+\omega+\omega^2=0$ implies

$$\begin{aligned} 3X_t &= I_t + J_t + K_t, \\ 3\lambda^{1/3}L_t &= I_t + \omega^2 J_t + \omega K_t, \\ 3\lambda^{2/3}A_t &= I_t + \omega J_t + \omega^2 K_t. \end{aligned}$$

The real parts of ω and ω^2 are $-1/2$. Although the results for $E|\tilde{J}_t|^2$ and $E|\tilde{K}_t|^2$ show that the martingales $\tilde{J}_t$ and $\tilde{K}_t$ are not L^2 bounded, it is easy to show that $\exp\left(-\lambda^{1/3}t\right) J_t$ and $\exp\left(-\lambda^{1/3}t\right) K_t \to 0$ a.s. and in L^2, and Theorem 1 then follows from $M_t = \exp\left(-\lambda^{1/3}t\right) I_t \to M$.

Recall that $\lambda_N = N^{-\alpha}$ and let

$$a(t) = (1/3)N^{2\alpha/3}\exp(N^{-\alpha/3}t), \quad l(t) = N^{-\alpha/3}a(t), \quad x(t) = N^{-2\alpha/3}a(t), \tag{1.7}$$ a

so that $A_t/a(t), L_t/l(t), X_t/x(t) \to M$a.s.. Let

$$S(\varepsilon) = N^{\alpha/3}[(2-2\alpha/3)\log N + \log(3\varepsilon)], \tag{1.8}$$ S

so $a(S(\varepsilon)) = \varepsilon N^2$. Let

$$\sigma(\varepsilon) = \inf\{t : A_t \geq \varepsilon N^2\} \quad \text{and} \quad \tau(\varepsilon) = \inf\{t : C_t \geq \varepsilon N^2\}. \tag{1.9}$$ sigtau

The first of these is easy to study.

th2 **Theorem 2.** *If $0<\varepsilon<1$, then as $N\to\infty$*

$$N^{-\alpha/3}(\sigma(\varepsilon) - S(\varepsilon)) \xrightarrow{P} -\log(M).$$

The coupling in (1.1) *implies $\tau(\varepsilon) \geq \sigma(\varepsilon)$. In the other direction, for any $\gamma>0$*

$$\limsup_{N\to\infty} P[\tau(\varepsilon) > \sigma((1+\gamma)\varepsilon)] \leq P\left(M \leq (1+\gamma)\varepsilon^{1/3}\right) + 11\frac{\varepsilon^{1/3}}{\gamma}.$$

The last result implies that for $\varepsilon<1$

$$\tau(\varepsilon) \sim (2-2\alpha/3)N^{\alpha/3}\log N. \tag{1.10}$$ tauLLN

Our next goal is to obtain more precise information about $\tau(\varepsilon)$ and about how $|C_t|/N^2$ increases from a small positive level to reach 1.

The first result in Theorem 2 shows that $(\sigma(\varepsilon)-S(\varepsilon))/N^{\alpha/3}$ is determined by the random variable M from Theorem 1, which in turn is determined by what happens early in the growth of the branching balloon process. Let

$$R = N^{\alpha/3}[(2-2\alpha/3)\log N - \log(M)], \tag{1.11}$$ R

R is defined so that $a(R) = (1/3)N^2/M$, and hence $A_R/N^2 \xrightarrow{P} 1/3$. Define

$$\psi(t) \equiv R + N^{\alpha/3}t, \quad W \equiv \psi(\log(3\varepsilon)), \quad \text{and} \quad I_{\varepsilon,t} = [\log(3\varepsilon), t] \tag{1.12}$$

psiWI

for $\log(3\varepsilon) \le t$. W is defined so that $a(W) = \varepsilon N^2/M$ and hence $A_W/N^2 \xrightarrow{P} \varepsilon$. The arguments that led to Theorem 2 will show that if ε is small then C_W/A_W is close to 1 with high probability.

To get a lower bound on the growth of C_t after time W we declare that the centers in $\mathcal{C}_W$ and $\mathcal{A}_W$ to be generation 0 in $\mathcal{C}_t$ and $\mathcal{A}_t$ respectively, and we number the succeeding generations in the obvious way, a center born from an area of generation k is in generation $k+1$. For $t \ge \log(3\varepsilon)$, let $C^k_{W,\psi(t)}$ and $A^k_{W,\psi(t)}$ denote the areas covered at time $\psi(t)$ by respective centers of generations $j \in \{0, 1, \dots, k\}$ and let

$$g_0(t) = \varepsilon\left[1 + (t - \log(3\varepsilon)) + \frac{(t-\log(3\varepsilon))^2}{2}\right], \quad f_0(t) = g_0(t) - \varepsilon^{7/6}. \tag{1.13}$$

gfdef

To explain these definitions, we note that Lemma 4.3 will show that for any t, there is an $\varepsilon_0 = \varepsilon_0(t)$ so that for any $0 < \varepsilon < \varepsilon_0$

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}} \left|N^{-2}A^0_{W,\psi(s)} - g_0(s)\right| > \eta\right) = 0 \quad \text{for any } \eta > 0,$$

$$P\left(\inf_{s\in I_{\varepsilon,t}} N^{-2}(C^0_{W,\psi(s)} - A^0_{W,\psi(s)}) < -\varepsilon^{7/6}\right) \le P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}.$$

Since $C^0_{W,\psi(t)} \le A^0_{W,\psi(t)}$, if ε is small, with high probability $g_0(t)$ and $f_0(t)$ provide upper and lower bounds respectively for $C^0_{W,\psi(t)}$.

To begin to improve these bounds we let

$$f_1(t) = 1 - (1 - f_0(t))\exp\left(-\int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} f_0(s)\,ds\right),$$

and define g_1 similarly. To explain this equation note that an $x \notin C^0_{W,\psi(t)}$ will not be in $C^1_{W,\psi(t)}$ if and only if no generation 1 center is born in the space-time cone

$$K^\varepsilon_{x,t} \equiv \left\{(y,s) \in \Gamma(N) \times [W, \psi(t)] : |y - x| \le (\psi(t) - s)/\sqrt{2\pi}\right\}.$$

Lemma 4.4 shows that for $0 < \varepsilon < \varepsilon_0$ and $\delta > 0$,

$$\limsup_{N\to\infty} P\left(\inf_{s\in I_{\varepsilon,t}} N^{-2}C^1_{W,\psi(s)} - f_1(s) < -\delta\right) \le P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}.$$

To iterate this we will let

$$f_{k+1}(t) = 1 - (1 - f_k(t))\exp\left(-\int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2}(f_k(s) - f_{k-1}(s))\,ds\right)$$

for $k \geq 1$. The difference $f_k(s) - f_{k-1}(s)$ in the integral comes from the fact that a new point in generation $k+1$ must come from a point that is in generation k but not in generation $k-1$. Combining these equations we have

$$f_{k+1}(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} f_k(s)\, ds \right). \tag{1.14}$$

fkinteq

Since $f_1(t) \geq f_0(t)$, letting $k \to \infty$, $f_k(t) \uparrow f_\varepsilon(t)$, where f_ε is the unique solution of

$$f_\varepsilon(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} f_\varepsilon(s)\, ds \right) \tag{1.15}$$

fepinteq

with $f_\varepsilon(\log(3\varepsilon)) = \varepsilon - \varepsilon^{7/6}$. $g_k(t)$ and $g_\varepsilon(t)$ are defined similarly.

$g_\varepsilon(t)$ and $f_\varepsilon(t)$ provide upper and lower bounds on the growth of $C_{\psi(t)}$ for $t \geq \log(3\varepsilon)$. To close the gap between these bounds we let $\varepsilon \to 0$.

h **Lemma 1.1.** *For any $t < \infty$, if $I_{\varepsilon,t} = [\log(3\varepsilon), t]$, then as $\varepsilon \to 0$,*

$$\sup_{s \in I_{\varepsilon,t}} |f_\varepsilon(s) - h(s)|\,, \ \sup_{s \in I_{\varepsilon,t}} |g_\varepsilon(s) - h(s)| \to 0$$

for some nondecreasing h with (a) $\lim_{t\to-\infty} h(t) = 0$, (b) $\lim_{t\to\infty} h(t) = 1$,

(c) $$h(t) = 1 - \exp\left(- \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds \right),$$

and (d) $0 < h(t) < 1$ for all t.

If one removes the 2 from inside the exponential, this is equation (36) in Aldous (2007). Since there is no initial condition, the solution is only unique up to time translation.

th3 **Theorem 3.** *Let h be the function in Lemma 1.1. For any $t < \infty$ and $\delta > 0$,*

$$\lim_{N\to\infty} P\left(\sup_{s \leq t} |N^{-2} C_{\psi(s)} - h(s)| \leq \delta \right) = 1.$$

This result shows that the displacement of $\tau(\varepsilon)$ from $(2 - 2\alpha/3)N^{\alpha/3} \log N$ on the scale $N^{\alpha/3}$ is dictated by the random variable M that gives the rate of growth of the branching balloon process, and that once C_t reaches εN^2, the growth is deterministic.

The solution $h(t)$ never reaches 1, so we need a little more work to show that

Theorem 4. *Let T be the first time the torus is covered. As $N \to \infty$*

$$T/(N^{\alpha/3} \log N) \xrightarrow{P} 2 - 2\alpha/3.$$

Proof. Theorem 3 implies that if $\delta > 0$ and N is large, then the number of centers in $\mathcal{C}_{\psi(0)}$ with high probability dominate a Poisson random variable with mean $\lambda(\delta)N^{2-(2\alpha/3)}$, where

$$\lambda(\delta) = \int_{-\infty}^{0} (h(s) - \delta)^+ \, ds.$$

If δ_0 is small enough, $\lambda_0 \equiv \lambda(\delta_0) > 0$. Dividing the torus into disjoint squares of size $\kappa N^{\alpha/3}\sqrt{\log N}$, the probability a given square is vacant is $\exp(-\lambda_0 \kappa \log N)$. If N is large, the number of squares is $\leq N^{2-(2\alpha/3)}$ So if $\lambda_0 \kappa \geq 2$, then with high probability none of our squares is vacant. Thus even if no more births of new centers occur then the entire square will be covered by a time $\psi(0) + O(N^{\alpha/3}\sqrt{\log N})$. $\square$

2 Proof of Theorem 1

We begin with some calculus

conv **Lemma 2.1.** $\int_0^t s^m (t-s)^n ds = \frac{m!n!}{(m+n+1)!} t^{m+n+1}$.

Proof. Integrating by parts

$$\begin{aligned}
\int_0^t \frac{s^m}{m!} \frac{(t-s)^n}{n!} \, ds &= \int_0^t \frac{s^{m+1}}{(m+1)!} \frac{(t-s)^{n-1}}{(n-1)!} \, ds \\
\ldots &= \int_0^t \frac{s^{m+n}}{(m+n)!} \, ds = \frac{t^{m+n+1}}{(m+n+1)!},
\end{aligned}$$

which proves the desired result. $\square$

Let $F(t) = \lambda t^3/3!$ for $t \geq 0$, and $F(t) = 0$ for $t < 0$. Let $V(t) = \sum_{k=0}^{\infty} F^{*k}(t)$, where $*k$ indicates the k-fold convolution.

V **Lemma 2.2.** *If* $\omega = \left(-1 + i\sqrt{3}\right)/2$, *then*

$$V(t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!} = \frac{1}{3}\left[\exp\left(\lambda^{1/3} t\right) + \exp\left(\lambda^{1/3} \omega t\right) + \exp\left(\lambda^{1/3} \omega^2 t\right)\right].$$

Proof. We first use induction to show that

$$F^{*k}(t) = \begin{cases} \lambda^k t^{3k}/(3k)! & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2.1}$$ Fconv

This holds for $k = 0, 1$ by our assumption. If the equality holds for $k = n$, then using Lemma 2.1 we have for $t \geq 0$

$$F^{*(n+1)}(t) = \int_0^t F^{*n}(t-s) \, dF(s) = \int_0^t \frac{\lambda^n (t-s)^{3n}}{(3n)!} \frac{\lambda s^2}{2} \, ds = \frac{\lambda^{n+1} t^{3n+3}}{(3n+3)!}.$$

It follows by induction that $V(t) = \sum_{k=0}^{\infty} \lambda^k t^{3k}/(3k)!$. To evaluate the sum we note that setting $\lambda = 1$, $U(t) = \sum_{k=0}^{\infty} t^{3k}/(3k)!$ solves

$$U'''(t) = U(t) \quad \text{with } U(0) = 1 \text{ and } U'(0) = U''(0) = 0.$$

This differential equation has solutions of the from $e^{\gamma t}$, where $\gamma^3 = 1$, i.e. $\gamma = 1, \omega$ and ω^2. This leads to the general solution

$$U(t) = Ae^t + Be^{\omega t} + Ce^{\omega^2 t}$$

for some constants A, B, C. Using the initial conditions for $U(t)$ we have

$$A + B + C = 1, \quad A + B\omega + C\omega^2 = 0, \quad A + B\omega^2 + C\omega = 0.$$

Since $1 + \omega + \omega^2 = 0$, we have $A = B = C = 1/3$. Since $V(t) = U(\lambda^{1/3} t)$, we have proved the desired result. $\square$

XLAlem **Lemma 2.3.** $E(X_t, L_t, A_t) = (V(t), V''(t)/\lambda, V'(t)/\lambda)$.

