Miscellany

David Aldous

October 28, 2016

Some little math topics

- Buffon's needle and stochastic geometry.
- What is the chance your vote will make a difference?
- A bet that's favorable to both parties.
- Bayes rule, illustrated by the following:
- Blue or Green taxi?
- The two envelope paradox.
- The frog riddle.

Buffon's needle

[Different viewpoint from textbook or Wikipedia accounts.]

Take a object (pencil, circular wire, electrical cord) which is 1-dimensional, and so has some length L. Throw it "at random" onto a floor on which there are parallel lines, 1 unit distance apart.

[board]

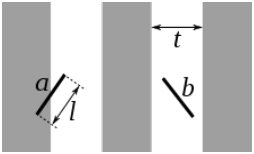
Random variable $N =$ number of times the object crosses a line.

Formula: $\mathbb{E}N = 2\pi^{-1}L$.

So – surprisingly – the shape of the object does not matter.

Here's the math argument.

- Consider a very short line segment, length δ. Here $\mathbb{E}N$ will be very small – write it as $c\delta$ for some c.
- Any longer length-L object is like L/δ such short segments, so (linearity of expectation) $\mathbb{E}N = cL$; with the same c for all objects.
- But for the special object which is a diameter-1 circle, we have $N = 2$, non-random.
- So solve the equation $2 = c \times \pi$ to find c.



The usual name **needle** refers to the case of a line segment of length $L \leq 1$. In this case $N = 0$ or 1 and so

$$\mathbb{P}(N = 1) = \mathbb{E}N = 2L/\pi.$$

The story may seem artificial, but this is related to basic math formulas in the field of **stochastic geometry**, which is the study of random lines, triangles, etc in the plane.

For instance in the study of road networks, a basic statistic is

$$L = \text{ length of network per unit area.}$$

If we daw an imaginary random line on the map, then we get an analogous formula

average number of intersection of roads with this line $= 2L/\pi$
per unit length.


Brownlee
Plainview
Wayne
Pierce
Neligh
Tilden
Norfolk
Mullen
Seneca
Thedford
Halsey
Dunning
Brewster
Taylor
Burwell
Bartlett
Madison
West Point
Oakland
Albion
St Edward
Ord
Tryon
Ringgold
Arnold
Merna
Broken Bow
Scotia
Columbus
Fremont
Platte River
Fullerton
Callaway
Loup City
St Paul
David City
Paxton
North Platte
Central City
Ravenna
Gothenburg
Grand Island
Seward
Aurora
York
Lincoln
Cozad
Lexington
Gibbon
Kearney
Wellfleet
Maywood
Curtis
Hastings
Crete
Geneva
Wauneta
Holdrege
Google
Map data ©2016 Google
20 mi

Google Maps

What is the chance your vote will make a difference in an upcoming election?

The answer depends very much on the current available information. The paper *What is the probability your vote will make a difference?* by Andrew Gelman and Nate Silver and Aaron Edlin was written a few weeks before the 2008 US Presidential election – depends on your State.

I will consider two simpler cases. First, a small semantic point. With N votes between two leading candidates (A and B) there are two possibilities to consider:

- (N even): both get N/2 votes.
- (N odd): one gets (N+1)/2 votes, the other gets (N-1)/2 votes.

The chance (that your vote makes a difference) is 1/2 in each case.

Setting 1: a small club. N members will vote for a president. You know each candidate has some support but you have no idea how much support. So you guess a distribution on "number of votes", say uniform on $[N/4, 3N/4]$. Then the probability of the event above is about $2/N$, and so the chance (that your vote makes a difference) is about $1/N$.

Setting 2: a large State election for Governor, which according to opinion polls is too close to call.

Well organized opinion polls have historically been quite accurate.
[show Field Poll track record]

So let's suppose that (number of votes for A) will be random with mean 50% and s.d. 2.5%. We have no good reason to assume Normal (many errors other than sampling variation) but let's do so anyway.

The Normal approximation for chance A gets exactly N/2 votes is

$$\frac{1}{0.025N} \times \phi(0) \approx 16/N$$

and we conclude

> the chance (that your vote makes a difference) is about $8/N$.

In major California elections there are about 13 million votes, so (if opinions polls say "too close to call") the chance is about 1 in 1.6 million.

If instead opinion polls said "60%" then you have to multiply by chance of this large error; no theory here, need to guess/extrapolate from historical data for poll errors.

A bet that is advantageous to both parties.

In February 2012, I (resident in the U.S.) and my friend Sir Jonathan (resident in the U.K.) agreed that there is at 50-50 chance that, one year ahead of 2013, the pound/dollar exchange rate will be on either side of 1.60 dollars per pound. (The figure didn't matter, just that we agreed on the figure).

Agreed that there is at 50-50 chance that, one year ahead of 2013, the pound/dollar exchange rate will be on either side of 1.60 dollars per pound.

We then made a bet:

- if 1 pound is at that time worth more than 1.60 dollars then he will pay me 1 pound;
- if not then I will pay him 1.60 dollars.

So what could happen?

- From my viewpoint I either lose 1.60 dollars or win an amount that by definition is worth more than 1.60 dollars;
- From his viewpoint he either loses 1 pound or wins an amount that by definition is worth more than 1 pound;

and since we agreed it was a 50-50 chance, to each of us this is a favorable bet.

Another common cognitive error is **base rate neglect**, which is the psychologist's phrase for not appreciating Bayes formula. In a famous example, subjects are asked the following hypothetical question.

> *A taxi was involved in a hit and run accident at night. Two taxi companies, the Green and the Blue, operate in the city. 85% of the taxis in the city are Green and 15% are Blue.*
>
> *A witness identified the taxi as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.*
>
> *What is the probability that the taxi involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?*

Most people answer either 80% or make some guess over 50%. The correct answer, via Bayes formula, is
[board]