Proof. $F(t) = \lambda t^3/3!$. In the balloon branching process, the initial center x gives birth to new centers at rate $F'(t) = \lambda t^2/2$, and all the centers behave independently and with the same distribution as the one at x. So

$$EX_t = 1 + \int_0^t EX_{t-s}\, dF(s).$$

Using (4.5) from Chapter 3 of Durrett (2005) and then (1.2):

$$\begin{aligned} EX_t &= V(t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!}, \\ EL_t &= \int_0^t EX_s\, ds = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+1}}{(3k+1)!}, \\ EA_t &= \int_0^t EL_s\, ds = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k+2}}{(3k+2)!}. \end{aligned} \tag{2.2}$$ meanXLA

Since $V(t) = 1 + \sum_{k=0}^{\infty} \lambda^{k+1} t^{3k+3}/(3k+3)!$, it is easy to see that $EA_t = V'(t)/\lambda$ and $EL_t = V''(t)/\lambda$. $\square$

Our next step is to compute second moments.

renewaleq **Lemma 2.4.** *Let $\{N_t : t \geq 0\}$ be a Poisson process on $[0, \infty)$ with intensity $\lambda(\cdot)$ and let Π_t be the set of points at time t. If $\{Y_t, Z_t : t \geq 0\}$ are two complex valued stochastic processes satisfying*

$$Y_t = y(t) + \sum_{s_i \in \Pi_t} Y^i_{t-s_i}, \qquad Z_t = z(t) + \sum_{s_i \in \Pi_t} Z^i_{t-s_i},$$

where (Y^i, Z^i), $i = 1, 2, \ldots$ *are i.i.d. copies of* (Y, Z), *and independent of* N, *then*

$$EY_t = y(t) + \int_0^t EY_{t-s}\lambda(s)\,ds,$$
$$E(Y_t Z_t) = (EY_t)(EZ_t) + \int_0^t E(Y_{t-s}Z_{t-s})\lambda(s)\,ds.$$

Proof. N_t has Poisson distribution with mean $\Lambda_t = \int_0^t \lambda(s)ds$. Given $N_t = n$, the conditional distribution of Π_t is same as the distribution of $\{t_1, \ldots, t_n\}$, where $t_1, \ldots, t_n$ are i.i.d. from $[0, t]$ with density $\beta(\cdot) = \lambda(\cdot)/\Lambda_t$. Hence

$$E(Y_t|N_t) = y(t) + \sum_{i=1}^{N_t} EY^i_{t-t_i} = y(t) + N_t \int_0^t EY_{t-s}\,\beta(s)\,ds,$$

and taking expected values $EY_t = y(t) + \int_0^t EY_{t-s}\lambda(s)\,ds$.

Similarly $EZ_t = z(t) + \int_0^t EZ_{t-s}\lambda(s)ds$. Using the conditional distribution of Π_t given N_t, $E(Y_t Z_t|N_t)$ is

$$\begin{aligned}
&= y(t)z(t) + y(t)E\sum_{i=1}^{N_t} Z^i_{t-t_i} + z(t)E\sum_{i=1}^{N_t} Y^i_{t-t_i} + E\left[\sum_{i=1}^{N_t} Y^i_{t-t_i}Z^i_{t-t_i} + \sum_{i\neq j} Y^i_{t-t_i}Z^j_{t-t_j}\right]\\
&= y(t)z(t) + y(t)N_t\int_0^t EZ_{t-s}\,\beta(s)\,ds + z(t)N_t\int_0^t EY_{t-s}\,\beta(s)\,ds\\
&\quad + N_t\int_0^t E(Y_{t-s}Z_{t-s})\,\beta(s)\,ds + N_t(N_t-1)\int_0^t EY_{t-s}\,\beta(s)\,ds\int_0^t EZ_{t-s}\beta(s)ds.
\end{aligned}$$

Taking expectation on both sides and using $EN_t(N_t - 1) = \Lambda_t^2$, we get

$$E(Y_t Z_t) = (EY_t)(EZ_t) + \int_0^t E(Y_{t-s}Z_{t-s})\lambda(s)ds,$$

which completes the proof. □

mart **Lemma 2.5.** *If* $M_t = \exp(-\lambda^{1/3}t)[X_t + \lambda^{1/3}L_t + \lambda^{2/3}A_t]$, *then* $\{M_t : t \geq 0\}$ *is a square integrable martingale with respect to the filtration* $\{\mathcal{F}_t : t \geq 0\}$. $EM_t = 1$ *and*

$$EM_t^2 = \frac{8}{7} - \frac{1}{3}\exp\left(-\lambda^{1/3}t\right) + \theta_t \quad \textit{where} \quad |\theta_t| \leq \frac{4}{15}\exp(-5\lambda^{1/3}t/2).$$

and hence $(8/7) - EM_t^2 \leq \exp(-\lambda^{1/3}t)$.

Proof. Let $h(t, x, \ell, a) = \exp(-\lambda^{1/3}t)[x + \lambda^{1/3}\ell + \lambda^{2/3}a]$, and let $\mathcal{L}$ be the generator of the Markov process (t, X_t, L_t, A_t). (1.6) implies $\mathcal{L}h = 0$, so the desired result follows from Dynkin's formula. $EM_t = EM_0 = 1$.

To compute EM_t^2 we use Lemma 2.4. Let $Y_t = Z_t = X_t + \lambda^{1/3} L_t + \lambda^{2/3} A_t$ and $g(t) \equiv (EY_t)^2$. Since $EM_t = 1$, $g(t) = \exp\left(2\lambda^{1/3} t\right)$. Then using Lemma 2.4

$$EY_t^2 = g(t) + \int_0^t EY_{t-s}^2 \, dF(s).$$

Solving the renewal equation using (4.8) in Chapter 3 of Durrett (2005),

$$EY_t^2 = g * V(t) = \exp\left(2\lambda^{1/3} t\right) + \int_0^t \exp(2\lambda^{1/3}(t-s)) V'(s) \, ds,$$

where $V = \sum_{k=0}^{\infty} F^{*k}$. To evaluate the integral we use Lemma 2.2 to conclude

$$\begin{aligned}
&\int_0^t \exp\left(-2\lambda^{1/3} s\right) V'(s) \, ds \\
&= \frac{1}{3} \int_0^t \exp\left(-2\lambda^{1/3} s\right) \cdot \lambda^{1/3} \left[\exp\left(\lambda^{1/3} s\right) + \omega \exp\left(\lambda^{1/3} \omega s\right) + \omega^2 \exp\left(\lambda^{1/3} \omega^2 s\right)\right] ds \\
&= \frac{1}{3} \left[\frac{1}{1-2} \left\{\exp\left(-\lambda^{1/3} t\right) - 1\right\} + \frac{\omega}{\omega - 2} \left\{\exp\left((\omega - 2)\lambda^{1/3} t\right) - 1\right\} \right. \\
&\qquad\qquad \left. + \frac{\omega^2}{\omega^2 - 2} \left\{\exp\left((\omega^2 - 2)\lambda^{1/3} t\right) - 1\right\}\right].
\end{aligned}$$

Now using $1 = -\omega - \omega^2$ and $\omega^3 = 1$,

$$1 - \frac{\omega}{\omega - 2} - \frac{\omega^2}{\omega^2 - 2} = 1 - \frac{\omega^3 - 2\omega + \omega^3 - 2\omega^2}{\omega^3 - 2\omega^2 - 2\omega^2 + 4} = 1 - \frac{4}{7} = \frac{3}{7}.$$

Since $\omega = \left(-1 + i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1 - i\sqrt{3}\right)/2$, the remaining error satisfies

$$\begin{aligned}
3|\theta_t| &= \left|\frac{\omega}{\omega - 2} \exp\left((\omega - 2)\lambda^{1/3} t\right)\right| + \left|\frac{\omega^2}{\omega^2 - 2} \exp\left((\omega^2 - 2)\lambda^{1/3} t\right)\right| \\
&= \left(\frac{1}{|\omega - 2|} + \frac{1}{|\omega^2 - 2|}\right) \exp\left(-5\lambda^{1/3} t/2\right) \le 2 \cdot \frac{2}{5} \exp\left(-5\lambda^{1/3} t/2\right),
\end{aligned}$$

since $\omega - 2$ and $\omega^2 - 2$ each have real part $-5/2$. Putting all together

$$\int_0^t \exp\left(-2\lambda^{1/3} s\right) V'(s) \, ds = \frac{1}{7} - \frac{1}{3} \exp\left(-\lambda^{1/3} t\right) + \theta_t, \tag{2.3}$$

intbd

Since $EM_t^2 = \exp\left(-2\lambda^{1/3} t\right) EY_t^2$, the desired result follows. □

We use the previous calculation to get bounds for EA_t^2, EL_t^2 and EX_t^2, which will be useful later.

sqbound **Lemma 2.6.** *Let $a(\cdot), l(\cdot)$ and $x(\cdot)$ be as in (1.7). Then*

$$EA_t^2 \le \frac{27}{2}a^2(t), \quad EL_t^2 \le \frac{27}{2}l^2(t), \quad EX_t^2 \le \frac{27}{2}x^2(t).$$

Proof. By (2.3) we have

$$\int_0^t \exp\left(-2\lambda^{1/3}s\right) V'(s)\, ds \le \frac{1}{7} + \frac{4}{15} = \frac{43}{105} \le \frac{1}{2}. \tag{2.4}$$ intbd1

Now using Lemma 2.4

$$EA_t^2 = (EA_t)^2 + \int_0^t EA_{t-s}^2\, dF(s), \quad EL_t^2 = (EL_t)^2 + \int_0^t EL_{t-s}^2\, dF(s),$$
$$EX_t^2 = (EX_t)^2 + \int_0^t EX_{t-s}^2\, dF(s).$$

Solving the renewal equations $EA_t^2 = \phi_a * V(t), EL_t^2 = \phi_l * V(t)$ and $EX_t^2 = \phi_x * V(t)$, where $V(\cdot)$ is as in Lemma 2.2 and $\phi_a(t) = (EA_t)^2, \phi_l(t) = (EL_t)^2$ and $\phi_x(t) = (EX_t)^2$. A crude upper bound for $\phi_a(t)$ is $9a^2(t)$. Since $a(t-s) = a(t)\exp\left(-\lambda^{1/3}s\right)$,

$$a^2 * V(t) = a^2(t)\left[1 + \int_0^t \exp\left(-\lambda^{1/3}s\right) V'(s)\, ds\right] \le \frac{3a^2(t)}{2} \tag{2.5}$$ a2bd

by (2.4). Hence $EA_t^2 \le 9a^2 * V(t) \le (27/2)a^2(t)$.

Similarly using the bounds $9l^2(t)$ and $9x^2(t)$ for $\phi_l(t)$ and $\phi_x(t)$ respectively and noting that $l(t-s)/l(t) = x(t-s)/x(t) = \exp\left(-\lambda^{1/3}s\right)$, we get the desired bounds for EL_t^2 and EX_t^2. □

JKbds **Lemma 2.7.** *Let $\tilde{J}_t, \tilde{K}_t = e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t)$ when $\eta = \omega\lambda^{1/3}$, $\omega^2\lambda^{1/3}$ respectively. Then $\tilde{J}_t$ and $\tilde{K}_t$ are complex martingales with respect to the filtration $\mathcal{F}_t$, and*

$$E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 = \frac{1}{6}\exp(2\lambda^{1/3}t) + \frac{1}{2} + \theta_t, \quad \textit{where } |\theta_t| \le \frac{2}{3}\exp\left(\lambda^{1/3}t/2\right),$$

and hence $E|\tilde{J}_t|^2, E|\tilde{K}_t|^2 \le (4/3)\exp\left(2\lambda^{1/3}t\right)$.

Proof. Let $h(t,x,\ell,a) = e^{-\eta t}(x + \eta\ell + \eta^2 a)$, and let $\mathcal{L}$ be the generator of the Markov process (t, X_t, L_t, A_t). (1.6) implies $\mathcal{L}h = 0$, where $\eta = \lambda^{1/3}\omega, \lambda^{1/3}\omega^2$, so that $\tilde{J}_t$ and $\tilde{K}_t$ are complex martingales from Dynkin's formula.

First we compute $E|J_t|^2$, where $J_t = \exp\left(\lambda^{1/3}\omega t\right)\tilde{J}_t$. For that we use Lemma 2.4 with $Y_t = J_t$ and $Z_t = \bar{J}_t$, the complex conjugate. Since $\tilde{J}_t$ is a complex martingale with $\tilde{J}_0 = 1$ and $\omega = \left(-1 + i\sqrt{3}\right)/2$, $E\tilde{J}_t = 1$ and hence

$$|EJ_t|^2 = \exp(-\lambda^{1/3}t).$$

Using Lemma 2.4 $E|J_t|^2 = |EJ_t|^2 + \int_0^t E|J_{t-s}|^2\,dF(s)$. Solving the renewal equation as we have done twice before

$$E|J_t|^2 = \exp(-\lambda^{1/3}t) + \int_0^t \exp(-\lambda^{1/3}(t-s))V'(s)\,ds.$$

Repeating the first part of the proof for $K_t = \exp\left(\lambda^{1/3}\omega^2 t\right)\tilde{K}_t$, we see that $E|K_t|^2$ is also equal to the right-hand side above.

The integral is $\exp(-\lambda^{1/3}t)$ times

$$\begin{aligned}
&\frac{1}{3}\int_0^t \exp\left(\lambda^{1/3}s\right)\cdot\lambda^{1/3}\left[\exp\left(\lambda^{1/3}s\right) + \omega\exp\left(\lambda^{1/3}\omega s\right) + \omega^2\exp\left(\lambda^{1/3}\omega^2 s\right)\right]\,ds \\
&= \frac{1}{3}\left[\frac{1}{1+1}\left\{\exp\left(2\lambda^{1/3}t\right) - 1\right\} + \frac{\omega}{\omega+1}\left\{\exp\left((\omega+1)\lambda^{1/3}t\right) - 1\right\}\right. \\
&\qquad\qquad\qquad\qquad\left. + \frac{\omega^2}{\omega^2+1}\left\{\exp\left((\omega^2+1)\lambda^{1/3}t\right) - 1\right\}\right].
\end{aligned}$$

Now using $1 = -\omega - \omega^2$ and $\omega^3 = 1$,

$$-\frac{1}{2} - \frac{\omega}{\omega+1} - \frac{\omega^2}{\omega^2+1} = -\frac{1}{2} - \frac{\omega^3+\omega+\omega^3+\omega^2}{\omega^3+\omega^2+\omega+1} = -\frac{3}{2}.$$

Since $\omega = \left(-1+i\sqrt{3}\right)/2$ and $\omega^2 = \left(-1-i\sqrt{3}\right)/2$, if we take

$$\begin{aligned}
\theta_t &= \frac{1}{3}\left[\frac{\omega}{\omega+1}\exp\left((\omega+1)\lambda^{1/3}t\right) + \frac{\omega^2}{\omega^2+1}\exp\left((\omega^2+1)\lambda^{1/3}t\right)\right], \text{ then} \\
3|\theta_t| &\le \left(\frac{1}{|\omega+1|} + \frac{1}{|\omega^2+1|}\right)\exp\left(\lambda^{1/3}t/2\right) \le 2\exp\left(\lambda^{1/3}t/2\right),
\end{aligned}$$

since each of $\omega+1$ and ω^2+1 has real part $1/2$. Putting all together

$$E|J_t|^2 \le \frac{1}{6}\exp\left(\lambda^{1/3}t\right) + \frac{1}{2}\exp\left(-\lambda^{1/3}t\right) + \frac{2}{3}\exp\left(-\lambda^{1/3}t/2\right), \tag{2.6}$$ Jbd

which completes the proof, since $E|\tilde{J}_t|^2/E|J_t|^2 = \exp(\lambda^{1/3}t) = E|\tilde{K}_t|^2/E|K_t|^2$. □

Lemma 2.8. *If $M = \lim_{t\to\infty} M_t$, we have $P(M>0) = 1$ and*

$$\exp(-\lambda^{1/3}t)X_t,\ \lambda^{1/3}\exp(-\lambda^{1/3}t)L_t,\ \lambda^{2/3}\exp(-\lambda^{1/3}t)A_t \to \frac{M}{3} \quad \textit{a.s. and in } L^2.$$

Proof. $M = \lim_{t\to\infty} M_t$ exists a.s. and in L^2, since M_t is an L^2 bounded martingale. Recall that

$$\begin{aligned}
I_t &= X_t + \lambda^{1/3}L_t + \lambda^{2/3}A_t, \\
J_t &= X_t + \omega\lambda^{1/3}L_t + \omega^2\lambda^{2/3}A_t, \\
K_t &= X_t + \omega^2\lambda^{1/3}L_t + \omega\lambda^{2/3}A_t.
\end{aligned}$$

Since $1+\omega+\omega^2=0$ and $\omega^3=1$,

$$\begin{aligned} 3X_t &= I_t + J_t + K_t, \\ 3\lambda^{1/3} L_t &= I_t + \omega^2 J_t + \omega K_t, \\ 3\lambda^{2/3} A_t &= I_t + \omega J_t + \omega^2 K_t. \end{aligned} \tag{2.7}$$

lincomb

Since $\exp(-\lambda^{1/3}t)I_t \to M$, it suffices to show that $\exp(-\lambda^{1/3}t)J_t$ and $\exp(-\lambda^{1/3}t)K_t$ go to 0 a.s. and in L^2. We will only prove this for J_t, since the argument for K_t is almost identical. $\tilde{J}_t$ is a complex martingale, so $|\tilde{J}_t|$ is a real submartingale. Using the L^2 maximal inequality, (4.3) in Chapter 4 of Durrett (2005), and Lemma 2.7,

$$E\left(\max_{0\le s\le t} |\tilde{J}_s|^2\right) \le 4E|\tilde{J}_t|^2 \le \frac{16}{3}\exp(2\lambda^{1/3}t). \tag{2.8}$$

L2max

The real part of ω is $-1/2$. So writing $\tilde{J}_s = \exp(\lambda^{1/3}(1-\omega)s)\cdot\exp(-\lambda^{1/3}s)J_s$, we see that

$$E\left(\max_{u\le s\le t} |\tilde{J}_s|^2\right) \ge \exp(3\lambda^{1/3}u)E\left(\max_{u\le s\le t} \left|\exp(-\lambda^{1/3}s)J_s\right|^2\right). \tag{2.9}$$

hammer

Combining these bounds with Chebyshev inequality, and taking $t_n = 2\lambda^{-1/3}\log n$ for $n=1,2,\ldots$

$$\begin{aligned} P\left(\max_{t_n\le s\le t_{n+1}} \left|\exp\left(-\lambda^{1/3}s\right)J_s\right|^2 \ge \varepsilon\right) &\le \varepsilon^{-2}E\left(\max_{t_n\le s\le t_{n+1}} \left|\exp(-\lambda^{1/3}s)J_s\right|^2\right) \\ &\le \frac{16}{3}\varepsilon^{-2}\exp\left(\lambda^{1/3}(2t_{n+1}-3t_n)\right) = \frac{16}{3}\varepsilon^{-2}\frac{(n+1)^4}{n^6} \end{aligned} \tag{2.10}$$

supJbd

for any $\varepsilon>0$. Summing over n, and using the Borel-Cantelli lemma

$$|\exp(-\lambda^{1/3}s)J_s| \to 0 \quad \text{a.s.}$$

To get convergence in L^2 we use (2.6).

$$E\left|\exp\left(-\lambda^{1/3}t\right)J_t\right|^2 \le \frac{4}{3}\exp\left(-\lambda^{1/3}t\right) \to 0 \text{ as } t\to\infty.$$

To prove that $P(M>0)=1$ we begin by noting that convergence in L^2 implies that $P(M>0)>0$. Every time a new balloon is born it has positive probability of starting a process with a positive limit, so this will happen eventually and $P(M>0)=1$. □

3 Proof of Theorem 2

Recall that $\sigma(\varepsilon)=\inf\{t : A_t \ge \varepsilon N^2\}$ and $\tau(\varepsilon)=\inf\{t : C_t\ge \varepsilon N^2\}$. Also recall the definitions of $a(\cdot), l(\cdot), x(\cdot)$ and $S(\cdot)$ from (1.7) and (1.8). Note that $a(S(\varepsilon))=\varepsilon N^2$ and $A_t/a(t), L_t/l(t), X_t/x(t)\to M$ a.s. by Theorem 1. We begin by estimating the difference between them.

supbound **Lemma 3.1.** *For any* $\gamma, u > 0$

$$P\left(\sup_{t\geq u}|A_t/a(t) - M| \geq \gamma^2\right) \leq C\gamma^{-4}\exp\left(-\lambda^{1/3}u\right)$$

for some constant C*. The same bound holds for* $P\left(\sup_{t\geq u}|L_t/l(t) - M| \geq \gamma^2\right)$ *and* $P\left(\sup_{t\geq u}|X_t/x(t) - M| \geq \gamma^2\right)$*.*

Proof. Using (2.7) $A_t/a(t) = M_t + \omega\exp\left(-\lambda^{1/3}t\right)J_t + \omega^2\exp\left(-\lambda^{1/3}t\right)K_t$. For $0 < u \leq t$ the triangle inequality implies

$$|A_t/a(t) - M| \leq |M_t - M| + \left|\exp\left(-\lambda^{1/3}t\right)J_t\right| + \left|\exp\left(-\lambda^{1/3}t\right)K_t\right|. \tag{3.1}$$ bd1

Taking the supremum over t,

$$\begin{aligned}
&P\left(\sup_{t\geq u}|A_t/a(t) - M| \geq \gamma^2\right) \\
\leq &P\left(\sup_{t\geq u}|M_t - M| \geq \gamma^2/3\right) + P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right| \geq \gamma^2/3\right) \\
&\qquad + P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)K_t\right| \geq \gamma^2/3\right).
\end{aligned} \tag{3.2}$$ supbd

To bound the first term in the right hand side of (3.2) we note that

$$E\left(\sup_{t\geq u}|M_t - M|^2\right) = \lim_{U\to\infty}E\left(\max_{u\leq t\leq U}|M_t - M|^2\right).$$

Using triangle inequality $|M_t - M| \leq |M_t - M_u| + |M_u - M|$. Taking supremum over $t \in [u, U]$ and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$E\left(\max_{u\leq t\leq U}|M_t - M|^2\right) \leq 2\left(E\left(\max_{u\leq t\leq U}|M_t - M_u|^2\right) + E|M_u - M|^2\right).$$

Using the L^2 maximal inequality, (4.3) in Chapter 4 of Durrett (2005), and orthogonality of martingale increments,

$$E\left(\max_{u\leq t\leq U}|M_t - M_u|^2\right) \leq 4E(M_U - M_u)^2 = 4(EM_U^2 - EM_u^2).$$

Since the martingale M_t converges to M in L^2, $EM^2 = \lim_{t\to\infty}EM_t^2 = 8/7$. Then using orthogonality of martingale increments and Lemma 2.5,

$$E(M_u - M)^2 = EM^2 - EM_u^2 \leq \exp\left(-\lambda^{1/3}u\right).$$

Combining the last four bounds with Lemma 2.5, and using Chebyshev inequality

$$P\left(\sup_{t\geq u}|M_t - M| \geq \gamma^2/3\right) \leq 9\gamma^{-4}\cdot 10\exp\left(-\lambda^{1/3}u\right). \tag{3.3}$$ supbd1

To bound the second term in the right hand side of (3.2) we take $t_n = u + 2\lambda^{-1/3}\log n$ for $n = 1, 2, \ldots$ and use an argument similar to the one leading to (2.10) together with Chebyshev inequality to get

$$\begin{aligned}
P\left(\sup_{t\geq u}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right| \geq \gamma^2/3\right) &\leq \sum_{n=1}^{\infty} P\left(\max_{t_n\leq t\leq t_{n+1}}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right| \geq \gamma^2/3\right)\\
&\leq 9\gamma^{-4}\sum_{n=1}^{\infty} E\left(\max_{t_n\leq t\leq t_{n+1}}\left|\exp\left(-\lambda^{1/3}t\right)J_t\right|\right)^2\\
&\leq 9\cdot\frac{16}{3}\gamma^{-4}\sum_{n=1}^{\infty}\exp(\lambda^{1/3}(2t_{n+1}-3t_n))\\
&= 48\gamma^{-4}\exp(-\lambda^{1/3}u)\sum_{n=1}^{\infty}\frac{(n+1)^4}{n^6}. \qquad (3.4)
\end{aligned}$$

supbd2

Repeating the previous argument for the third term in the right hand side of (3.2) we get the same upper bound as in (3.4). Combining (3.2), (3.3) and (3.4) we get the desired bound for $A_t/a(t)$.

The same bound also works for both $L_t/l(t)$ and $X_t/x(t)$, since using (2.7)

$$\begin{aligned}
L_t/l(t) &= M_t + \omega^2\exp(-\lambda^{1/3}t)J_t + \omega\exp(-\lambda^{1/3}t)K_t,\\
X_t/x(t) &= M_t + \exp(-\lambda^{1/3}t)J_t + \exp(-\lambda^{1/3}t)K_t,
\end{aligned}$$

and so the upper bound in (3.1) also works for $L_t/l(t)$ and $X_t/x(t)$. □

We now use Lemma 3.1 to study the limiting behavior of $\sigma(\varepsilon)$.

ALXbd **Lemma 3.2.** *Let $W_\varepsilon = S(\varepsilon/M)$, where $S(\cdot)$ is as in* (1.8) *and M is the limit random variable in Theorem 1. Then for any $\eta > 0$*

$$\begin{aligned}
\lim_{N\to\infty} P(|A_{W_\varepsilon} - \varepsilon N^2| > \eta N^2) &= \lim_{N\to\infty} P(|L_{W_\varepsilon} - \varepsilon N^{2-\alpha/3}| > \eta N^{2-\alpha/3})\\
&= \lim_{N\to\infty} P(|X_{W_\varepsilon} - \varepsilon N^{2-2\alpha/3}| > \eta N^{2-2\alpha/3}) = 0.
\end{aligned}$$

Proof. Since $P(M > 0) = 1$, given $\theta > 0$, we can choose $\gamma = \gamma(\theta) > 0$ so that $\gamma < \eta/\varepsilon$ and

$$P(M < \gamma) < \theta. \qquad (3.5)$$

Mnot0

Using Lemma 3.1 we can choose a constant $b = b(\gamma, \theta)$ such that

$$P\left(\sup_{t\geq bN^{\alpha/3}}|A_t/a(t) - M| > \gamma^2\right) < \theta.$$

Combining with (3.5)

$$P\left(\sup_{t\geq bN^{\alpha/3}}|A_t/a(t) - M| > \gamma M\right) < 2\theta.$$

Since $a(W_\varepsilon) = \varepsilon N^2/M$, by the choices of γ and b,

$$\begin{aligned} P(|A_{W_\varepsilon} - \varepsilon N^2| \geq \eta N^2) &\leq P(|A_{W_\varepsilon} - \varepsilon N^2| \geq \varepsilon\gamma N^2) \\ &= P(|A_{W_\varepsilon}/a(W_\varepsilon) - M| \geq \gamma M) < 2\theta + P\left(W_\varepsilon < bN^{\alpha/3}\right). \end{aligned}$$

By the definition of $S(\cdot)$,

$$P\left(W_\varepsilon < bN^{\alpha/3}\right) = P\left(M > \frac{3\varepsilon}{b} N^{2-2\alpha/3}\right) \to 0$$

as $N \to \infty$, and so $\limsup_{N\to\infty} P\left(|A_{W_\varepsilon} - \varepsilon N^2| > \eta N^2\right) \leq 2\theta$. Since $\theta > 0$ is arbitrary, we have shown that

$$\lim_{N\to\infty} P\left(|A_{W_\varepsilon} - \varepsilon N^2| \geq \eta) N^2\right) = 0.$$

Repeating the argument for L_{W_ε} and X_{W_ε}, and noting that $l(W_\varepsilon) = \varepsilon N^{2-\alpha/3}/M$ and $x(W_\varepsilon) = \varepsilon N^{2-2\alpha/3}/M$, we get the other two assertions. □

As a corollary of Lemma 3.2 we get the first conclusion of Theorem 2.

th2part1

Corollary 1. *As $N \to \infty$, $N^{-\alpha/3}(\sigma(\varepsilon) - S(\varepsilon)) \xrightarrow{P} -\log(M)$.*

Proof. For any $\eta > 0$ choose $\gamma > 0$ so that $\log(1+\gamma) < \eta$ and $\log(1-\gamma) > -\eta$. Let W_ε be as in Lemma 3.2. Clearly $W_{(1+\gamma)\varepsilon} = S(\varepsilon) + N^{\alpha/3}[\log(1+\gamma) - \log M]$ and $W_{(1-\gamma)\varepsilon} = S(\varepsilon) + N^{\alpha/3}[\log(1-\gamma) - \log M]$. Using Lemma 3.2

$$\begin{aligned} &P\left[N^{-\alpha/3}(\sigma(\varepsilon) - S(\varepsilon)) > -\log M + \eta\right] \\ &\qquad \leq P\left(\sigma(\varepsilon) > W_{(1+\gamma)\varepsilon}\right) = P\left(A_{W_{(1+\gamma)\varepsilon}} < \varepsilon N^2\right) \to 0, \\ &P\left[N^{-\alpha/3}(\sigma(\varepsilon) - S(\varepsilon)) < -\log M - \eta\right] \\ &\qquad \leq P\left(\sigma(\varepsilon) < W_{(1-\gamma)\varepsilon}\right) = P\left(A_{W_{(1-\gamma)\varepsilon}} > \varepsilon N^2\right) \to 0 \end{aligned}$$

as $N \to \infty$, and the proof is complete. □

The second conclusion in Theorem 2 follows from $C_t \leq A_t$. To get the third we have to show that when A_t/N^2 is small, C_t/N^2 is not very much smaller. To prepare for that we need the following result.

renewalineq

Lemma 3.3. *Let $F(t) = \lambda t^3/3!$. If $u(\cdot)$ and $\beta(\cdot)$ are functions such that $u(t) \leq \beta(t) + \int_0^t u(t-s)dF(s)$ for all $t \geq 0$, then*

$$u(t) \leq \beta * V(t) = \beta(t) + \int_0^t \beta(t-s)dV(s),$$

where $V(\cdot)$ is as in Lemma 2.2.

Proof. Define $\tilde{\beta}(t) \equiv \beta(t) + \int_0^t u(t-s)dF(s) - u(t)$. So $\tilde{\beta}(t) \geq 0$ for all $t \geq 0$. If $\hat{\beta}(t) \equiv \beta(t) - \tilde{\beta}(t)$, then

$$u(t) = \hat{\beta}(t) + \int_0^t u(t-s)dF(s).$$

Solving the renewal equation we get $u(t) = \hat{\beta} * V(t)$, where $V(\cdot)$ is as in Lemma 2.2. Since $\hat{\beta}(t) \leq \beta(t)$ for all $t \geq 0$, we get the result. $\square$

We now apply Lemma 3.3 to estimate the difference between EA_t and EC_t.

compare1 **Lemma 3.4.** *For any $t \geq 0$ and $a(\cdot)$ as in* (1.7),

$$EC_t \geq EA_t - \frac{11a^2(t)}{N^2}.$$

Proof. In either of our processes, if a center is born at time s, then radius of the corresponding disk at time $t > s$ will be $(t-s)/\sqrt{2\pi}$. Thus x will be covered at time t if and only if there is a center in the space-time cone

$$K_{x,t} \equiv \left\{(y,s) \in \Gamma(N) \times [0,t] : |y-x| \leq (t-s)/\sqrt{2\pi}\right\}. \tag{3.6}$$ cone

If $0 = s_0, s_1, s_2, ...$ are the birth times of new centers in $\mathcal{C}_t$, then

$$P(x \notin \mathcal{C}_t | s_0, s_1, s_2, \ldots) = \prod_{i: s_i \leq t} \left[1 - \frac{(t-s_i)^2}{2N^2}\right] \leq \exp\left[-\sum_{i: s_i \leq t} \frac{(t-s)^2}{2N^2}\right],$$

since $1 - x \leq e^{-x}$. Let $q(t) \equiv P(x \notin \mathcal{C}_t)$, which does not depend on x, since we have a random chosen starting point. Recall that $\tilde{X}_t$ is the number of centers born by time t in $\mathcal{C}_t$. Using the last inequality

$$q(t) \leq E \exp\left[-\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right],$$

and $EC_t = N^2(1 - q(t))$. Integrating $e^{-y} \geq 1 - y$ gives $1 - e^{-x} \geq x - x^2/2$ for $x \geq 0$. So

$$\begin{aligned} EC_t &\geq N^2 E\left[1 - \exp\left(-\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right)\right] \qquad (3.7) \\ &\geq N^2 E\left[\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s - \frac{1}{2}\left(\int_0^t \frac{(t-s)^2}{2N^2} d\tilde{X}_s\right)^2\right]. \end{aligned}$$ eq7

For the first term on the right we use $E\tilde{X}_t = 1 + \lambda \int_0^t EC_s ds$. For the second term on the right, we use the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$ described in the introduction, see (1.1), so that we have $\int_0^t (t-s)^2 d\tilde{X}_s \le \int_0^t (t-s)^2 dX_s$. Combining these two facts

$$\begin{aligned} EC_t &\ge \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{1}{2N^2} E\left[\int_0^t \frac{(t-s)^2}{2} dX_s\right]^2 \\ &= \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EC_s ds - \frac{EA_t^2}{2N^2}. \end{aligned} \tag{3.8}$$

eq6

The last equality follows from (1.2), as does the next equation for EA_t.

$$EA_t = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} V'(s)\, ds = \frac{t^2}{2} + \int_0^t \frac{(t-s)^2}{2} \lambda EA_s ds. \tag{3.9}$$

eq13

Here $V(\cdot)$ is as in Lemma 2.2 and $EA_t = V'(t)/\lambda$ by Lemma 2.3. Combining (3.8) and (3.9), if $u(t) \equiv EA_t - EC_t$, and $F(s) = \lambda s^3/3!$, then

$$u(t) \le \frac{EA_t^2}{2N^2} + \int_0^t \frac{(t-s)^2}{2} \lambda u(s)\, ds = \frac{EA_t^2}{2N^2} + \int_0^t u(t-r)\, dF(r),$$

where the last step is obtained by changing variables $s \mapsto t - r$. If $\beta(t) = EA_t^2/2N^2$, then by Lemma 2.6 $\beta(t) \le 27a^2(t)/4N^2$, and using Lemma 3.3 and (2.5)

$$u(t) \le \beta * V(t) \le \frac{27}{4N^2}(a^2) * V(t) \le \frac{27}{4N^2}\frac{3}{2}a^2(t),$$

which gives the result, since $81/8 \le 11$. □

We now use Lemma 3.4 to get the last conclusion of Theorem 2.

tausigma **Lemma 3.5.** *For any* $\gamma > 0$

$$\limsup_{N\to\infty} P(\tau(\varepsilon) > \sigma((1+\gamma)\varepsilon)) \le P\left(M \le (1+\gamma)\varepsilon^{1/3}\right) + 11\frac{\varepsilon^{1/3}}{\gamma}.$$

Proof. Let $U = \sigma((1+\gamma)\varepsilon)$ and $T = S(\varepsilon^{2/3})$, where $S(\cdot)$ is as in (1.8). Now

$$S(\varepsilon^{2/3}) - S((1+\gamma)\varepsilon) = N^{\alpha/3}\left[-\frac{1}{3}\log(\varepsilon) - \log(1+\gamma)\right].$$

It follows from Corollary 1 that $\limsup_{N\to\infty} P(U \ge T)$

$$\le P\left(-\log(M) \ge -\frac{1}{3}\log(\varepsilon) - \log(1+\gamma)\right) = P\left(M \le (1+\gamma)\varepsilon^{1/3}\right).$$

Using Markov inequality, Lemma 3.4, and $a(T) = \varepsilon^{2/3}N^2$,

$$P\left(|A_T - C_T| > \gamma\varepsilon N^2\right) \le \frac{E(A_T - C_T)}{\gamma\varepsilon N^2} \le \frac{6(a(T))^2}{\gamma\varepsilon N^4} \le 11 \cdot \frac{\varepsilon^{1/3}}{\gamma}. \tag{3.10}$$

b3

Using these two bounds and the fact that $|A_t - C_t|$ is nondecreasing in t, we get

$$\begin{aligned}
&\limsup_{N\to\infty} P[\tau(\varepsilon) > \sigma((1+\gamma)\varepsilon)] = \limsup_{N\to\infty} P\left[|A_U - C_U| > \gamma\varepsilon N^2\right] \\
&\le \limsup_{N\to\infty} P(U \ge T) + \limsup_{N\to\infty} P\left[|A_U - C_U| > \gamma\varepsilon N^2, U < T\right] \\
&\le \limsup_{N\to\infty} P(U \ge T) + P\left(|A_T - C_T| > \gamma\varepsilon N^2\right),
\end{aligned}$$

which completes the proof. $\square$

4 Proof of Theorem 3

Let $\mathcal{C}^0_{s,t}$ be the set of points covered in $\mathcal{C}_t$ at time t by the balloons born before time s. If we number the generations of centers in $\mathcal{C}_t$ starting with those existing at time s as $\mathcal{C}_t$-centers of generation 0, then $\mathcal{C}^0_{s,t}$ is the set of points covered at time t by the generation 0 centers of $\mathcal{C}_t$. Let $\mathcal{C}^1_{s,t}$ be the set of points, which are either in $\mathcal{C}^0_{s,t}$, or are covered at time t by a balloon born from this area. This is the set of points covered by $\mathcal{C}_t$-centers of generations ≤ 1 at time t , ignoring births from $\mathcal{C}^1_{s,t} \setminus \mathcal{C}^0_{s,t}$, which are second generation centers. Continuing by induction, we let $\mathcal{C}^k_{s,t}$ be the set of points and $C^k_{s,t} = \left|\mathcal{C}^k_{s,t}\right|$ be the total area covered by $\mathcal{C}_t$-centers of generations $0 \le j \le k$ at time t. Similarly $A^k_{s,t}$ denotes the total area of the balloons in $\mathcal{A}_t$ of generations $j \in \{0, 1, \dots, k\}$ at time t, where generation 0 centers are those existing at time s.

Recall the following definitions from (1.7), (1.8), (1.11) and (1.12).

$$\begin{aligned}
a(t) &= (1/3)N^{2\alpha/3}\exp\left(N^{-\alpha/3}t\right), \\
S(\varepsilon) &= N^{\alpha/3}[(2-2\alpha/3)\log N + \log(3\varepsilon)], \\
R &= N^{\alpha/3}[(2-2\alpha/3)\log N - \log(M)],
\end{aligned}$$

where M is the limit random variable in Theorem 1, and for $\log(3\varepsilon) \le t$,

$$\psi(t) \equiv R + N^{\alpha/3}t, \quad W \equiv \psi(\log(3\varepsilon)), \text{ and } I_{\varepsilon,t} = [\log(3\varepsilon), t].$$

Note that $\psi(t) \le 0$ only if $M \ge N^{2-2\alpha/3}t$.

Obviously $C^0_{s,t} \le A^0_{s,t}$. For the other direction we have the following lemma.

compare2 **Lemma 4.1.** *For any* $0 < s < t$,

$$EC^0_{s,t} \ge EA^0_{s,t} - \frac{a^2(s)}{N^2} p\left((t-s)\lambda^{1/3}\right),$$

where for some positive constants c_1, c_2 *and* c_4,

$$p(x) = c_1 + c_2x^2/2! + c_4x^4/4!. \tag{4.1}$$

pxdef

Proof. By the definition of $A^0_{s,t}$,

$$A^0_{s,t} = \int_0^s \frac{(t-r)^2}{2}\,dX_r = \frac{(t-s)^2}{2}X_s + (t-s)L_s + A_s. \qquad (4.2) \quad \boxed{\texttt{Ast}}$$

For the second equality we have written $(t-r)^2 = (t-s)^2 + 2(t-s)(s-r) + (s-r)^2$ and used (1.2). As in Lemma 3.4, a point x is not covered by time t by the balloons born before time s, if and only if no center is born in the truncated space-time cone

$$K_{x,s,t} \equiv \left\{(y,r) \in \Gamma(N) \times [0,s] : |y-x| \le (t-r)/\sqrt{2\pi}\right\}.$$

So using arguments similar to the ones for (3.7) and the inequality $1-e^{-x} \ge x - x^2/2$ for $x \ge 0$, which comes from integrating $e^{-y} \ge 1-y$,

$$\begin{aligned} EC^0_{s,t} &\ge N^2 E\left[1 - \exp\left(-\int_0^s \frac{(t-r)^2}{2N^2}\,d\tilde{X}_r\right)\right] \\ &\ge N^2\left[E\int_0^s \frac{(t-r)^2}{2N^2}\,d\tilde{X}_r - \frac{1}{2}E\left(\int_0^s \frac{(t-r)^2}{2N^2}\,d\tilde{X}_r\right)^2\right]. \end{aligned}$$

For the first term on the right, we use $E\tilde{X}_t = 1 + \lambda\int_0^t EC_s ds$. For the second term on the right, we use the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$ described in the introduction, see (1.1), to conclude that

$$\int_0^s (t-r)^2\,d\tilde{X}_r \le \int_0^s (t-r)^2\,dX_r = 2A^0_{s,t}.$$

Combining these two facts, using the first equality in (4.2), $EX_t = 1 + \lambda\int_0^t EA_s\,ds$, and Lemma 3.4,

$$\begin{aligned} EC^0_{s,t} &\ge \frac{t^2}{2} + \int_0^s \frac{(t-r)^2}{2}\lambda EC_r\,dr - \frac{E(A^0_{s,t})^2}{2N^2} \\ &\ge \frac{t^2}{2} + \int_0^s \frac{(t-r)^2}{2}\lambda EA_r\,dr - 11\int_0^s \frac{(t-r)^2}{2}\frac{\lambda a^2(r)}{N^2}\,dr - \frac{E(A^0_{s,t})^2}{2N^2} \\ &= EA^0_{s,t} - 11\int_0^s \frac{(t-r)^2}{2}\frac{\lambda a^2(r)}{N^2}dr - \frac{E(A^0_{s,t})^2}{2N^2}. \qquad (4.3) \quad \boxed{\texttt{eq1}} \end{aligned}$$

To estimate the second term in the right side of (4.3), we write

$$(t-r)^2/2 = (t-s)^2/2 + (t-s)(s-r) + (s-r)^2/2,$$

change variables $r = s-q$, and note $a(s-q) = a(s)\exp\left(-\lambda^{1/3}q\right)$, to get

$$\begin{aligned} \int_0^s \frac{(t-r)^2}{2}\lambda a^2(r)\,dr &= a^2(s)\Bigg[\frac{(t-s)^2}{2}\lambda^{2/3}\int_0^s \lambda^{1/3}\exp\left(-2\lambda^{1/3}q\right)\,dq \\ &\quad + (t-s)\lambda^{1/3}\int_0^s \lambda^{2/3}q\exp\left(-2\lambda^{1/3}q\right)\,dq + \int_0^s \lambda\frac{q^2}{2}\exp\left(-2\lambda^{1/3}q\right)\,dq\Bigg] \\ &\le \frac{a^2(s)}{2}\left[\frac{(t-s)^2}{2}\lambda^{2/3} + (t-s)\lambda^{1/3} + 1\right]. \qquad (4.4) \quad \boxed{\texttt{2nd}} \end{aligned}$$

For the last inequality we have used

$$\int_0^s r^k \exp(-\mu r)\,dr \le \int_0^\infty r^k \exp(-\mu r)\,dr = \frac{k!}{\mu^{k+1}}.$$

To estimate the third term in the right side of (4.3) we use (4.2) to get

$$E(A^0_{s,t})^2 \le 3[EX_s^2(t-s)^4/4 + EL_s^2(t-s)^2 + EA_s^2].$$

Applying Lemma 2.6 and using the fact that $a(s) = \lambda^{-1/3} l(s) = \lambda^{-2/3} x(s)$,

$$\begin{aligned} E(A^0_{s,t})^2 &\le 3 \cdot \frac{27}{2}\left[x^2(s)\frac{(t-s)^4}{4} + l^2(s)(t-s)^2 + a^2(s)\right] \\ &\le 243 a^2(s)\left[\frac{(t-s)^4}{4!}\lambda^{4/3} + \frac{(t-s)^2}{2!}\lambda^{2/3} + 1\right]. \qquad (4.5) \end{aligned}$$

3rd

Combining (4.3), (4.4) and (4.5) we get the result. □

To show uniform convergence of $C^k_{W,\psi(\cdot)}$ to $C_{\psi(\cdot)}$, we also need to bound the difference A_t and $A^k_{s,t}$ for suitable choices of s and t.

Abound

Lemma 4.2. *If $T = S(\varepsilon^{2/3})$, where $S(\cdot)$ is as in* (1.8)*, then for any $t > 0$*

$$EA_{T+tN^{\alpha/3}} - EA^k_{T,T+tN^{\alpha/3}} \le \varepsilon^{2/3} N^2 \sum_{j=k+1}^{\infty} \frac{t^j}{j!}.$$

Proof. Using (4.2) $EA^0_{s,t} = EA_s + EL_s(t-s) + EX_s(t-s)^2/2$. If $X^k_{s,t}$ and $L^k_{s,t}$ denote the number of centers and sum of radii of all the balloons in $\mathcal{A}_t$ of generations $j \in \{1, 2, \ldots, k\}$ at time t, where generation 0 centers are those which are born before time s, then for $t > s$,

$$\frac{d}{dt} EX^1_{s,t} = N^{-\alpha} EA^0_{s,t}, \; \frac{d}{dt} EL^1_{s,t} = EX^1_{s,t}, \; \frac{d}{dt} EA^1_{s,t} = EL^1_{s,t}.$$

Integrating and using (4.2) we have

$$\begin{aligned} EX^1_{s,t} &= N^{-\alpha}\left[(t-s)EA_s + \frac{(t-s)^2}{2!}EL_s + \frac{(t-s)^3}{3!}EX_s\right], \\ EL^1_{s,t} &= N^{-\alpha}\left[\frac{(t-s)^2}{2!}EA_s + \frac{(t-s)^3}{3!}EL_s + \frac{(t-s)^4}{4!}EX_s\right], \\ EA^1_{s,t} &= N^{-\alpha}\left[\frac{(t-s)^3}{3!}EA_s + \frac{(t-s)^4}{4!}EL_s + \frac{(t-s)^5}{5!}EX_s\right]. \end{aligned}$$

Turning to other generations, for $k \geq 2$ and $t > s$,

$$\begin{aligned}
\frac{d}{dt}\left(EX_{s,t}^k - EX_{s,t}^{k-1}\right) &= N^{-\alpha}\left(EA_{s,t}^{k-1} - EA_{s,t}^{k-2}\right), \\
\frac{d}{dt}\left(EL_{s,t}^k - EL_{s,t}^{k-1}\right) &= \left(EX_{s,t}^k - EX_{s,t}^{k-1}\right), \\
\frac{d}{dt}\left(EA_{s,t}^k - EA_{s,t}^{k-1}\right) &= \left(EL_{s,t}^k - EL_{s,t}^{k-1}\right),
\end{aligned}$$

and using induction on k we have

$$EA_{s,t}^k = \sum_{j=0}^{k} N^{-\alpha j}\left[\frac{(t-s)^{3j}}{(3j)!}EA_s + \frac{(t-s)^{3j+1}}{(3j+1)!}EL_s + \frac{(t-s)^{3j+2}}{(3j+2)!}EX_s\right].$$

Since $A_{s,t}^k \uparrow A_t$, $EA_t = \lim_{k\to\infty} EA_{s,t}^k$. Replacing s by T and t by $T + tN^{\alpha/3}$,

$$\begin{aligned}
&EA_{T+tN^{\alpha/3}} - EA_{T,T+tN^{\alpha/3}}^k \\
&= \sum_{j=k+1}^{\infty}\left[\frac{t^{3j}}{(3j)!}EA_T + \frac{t^{3j+1}}{(3j+1)!}N^{\alpha/3}EL_T + \frac{t^{3j+2}}{(3j+2)!}N^{2\alpha/3}EX_T\right].
\end{aligned} \tag{4.6}$$
eq5

Using the fact that $EA_T + N^{\alpha/3}EL_T + N^{2\alpha/3}EX_T - 3a(T) = 0$ and $a(T) = \varepsilon^{2/3}N^2$, the right hand side of (4.6) is $\leq \varepsilon^{2/3}N^2 \sum_{j=k+1}^{\infty} t^j/j!$, which completes the proof. $\square$

Recall that for $\log(3\varepsilon) \leq t$,

$$g_0(t) = \varepsilon\left[1 + (t - \log(3\varepsilon) + \frac{(t-\log(3\varepsilon))^2}{2}\right], \quad f_0(t) = g_0(t) - \varepsilon^{7/6}. \tag{4.7}$$
gdef2

B0bounds **Lemma 4.3.** *For any $t < \infty$, there is an $\varepsilon_0 = \varepsilon_0(t) > 0$ so that for $0 < \varepsilon < \varepsilon_0$,*

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}}\left|N^{-2}A_{W,\psi(s)}^0 - g_0(s)\right| > \eta\right) = 0 \quad \textit{for any } \eta > 0,$$

$$P\left(\inf_{s\in I_{\varepsilon,t}} N^{-2}\left(C_{W,\psi(s)}^0 - A_{W,\psi(s)}^0\right) < -\varepsilon^{7/6}\right) \leq P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}.$$

Proof. To prove the first result we use (4.2) to conclude

$$A_{W,\psi(t)}^0 = \frac{(t-\log(3\varepsilon))^2}{2}N^{2\alpha/3}X_W + (t-\log(3\varepsilon))N^{\alpha/3}L_W + A_W.$$

Applying Lemma 3.2

$$\begin{aligned}
&\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}} \left|N^{-2}A^0_{W,\psi(s)} - g_0(s)\right| > \eta\right) \\
&\quad \le \lim_{N\to\infty} P\left(|N^{-(2-2\alpha/3)}X_W - \varepsilon| > \frac{2\eta}{3(t-\log(3\varepsilon))^2}\right) \\
&\quad\quad + \lim_{N\to\infty} P\left(|N^{-(2-\alpha/3)}L_W - \varepsilon| > \frac{\eta}{3(t-\log(3\varepsilon))}\right) \\
&\quad\quad + \lim_{N\to\infty} P\left(|N^{-2}A_W - \varepsilon| > \frac{\eta}{3}\right) = 0.
\end{aligned}$$

Take $\varepsilon_0 = \varepsilon_0(t)$ be such that $\varepsilon_0^{1/12} p(t - \log(3\varepsilon)) \le 1$, where $p(\cdot)$ is the polynomial in (4.1). Let $T = S(\varepsilon^{2/3})$, where $S(\cdot)$ is defined in (1.8), and $T' = T + (t - \log(3\varepsilon))N^{\alpha/3}$. Using the fact that $A^0_{s,s+t} - C^0_{s,s+t}$ is nondecreasing in s, Markov's inequality, and then Lemma 4.1 we see that

$$\begin{aligned}
&P\left(\sup_{s\in I_{\varepsilon,t}} \left|A^0_{W,\psi(s)} - C^0_{W,\psi(s)}\right| > \varepsilon^{7/6}N^2, W \le T\right) \\
&\quad \le P\left(|A^0_{T,T'} - C^0_{T,T'}| > \varepsilon^{7/6}N^2\right) \le \frac{E|A^0_{T,T'} - C^0_{T,T'}|}{\varepsilon^{7/6}N^2} \\
&\quad \le \frac{a^2(T)p(t-\log(3\varepsilon))}{\varepsilon^{7/6}N^4}.
\end{aligned}$$

Noting that $P(W > T) = P(M < \varepsilon^{1/3})$, $a(T) = \varepsilon^{2/3}N^2$, and $\varepsilon^{1/12}p(t - \log(3\varepsilon)) < 1$ for $\varepsilon < \varepsilon_0$ we have

$$P\left(\sup_{s\in I_{\varepsilon,t}} \left|A_{W,\psi(s)} - C_{W,\psi(s)}\right| > \varepsilon^{7/6}N^2\right) \le P\left(M < \varepsilon^{1/3}\right) + \varepsilon^{1/12}.$$

which completes the proof. $\square$

Our next step is to improve the lower bound in Lemma 4.3. Let

$$\rho^0_t = N^{-2}A_{W,\psi(t)} - \varepsilon^{7/6}.$$

On the event

$$F = \left\{\left|N^{-2}\mathcal{C}^0_{W,\psi(s)}\right| \ge \rho^0_s \text{ for all } s \in I_{\varepsilon,t}\right\}, \tag{4.8}$$

Fdef

which has probability tending to 1 as $\varepsilon \to 0$ by Lemma 4.3, $\mathcal{C}^0_{W,\psi(s)}$ can be coupled with a process $\mathcal{B}^0_{\psi(s)}$ so that $N^{-2}|\mathcal{B}^0_{\psi(s)}| = \rho^0_s$ and $\mathcal{C}^0_{W,\psi(s)} \supseteq \mathcal{B}^0_{\psi(s)}$ for $s \in I_{\varepsilon,t}$. If for $k \ge 1$ $\mathcal{B}^k_{\psi(t)}$ is obtained from $\mathcal{B}^0_{\psi(t)}$ in the same way as $\mathcal{C}^k_{W,\psi(t)}$ is obtained from $\mathcal{C}^0_{W,\psi(t)}$, then on F $\mathcal{C}^k_{W,\psi(s)} \supseteq \mathcal{B}^k_{\psi(s)}$ for $s \in I_{\varepsilon,t}$. For $k \ge 1$ let

$$\rho^k_s = N^{-2}|\mathcal{B}^k_{\psi(s)}|.$$

We begin with the case $k = 1$. For $f_0(t)$ as in (4.7), let

$$f_1(t) = 1 - (1 - f_0(t)) \exp\left(- \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} f_0(s)\, ds \right). \tag{4.9}$$ f1eq2

f1lb **Lemma 4.4.** *For any $t < \infty$ there is an $\varepsilon_0 = \varepsilon_0(t) > 0$ so that for $0 < \varepsilon < \varepsilon_0$ and any $\delta > 0$,*

$$\limsup_{N\to\infty} P\left[\inf_{s\in I_{\varepsilon,t}} (N^{-2} C^1_{W,\psi(s)} - f_1(s)) < -\delta \right] \le P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}.$$

Proof. As in Lemma 3.4, if $x \notin \mathcal{B}^0_{\psi(t)}$, then $x \notin \mathcal{B}^1_{\psi(t)}$ if and only if no generation 1 center is born in the space-time cone

$$K^\varepsilon_{x,t} \equiv \left\{ (y,s) \in \Gamma(N) \times [W, \psi(t)] : |y - x| \le (\psi(t) - s)/\sqrt{2\pi} \right\}.$$

Conditioning on $\mathcal{G}^0_t = \sigma\{\mathcal{B}^0_{\psi(s)} : s \in I_{\varepsilon,t}\}$, the locations of generation 1 centers in $\mathcal{B}^1_t$ is a Poisson point process on $\Gamma(N) \times [W, \psi(t)]$ with intensity

$$N^{-2} \times |\mathcal{B}^0_s| N^{-\alpha} = \rho^0_{\psi^{-1}(s)} N^{-\alpha},$$

Using this and then changing variables $s = \psi(r)$, where $\psi(r) = R + N^{\alpha/3} r$,

$$\begin{aligned} P\left(x \notin \mathcal{B}^1_{\psi(t)} \middle| \mathcal{G}^0_t \right) &= (1 - \rho^0_t) \exp\left(- \int_W^{\psi(t)} \frac{(\psi(t) - s)^2}{2} \rho^0_{\psi^{-1}(s)} N^{-\alpha}\, ds \right) \\ &= (1 - \rho^0_t) \exp\left(- \int_{\log(3\varepsilon)}^{t} \frac{(t-r)^2}{2} \rho^0_r\, dr \right). \end{aligned}$$

Let $E_{x,t} = \{x \notin \mathcal{B}^1_t\}$. Since $K^\varepsilon_{x,t}$ and $K^\varepsilon_{y,t}$ are disjoint if $|x - y| > 2(t - \log(3\varepsilon))N^{\alpha/3}/\sqrt{2\pi}$, the events $E_{x,t}$ and $E_{y,t}$ are conditionally independent given $\mathcal{G}^0_t$ if this holds. Define the random variables Y_x, $x \in \Gamma(N)$, so that $Y_x = 1$ if $E_{x,t}$ occurs, and $Y_x = 0$ otherwise. From (4.10)

$$E\left(Y_x \middle| \mathcal{G}^0_t \right) = (1 - \rho^0_t) \exp\left(- \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} \rho^0_s\, ds \right). \tag{4.10}$$ cmu1

Using independence of Y_x and Y_z for $|x - z| > 2(t - \log(3\varepsilon))N^{\alpha/3}/\sqrt{2\pi}$, and the fact that $\left\{ z : |x - z| \le 2(t - \log(3\varepsilon))N^{\alpha/3}/\sqrt{2\pi} \right\}$ has area $2(t - \log(3\varepsilon))^2 N^{2\alpha/3}$,

$$\begin{aligned} &\operatorname{var}\left(\int_{x\in\Gamma(N)} Y_x\, dx \middle| \mathcal{G}^0_t \right) \\ &= \int_{x,z\in\Gamma(N)} \left[E\left(Y_x Y_z \middle| \mathcal{G}^0_t \right) - E\left(Y_x \middle| \mathcal{G}^0_t \right) E\left(Y_z \middle| \mathcal{G}^0_t \right) \right] dx\, dz \\ &\qquad \le N^2 \cdot 2(t - \log(3\varepsilon))^2 N^{2\alpha/3}. \end{aligned} \tag{4.11}$$ cvar1

Using Chebyshev's inequality, we see that

$$P\left(\left|\int_{x\in\Gamma(N)} \left(Y_x - E\left(Y_x \middle| \mathcal{G}_t^0\right)\right)\, dx\right| > \frac{\eta}{2}N^2 \middle| \mathcal{G}_t^0\right) \le \frac{4\mathrm{var}\left(\int_{x\in\Gamma(N)} Y_x\, dx \middle| \mathcal{G}_t^0\right)}{\eta^2 N^4}. \tag{4.12}$$

cch1

Combining (4.10), (4.11), and (4.12) gives

$$P\left(\left|\left(1-\rho_t^1\right) - (1-\rho_t^0)\exp\left(-\int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2}\rho_s^0\, ds\right)\right| > \frac{\eta}{2} \middle| \mathcal{G}_t^0\right) \le \frac{8(t-\log(3\varepsilon))^2}{\eta^2 N^{2-2\alpha/3}}.$$

The same bound holds for the unconditional probability. By Lemma 4.3 if $\eta > 0$ and

$$F_{0,\eta} \equiv \left\{\sup_{s\in I_{\varepsilon,t}} |\rho_s^0 - f_0(s)| \le \eta\right\}, \quad \text{then} \quad \lim_{N\to\infty} P(F_{0,\eta}^c) = 0.$$

Let $\eta' = \eta\left[1 + (t-\log(3\varepsilon))^3/3!\right]^{-1}/2$. Using (4.9) and the fact that for $x, y \ge 0$

$$|e^{-x} - e^{-y}| = \left|\int_x^y e^{-z}\, dz\right| \le |x-y|, \tag{4.13}$$

eineq

we see that on the event $F_{0,\eta'}$, we have for any $s \in I_{\varepsilon,t}$

$$\begin{aligned}
&\left|\left(1-\rho_s^0\right)\exp\left(-\int_{\log(3\varepsilon)}^s \frac{(s-r)^2}{2}\rho_r^0\, dr\right) - (1-f_1(s))\right| \\
&\le \left|\left(1-\rho_s^0\right) - (1-f_0(s))\right| + \eta' \int_{\log(3\varepsilon)}^s \frac{(s-r)^2}{2}\, dr \le \eta' + \eta' \frac{(s-\log(3\varepsilon))^3}{3!} \le \frac{\eta}{2}.
\end{aligned}$$

So for any $s \in I_{\varepsilon,t}$

$$\begin{aligned}
&\lim_{N\to\infty} P\left(\left|\rho_s^1 - f_1(s)\right| > \eta\right) \le \lim_{N\to\infty} P\left(F_{0,\eta'}^c\right) \\
&\quad + \lim_{N\to\infty} P\left(\left|\left(1-\rho_s^1\right) - \left(1-\rho_s^0\right)\exp\left(-\int_{\log(3\varepsilon)}^s \frac{(s-r)^2}{2}\rho_r^0\, dr\right)\right| > \frac{\eta}{2}\right) = 0.
\end{aligned}$$

Since $\eta > 0$ is arbitrary, the two quantities being compared are increasing and continuous, and on the event F defined in (4.8) $N^{-2}C^1_{W,\psi(s)} \ge \rho_s^1$ for $s \in I_{\varepsilon,t}$,

$$\begin{aligned}
&\limsup_{N\to\infty} P\left[\inf_{s\in I_{\varepsilon,t}} (N^{-2}C^1_{W,\psi(s)} - f_1(s)) < -\delta\right] \\
\le\ & P(F^c) + \limsup_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}} |\rho_s^1 - f_1(s)| > \delta\right) \le P(F^c),
\end{aligned}$$

and the desired conclusion follows from Lemma 4.3. $\square$

To improve this we will let

$$f_{k+1}(t) = 1 - (1 - f_k(t)) \exp\left(- \int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} (f_k(s) - f_{k-1}(s))\, ds \right), \qquad (4.14) \quad \boxed{\texttt{fiter2}}$$

and recall that from (1.15) that as $k \uparrow \infty$, $f_k(t) \uparrow f_\varepsilon(t)$.

$\boxed{\texttt{fklb}}$ **Lemma 4.5.** *For any $t < \infty$ there is an $\varepsilon_0 = \varepsilon_0(t) > 0$ so that for $0 < \varepsilon < \varepsilon_0$ and any $\delta > 0$,*

$$\limsup_{N\to\infty} P\left[\inf_{s\in I_{\varepsilon,t}} (N^{-2} C_{\psi(s)} - f_\varepsilon(s)) < -\delta \right] \le P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}.$$

Proof. Conditioning on $\mathcal{G}_t^k = \sigma\left\{ \mathcal{B}^j_{\psi(s)} : 0 \le j \le k, s \in I_{\varepsilon,t} \right\}$, we have

$$P\left(x \notin \mathcal{B}^{k+1}_{\psi(t)} \middle| \mathcal{G}_t^k \right) = \left(1 - \rho_t^k\right) \exp\left(- \int_0^t \frac{(t-s)^2}{2} \left(\rho_s^k - \rho_s^{k-1} \right) \, ds \right).$$

Let $F_{k,\eta} = \{\sup_{s\in I_{\varepsilon,t}} |\rho_s^k - f_k(s)| \le \eta\}$, and $\eta' = \eta\left[1 + 2(t - \log(3\varepsilon))^3/3!\right]^{-1}/2$. Using (4.14) and $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$, we see that on the event $G_{k,\eta'} = F_{k,\eta'} \cap F_{k-1,\eta'}$, for any $s \in I_{\varepsilon,t}$

$$\begin{aligned}
&\left| \left(1 - \rho_t^k\right) \exp\left(- \int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} \left(\rho_s^k - \rho_s^{k-1} \right) \, ds \right) - (1 - f_{k+1}(t)) \right| \\
&\quad \le | \left(1 - \rho_t^k\right) - (1 - f_k(t))| + 2\eta' \int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} \, ds \\
&\quad = \eta' + 2\eta'(t - \log(3\varepsilon))^3/3 \le \eta/2.
\end{aligned}$$

Bounding the variance as before we can conclude by induction on k that for any $\eta > 0$

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}} \left| \rho_s^k - f_k(s) \right| > \eta \right) = 0. \qquad (4.15) \quad \boxed{\texttt{rhokbd}}$$

Next we bound the difference between $f_k(t)$ and $f_\varepsilon(t)$. Let $G(t) = t^3/3!$ for $t \ge 0$, and $G(t) = 0$ for $t < 0$. If $*k$ indicates the k-fold convolution, then for $k \ge 1$, using arguments similar to the ones in the proof of Lemma 2.2, $G^{*k}(t) = t^{3k}/(3k)!$ for $t \ge 0$, and $G^{*k}(t) = 0$ for $t < 0$. Now if $f * G^{*k}(t) = \int_0^t f(t-r)\, dG^{*k}(r)$, $\tilde{f}_k(\cdot) = f_k(\cdot + \log(3\varepsilon))$ and $\tilde{f}_\varepsilon(\cdot) = f_\varepsilon(\cdot + \log(3\varepsilon))$, then changing variables $s \mapsto t - r$ in (1.14) and (1.15), and using the inequality in (4.13),

$$\begin{aligned}
& |\tilde{f}_k(t - \log(3\varepsilon)) - \tilde{f}_\varepsilon(t - \log(3\varepsilon))| \\
\le\ & \left| \exp(-\tilde{f}_{k-1} * G(t - \log(3\varepsilon))) - \exp(-\tilde{f}_\varepsilon * G(t - \log(3\varepsilon))) \right| \\
\le\ & |\tilde{f}_{k-1} - \tilde{f}_\varepsilon| * G(t - \log(3\varepsilon)).
\end{aligned}$$

Iterating the above inequality and using $|\tilde{f}_\varepsilon(s) - \tilde{f}_0(s)| = \tilde{f}_\varepsilon(s) - \tilde{f}_0(s) \le 1$.

$$\begin{aligned}
|f_k(t) - f_\varepsilon(t)| &= |\tilde{f}_k(t - \log(3\varepsilon)) - \tilde{f}_\varepsilon(t - \log(3\varepsilon))| \\
&\le |\tilde{f}_0 - \tilde{f}_\varepsilon| * G^{*k}(t - \log(3\varepsilon)) \qquad (4.16) \\
&\le G^{*k}(t - \log(3\varepsilon)) = \frac{(t - \log(3\varepsilon))^{3k}}{(3k)!}.
\end{aligned}$$

fgap

where the last equality comes from (2.1).

Choose $K = K(\varepsilon, t)$ so that $(t - \log(3\varepsilon))^{3K}/(3K)! < \delta/2$. Since $C_{\psi(t)} \ge C^k_{W,\psi(t)}$ for any $k \ge 0$, and on the event F defined in (4.8), we have $C^k_{W,\psi(t)} \ge |\mathcal{B}^k_{\psi(t)}|$, we have

$$P\left(\inf_{s \in I_{\varepsilon,t}} \left(N^{-2} C_{\psi(s)} - f_\varepsilon(s) \right) < -\delta \right) \le P(F^c) + P\left(\sup_{s \in I_{\varepsilon,t}} \left| \rho^K_s - f_K(s) \right| > \delta/2 \right).$$

Using (4.15) and Lemma 4.3 we get the result. $\square$

It is now time to get upper bounds on $C_{\psi(s)}$. Recall $g_0(t)$ defined in (4.7), let $g_{-1}(t) = 0$ and for $k \ge 1$ let

$$g_k(t) = 1 - (1 - g_{k-1}(t)) \exp\left(- \int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} (g_{k-1}(s) - g_{k-2}(s))\, ds \right)$$

As in the case of $f_k(t)$, the equations above imply

$$g_k(t) = 1 - (1 - g_0(t)) \exp\left(- \int_{\log(3\varepsilon)}^t \frac{(t-s)^2}{2} g_{k-1}(s)\, ds \right),$$

so we have $g_k(t) \uparrow g_\varepsilon(t)$ as $k \uparrow \infty$.

glb **Lemma 4.6.** *For any $t < \infty$ there exists $\varepsilon_0 = \varepsilon_0(t) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and any $\delta > 0$,*

$$\limsup_{N \to \infty} P\left[\sup_{s \in I_{\varepsilon,t}} \left(N^{-2} C_{\psi(s)} - g_\varepsilon(s) \right) > \delta \right] \le P(M < \varepsilon^{1/3}) + \varepsilon^{2/3}.$$

Proof. $C^0_{W,\psi(t)} \le A^0_{W,\psi(t)}$. If $\phi^0_t = N^{-2} A^0_{W,\psi(t)}$ is the fraction of area covered by generation 0 balloons at time $\psi(t)$, generation 1 centers are born at rate $N^{2-\alpha}\phi^0_{\psi^{-1}(\cdot)}$. Let ϕ^1_t denotes the fraction of area covered by centers of generations ≤ 1 at time $\psi(t)$, then using an argument similar to the one for Lemma 4.4 gives

$$\lim_{N \to \infty} P\left(\sup_{s \in I_{\varepsilon,t}} \phi^1_s - g_1(s) > \eta \right) = 0$$

for any $\eta > 0$. Continuing by induction, if ϕ_t^k is the fraction of area covered by centers of generations $0 \le j \le k$, then by an argument similar to the one for Lemma 4.5,

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}} \left|\phi_s^k - g_k(s)\right| > \eta\right) = 0 \tag{4.17}$$

eq9

for any $\eta > 0$. Now using an argument similar to the one for (4.16)

$$\sup_{s\in I_{\varepsilon,t}} |g_k(s) - g_\varepsilon(s)| \le \frac{(t-\log(3\varepsilon))^{3k}}{(3k)!}. \tag{4.18}$$

eq8

Next we bound the difference between $C^k_{W,\psi(t)}$ and $C_{\psi(t)}$. Let $T = S(\varepsilon^{2/3})$, where $S(\cdot)$ is as in (1.8). Using the coupling between $\mathcal{C}_t$ and $\mathcal{A}_t$,

$$C_{\psi(t)} - C^k_{W,\psi(t)} \le A_{\psi(t)} - A^k_{W,\psi(t)}.$$

Using the fact that $EA_{s+t} - EA^k_{s,s+t}$ is nondecreasing in s, the definitions of W and T, Markov's inequality, and Lemma 4.2, we have for $T' = T + (t-\log(3\varepsilon))N^{\alpha/3}$,

$$\begin{aligned}
P\left(\sup_{s\in I_{\varepsilon,t}} \left(C_{\psi(s)} - \mathcal{C}^k_{W,\psi(s)}\right) > \frac{\delta N^2}{4}\right) &\le P(W > T) + P\left(A_{T'} - A_{T,T'} > \frac{\delta N^2}{4}\right)\\
&\le P(M < \varepsilon^{1/3}) + \frac{4}{\delta N^2} E(A_{T'} - A_{T,T'})\\
&\le P(M < \varepsilon^{1/3}) + \frac{4\varepsilon^{2/3}}{\delta} \sum_{j=k+1}^{\infty} \frac{(t-\log(3\varepsilon))^j}{j!}.
\end{aligned}$$

Choose $K = K(\varepsilon, t)$ large enough so that $\sum_{j=K+1}^{\infty}(t-\log(3\varepsilon))^j/j! < \delta/4$. If we let

$$F_K = \left\{\sup_{s\in I_{\varepsilon,t}} \left(C_{\psi(s)} - C^K_{W,\psi(s)}\right) \le (\delta/4)N^2\right\}, \quad \text{then} \quad P(F_K^c) \le P(M < \varepsilon^{1/3}) + \varepsilon^{2/3}.$$

By the choice of K and (4.18), $\sup_{s\in I_{\varepsilon,t}} |g_K(s) - g_\varepsilon(s)| \le \delta/2$. Combining the last two inequalities and using the fact that $N^{-2}C^K_{W,\psi(s)} \le \phi_s^K = N^{-2}A^K_{W,\psi(s)}$,

$$P\left(\sup_{s\in I_{\varepsilon,t}} N^{-2}C_{\psi(s)} - g_\varepsilon(s) > \delta\right) \le P(F_K^c) + P\left(\sup_{s\in I_{\varepsilon,t}} \left|\phi_s^K - g_K(s)\right| > \delta/4\right).$$

So using (4.17) we have the desired result. □

Our next goal is the

Proof of Lemma 1.1. We prove the result in two steps. To begin we consider a function $h_\varepsilon(\cdot)$ satisfying $h_\varepsilon(t) = e^t/3$ for $t < \log(3\varepsilon)$.

$$h_\varepsilon(t) = 1 - \exp\left(- \int_{-\infty}^{\log(3\varepsilon)} \frac{(t-s)^2}{2} \frac{e^s}{3}\, ds - \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} h_\varepsilon(s)\, ds \right) \qquad (4.19) \quad \boxed{\texttt{hep}}$$

for $t \geq \log(3\varepsilon)$, and prove that $h_\varepsilon(\cdot)$ converges to some $h(\cdot)$ with the desired properties.

$\boxed{\texttt{hepmono}}$ **Lemma 4.7.** *For fixed t, $h_\varepsilon(t)$ in (4.19) is monotone decreasing in ε.*

Proof. If we change variables $s = t - u$ and integrate by parts, or remember the first two moments of the exponential with mean 1, then

$$\begin{aligned} \int_{-\infty}^{t} (t-s) e^s\, ds &= \int_0^\infty u e^{t-u}\, du = e^t, \\ \int_{-\infty}^{t} \frac{(t-s)^2}{2} e^s\, ds &= \int_0^\infty \frac{u^2}{2} e^{t-u}\, du = e^t \int_0^\infty u e^{-u}\, du = e^t. \end{aligned} \qquad (4.20) \quad \boxed{\texttt{id1}}$$

Using $(t-s)^2/2 = (t-r)^2/2 + (t-r)(r-s) + (r-s)^2/2$ now gives the following identity.

$$\int_{-\infty}^{r} \frac{(t-s)^2}{2} e^s\, ds = e^r \left[\frac{(t-r)^2}{2} + (t-r) + 1 \right]. \qquad (4.21) \quad \boxed{\texttt{id}}$$

Using (4.19), the inequality $1 - e^{-x} \leq x$, (4.20), and changing variables $s = t - u$,

$$\begin{aligned} h_\varepsilon(t) - \frac{1}{3} e^t &\leq \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} \left(h_\varepsilon(s) - \frac{1}{3} e^s \right) ds \\ &= \int_0^{t - \log(3\varepsilon)} \left(h_\varepsilon(t-u) - \frac{1}{3} e^{t-u} \right) \frac{u^2}{2}\, du. \end{aligned}$$

Applying Lemma 3.3 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to $h_\varepsilon(\cdot + \log(3\varepsilon)) - \exp(\cdot + \log(3\varepsilon))/3$,

$$h_\varepsilon(t) - \frac{1}{3} e^t \leq 0 \ \text{ for any } t \geq \log(3\varepsilon).$$

This shows that if $0 < \varepsilon < \delta < 1$, then $h_\delta(t) \geq h_\varepsilon(t)$ for $t \leq \log(3\delta)$. To compare the exponentials for $t > \log(3\delta)$, we note that

$$\begin{aligned} &\int_{\log(3\varepsilon)}^{\log(3\delta)} \frac{(t-s)^2}{2} \left(h_\varepsilon(s) - \frac{1}{3} e^s \right) ds + \int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} \left(h_\varepsilon(s) - h_\delta(s) \right) ds \\ &\quad \leq 0 + \int_0^{t - \log(3\delta)} \left(h_\varepsilon(t-u) - h_\delta(t-u) \right) \frac{u^2}{2}\, ds. \end{aligned}$$

Applying Lemma 3.3 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to $h_\varepsilon(\cdot + \log(3\delta)) - h_\delta(\cdot + \log(3\delta))$, we see that $h_\varepsilon(t) - h_\delta(t) \leq 0$ for $t \geq \log(3\delta)$. $\square$

Lemma 4.8. *$h(t) = \lim_{\varepsilon\to 0} h_\varepsilon(t)$ exists. If $h \not\equiv 0$ then h has properties (a)–(d) in Lemma 1.1*

Proof. Lemma 4.7 implies that the limit exists. Since $0 \le h_\varepsilon(t) \le e^t/3$, $0 \le h(t) \le e^t/3$ and so $\lim_{t\to-\infty} h(t) = 0$. To show that

$$h(t) = 1 - \exp\left(-\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\,ds\right), \qquad (4.22) \quad \boxed{\texttt{hsatint}}$$

we need to show that as $\varepsilon \to 0$

$$\int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} h_\varepsilon(s)\,ds \to \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\,ds. \qquad (4.23) \quad \boxed{\texttt{eq10}}$$

Given $\eta > 0$, choose $\delta = \delta(\eta) > 0$ so that

$$\delta\left[1 + (t - \log(3\delta)) + (t-\log(3\delta))^2/2\right] < \eta/4.$$

By bounded convergence theorem, as $\varepsilon \to 0$,

$$\int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} h_\varepsilon(s)\,ds \to \int_{\log(3\delta)}^{t} \frac{(t-s)^2}{2} h(s)\,ds.$$

So we can choose $\varepsilon_0 = \varepsilon_0(\eta)$ so that the difference between the two integrals is at most $\eta/2$ for any $\varepsilon < \varepsilon_0$. Therefore if $\varepsilon < \varepsilon_0$, then

$$\begin{aligned}&\left|\int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} h_\varepsilon(s)\,ds - \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\,ds\right| \\ &\le \frac{\eta}{2} + 2\int_{-\infty}^{\log(3\delta)} \frac{(t-s)^2}{2}\frac{1}{3} e^s\,ds.\end{aligned}$$

Using the identity in (4.21) we conclude that second term is

$$\le 2\delta\left[1 + (t-\log(3\delta)) + (t-\log(3\delta))^2/2\right] \le \frac{\eta}{2}.$$

This shows (4.23) holds, and with (4.19) and (4.21) proves (4.22).

To prove $\lim_{t\to\infty} h(t) = 1$ note that if $h(\cdot) \not\equiv 0$, then there is an r with $h(r) > 0$, and so for $t > r$

$$\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\,ds \ge h(r)\int_{r}^{t} \frac{(t-s)^2}{2}\,ds = h(r)\frac{(t-r)^3}{3!} \to \infty$$

as $t \to \infty$. So in view of (4.22), $h(t) \to 1$ as $t \to \infty$, if $h(\cdot) \not\equiv 0$.

The last detail is to show if $h(\cdot) \not\equiv 0$, then $h(t) \in (0,1)$ for all t. Suppose, if possible, $h(t_0) = 0$. (4.22) implies $\int_{-\infty}^{t_0} h(s)[(t-s)^2/2]\,ds = 0$, and hence $h(s) = 0$

for $s \le t_0$. Changing variables $s \mapsto t - r$, and using (4.22) again with the inequality $1 - e^{-x} \le x$, imply that for any $t > t_0$

$$h(t) \le \int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds = \int_0^{t-t_0} h(t-r)\frac{r^2}{2}\, dr.$$

Applying Lemma 3.3 with $\lambda = 1$ and $\beta(\cdot) \equiv 0$ to the function $h(\cdot + t_0)$, we see that $h(t) \le 0$ for any $t > t_0$. But $h(t) \ge 0$ for any t, and hence $h \equiv 0$, a contradiction. $\square$

To complete the proof of Lemma 1.1 it suffices to show that $|f_\varepsilon(\cdot) - h_\varepsilon(\cdot)|$ and $|g_\varepsilon(\cdot) - h_\varepsilon(\cdot)|$ converge to 0 as $\varepsilon \to 0$. To do this, note that if

$$h_0(t) = 1 - \exp\left(-\int_{-\infty}^{\log(3\varepsilon)} \frac{(t-s)^2}{2}\frac{e^s}{3}\, ds\right),$$

then

$$h_\varepsilon(t) = 1 - (1 - h_0(t)) \exp\left(-\int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} h_\varepsilon(s)\, ds\right),$$

and so using the inequality $|e^{-x} - e^{-y}| \le |x - y|$ for $x, y \ge 0$,

$$|h_\varepsilon(t) - g_\varepsilon(t)| \le |h_0(t) - g_0(t)| + \int_{\log(3\varepsilon)}^{t} \frac{(t-s)^2}{2} |h_\varepsilon(s) - g_\varepsilon(s)|\, ds.$$

Now using the inequality $0 \le e^{-x} - 1 + x \le x^2/2$, and the identity in (4.21),

$$\begin{aligned} |h_0(t) - g_0(t)| &\le \frac{1}{2}\left[\varepsilon + \varepsilon(t - \log(3\varepsilon)) + \varepsilon\frac{(t-\log(3\varepsilon))^2}{2}\right]^2 \\ &\le \frac{3}{2}\varepsilon^2 \left[1 + (t - \log(3\varepsilon))^2 + \frac{(t-\log(3\varepsilon))^4}{4}\right]. \end{aligned}$$

Now applying Lemma 3.3 with $\lambda = 1$ and $\beta(t) = 1 + t^2 + t^4/4$ to the function

$$|h_\varepsilon(\cdot + \log(3\varepsilon)) - g_\varepsilon(\cdot + \log(3\varepsilon))|,$$

we have $|h_\varepsilon(t) - g_\varepsilon(t)| \le (3\varepsilon^2/2)\beta * V(t - \log(3\varepsilon))$, where $V(\cdot)$ is as in Lemma 2.2. Using $\lambda = 1$ in the expression of $V(\cdot)$ and Lemma 2.1,

$$\begin{aligned} \beta * V(t) &= \beta(t) + \int_0^t \beta(t-s)V'(s)\, ds \\ &= \sum_{k=0}^{\infty}\left[\frac{t^{3k}}{(3k)!} + 2\frac{t^{3k+2}}{(3k+2)!} + 6\frac{t^{3k+4}}{(3k+4)!}\right] \le 9e^t. \end{aligned}$$

So $|h_\varepsilon(t) - g_\varepsilon(t)| \le (3\varepsilon^2/2) \cdot 9\exp(t - \log(3\varepsilon))$, and so

$$\sup_{s \in I_{\varepsilon,t}} |h_\varepsilon(s) - g_\varepsilon(s)| \le 9\varepsilon e^t/2.$$

Repeating the argument for $f_\varepsilon(\cdot)$, and noting that $|h_0(t)-f_0(t)| = |h_0(t)-g_0(t)|+\varepsilon^{7/6}$,

$$\sup_{s\in I_{\varepsilon,t}} |h_\varepsilon(s) - f_\varepsilon(s)| \le \left(9\frac{3}{2}\varepsilon^2 + \varepsilon^{7/6}\right)\exp(t-\log(3\varepsilon)) = \left(\frac{1}{3}\varepsilon^{1/6} + \frac{9}{2}\varepsilon\right)e^t.$$

This completes the second step and we have proved Lemma 1.1. □

Now we have all the ingredients to prove Theorem 3.

Proof of Theorem 3. Let $h(\cdot)$ be as in Lemma 1.1. Choose $\varepsilon \in (0, \delta/6)$ small enough so that

$$\sup_{s\in I_{\varepsilon,t}} |g_\varepsilon(s) - h(s)| < \delta/2, \quad \sup_{s\in I_{\varepsilon,t}} |f_\varepsilon(s) - h(s)| < \delta/2.$$

Let $D = \left\{M \le 3\varepsilon N^{2-2\alpha/3}\right\}$. On the event D, $W = \psi(\log(3\varepsilon)) > 0$. So

$$\begin{aligned} P\left(\sup_{s\le t}\left|N^{-2}C_{\psi(s)} - h(s)\right| > \delta\right) &\le P(D^c) + P\left(N^{-2}C_W + h(\log(3\varepsilon)) > \delta\right) \\ &+ P\left(\sup_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - h(s)\right) > \delta\right) + P\left(\inf_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - h(s)\right) < -\delta\right). \end{aligned} \tag{4.24}$$

eq11

To estimate the second term in (4.24) note that $h(\log(3\varepsilon)) \le (1/3)\exp(\log(3\varepsilon)) < \delta/2$, and

$$P\left(N^{-2}C_W > \delta/2\right) \le P\left(A_W > (\delta/2)N^2\right) \to 0$$

as $N \to \infty$ by Lemma 3.2. To estimate the third term in (4.24) we use Lemma 4.6 to get

$$\begin{aligned} &\limsup_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - h(s)\right) > \delta\right) \\ &\le \limsup_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - g_\varepsilon(s)\right) > \delta/2\right) \le P(M < \varepsilon^{1/3}) + \varepsilon^{2/3}. \end{aligned}$$

For the fourth term in (4.24) use Lemma 4.5 to get

$$\begin{aligned} &\limsup_{N\to\infty} P\left(\inf_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - h(s)\right) < -\delta\right) \\ &\le \limsup_{N\to\infty} P\left(\inf_{s\in I_{\varepsilon,t}}\left(N^{-2}C_{\psi(s)} - f_\varepsilon(s)\right) < -\delta/2\right) \le P(M < \varepsilon^{1/3}) + \varepsilon^{1/12}. \end{aligned}$$

Letting $\varepsilon \to 0$, we see that for any $\delta > 0$,

$$\lim_{N\to\infty} P\left(\sup_{s\in I_{\varepsilon,t}}\left|N^{-2}C_{\psi(s)} - h(s)\right| > \delta\right) = 0 \tag{4.25}$$

eq2

It remains to show that $h(\cdot) \not\equiv 0$. Let ε, γ be such that

$$P[M \leq (1+\gamma)\varepsilon^{1/3}] + 11\frac{\varepsilon^{1/3}}{\gamma} < 1.$$

Fix any $\eta > 0$ and let $t_0 = \log(3\varepsilon(1+\gamma) + 3\eta)$. Using Lemma 3.2 and 3.5

$$\begin{aligned}
&\limsup_{N\to\infty} P\left(N^{-2}C_{\psi(t_0)} < \varepsilon\right) = \limsup_{N\to\infty} P(\tau(\varepsilon) > \psi(t_0)) \\
&\leq \limsup_{N\to\infty} P[\tau(\varepsilon) > \sigma(\varepsilon(1+\gamma))] + \limsup_{N\to\infty} P[\sigma(\varepsilon(1+\gamma)) > \psi(t_0)] \\
&\leq \limsup_{N\to\infty} P[\tau(\varepsilon) > \sigma(\varepsilon(1+\gamma))] + \limsup_{N\to\infty} P\left(\left|N^{-2}A_{W_{\varepsilon(1+\gamma)+\eta}} - \varepsilon(1+\gamma) - \eta\right| > \eta\right) \\
&\leq P[M \leq (1+\gamma)\varepsilon^{1/3}] + 11\frac{\varepsilon^{1/3}}{\gamma} < 1.
\end{aligned}$$

But if $h(t_0) = 0$, we get a contradiction to (4.25). This proves $h(\cdot) \not\equiv 0$. □

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