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# PERCOLATION AND MINIMAL SPANNING FORESTS IN INFINITE GRAPHS[1]

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The structure of a spanning forest that generalizes the minimal spanning tree is considered for infinite graphs with a value $f(b)$ attached to each bond $b$. Of particular interest are stationary random graphs; examples include a lattice with iid uniform values $f(b)$ and the Voronoi or complete graph on the sites of a Poisson process, with $f(b)$ the length of $b$. The corresponding percolation models are Bernoulli bond percolation and the "lily pad" model of continuum percolation, respectively. It is shown that under a mild "simultaneous uniqueness" hypothesis, with at most one exception, each tree in the forest has one topological end, that is, has no doubly infinite paths. If there is a tree in the forest, necessarily unique, with two topological ends, it must contain all sites of an infinite cluster at the critical point in the corresponding percolation model. Trees with zero, or three or more, topological ends are not possible. Applications to invasion percolation are given. If all trees are one-ended, there is a unique optimal (locally minimax for $f$) path to infinity from each site.

**1. Introduction.** For a finite set $V \subset \mathbb{R}^d$, a Euclidean *minimal spanning tree* (MST) of $V$ is a tree with site (that is, vertex) set $V$ and minimal total length of all bonds (that is, edges). More generally, given a finite graph with site set $V$ and bond set $\mathscr{B}$, and a labeling function $f\colon \mathscr{B} \to [0,\infty)$, a minimal spanning tree of $(V, \mathscr{B}, f)$ is a tree in $(V, \mathscr{B})$ spanning $V$ with $\sum_{b \in \mathscr{B}} f(b)$ minimal among all such trees; $(V, \mathscr{B}, f)$ determines a *labeled graph*. It is natural to ask whether there is a structure analogous to the MST when $(V, \mathscr{B}, f)$ is an infinite labeled graph, and if so to consider its properties, especially for random labeled graphs. Two particular cases of interest are:

the *lattice / uniform* model: $(V, \mathscr{B})$ is a lattice in $\mathbb{R}^d$ and $\{f(b), b \in \mathscr{B}\}$ are iid random variables uniform in $[0, 1]$

and

the *Poisson / Euclidean* model: $(V, \mathscr{B})$ is the complete graph on the set of sites of a Poisson process and $f$ is Euclidean length.

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In the latter case one can allow more general stationary point processes, yielding the *stationary / Euclidean* model.

The natural way to define such an MST analog is to find a property of bonds in a labeled graph which (i) in finite graphs, characterizes membership in the MST and (ii) makes sense in infinite graphs as well. This property then becomes the definition of membership in the MST analog. One such definition, based on Prim's inductive construction in [16] of the MST in finite graphs, was used by Aldous and Steele [1] for the stationary/Euclidean model; it yields a "minimal spanning forest" (MSF) in which every component is an infinite tree. Our definition, despite appearing quite different, is equivalent to theirs in a large class of models, which includes the lattice/uniform and Poisson/Euclidean.

The main structure of an infinite tree is given by its number of topological ends, which is the number of infinite self-avoiding paths from any fixed vertex. Thus a zero-ended infinite tree must contain sites of infinite degree, a one-ended tree is like an infinite system of river tributaries, a two-ended tree consists of a single doubly infinite path (the *trunk*) with finite branches emanating from it and a tree with three or more ends must contain at least one *branch point*, that is, a point from which there are at least three disjoint infinite self-avoiding paths. Aldous and Steele [1] conjectured that for the Poisson/Euclidean model, their MSF consists of a single one-ended tree.

Corresponding to any infinite random labeled graph $X = (V, \mathscr{B}, f)$, there is a percolation model as follows. We say a bond $b$ is *occupied at level* $r$ if $f(b) \le r$. Let $X_{\le r}$ denote the graph with site set $V$ and bond set $\{b \in \mathscr{B}: f(b) \le r\}$. An *r-cluster* is a connected component of $X_{\le r}$, and we say *percolation occurs* at level $r$ in $X$ if there is an infinite $r$-cluster. The *critical point* is

$$r_c(X) := \inf\{r: \text{percolation occurs in } X \text{ at level } r\}.$$

For the lattice/uniform model, the corresponding percolation model is Bernoulli bond percolation; for the stationary/Euclidean model, the corresponding percolation model is equivalent to the fixed-radius case of the standard *blob model* of continuum percolation, in which a closed ball of radius $r/2$ is centered at each point-process site, and one considers connected components of the union of the balls. This union is called the *occupied space*; its complement is the *vacant space*. The labeled-graph formulation enables one to couple all values of the order parameter $r$; see [2] for more examples of such coupling.

Let $X_{<r}$ denote the graph with site set $V$ and bond set $\{b \in \mathscr{B}: f(b) < r\}$. A *strict r-cluster* is a connected component of $X_{<r}$.

We say *simultaneous uniqueness* holds for a labeled graph if there is at most one infinite $r$-cluster for every value of $r$. Note that "simultaneous uniqueness a.s." is not the same as the property that for each fixed $r$, uniqueness holds a.s. at level $r$; see Example 1.9 of [2]. A sufficient condition from [2] for simultaneous uniqueness is given here in Proposition 2.2. We say that *strict simultaneous uniqueness* holds if there is at most one infinite strict $r$-cluster for every value of $r$. We will see that under mild assumptions, simultaneous uniqueness and strict simultaneous uniqueness are equivalent a.s.

We will examine here some relations between the structure of the MSF and properties of the corresponding percolation model for stationary random labeled graphs. In graphs for which the MSF is well defined, we will show that if strict simultaneous uniqueness holds, then the MSF contains at most one tree with other than one topological end; if it exists, this one exceptional tree has two topological ends and contains all sites of an infinite cluster at the critical point in the corresponding percolation model. Thus if there is no percolation at the critical point, as is believed to be true for most models of interest, then there are only one-ended trees. Even without strict simultaneous uniqueness, all trees in the MSF have one or two ends.

It would be desirable to also have implications in the other direction, that is, that percolation at the critical point implies something about the structure of the MSF, at least within restricted classes of models. Thus far we have not been able to obtain such results.

When all trees in the MSF are one-ended, there is a canonical one-to-one correspondence between sites and bonds in the MSF—one associates to each site the unique bond emanating "toward infinity" from that site. Aldous and Steele [1] used their MSF results to prove certain limit theorems about MST's of finite random point sets; proofs of some such limit theorems may be made easier by the use of this correspondence.

**2. Statement of main results.** The formal definition of a *stationary random labeled graph* $X$ in $\mathbb{R}^d$ can be found in [2]. Omitting technicalities, it is defined as a locally finite point process in the disjoint union of the site space $\mathbb{R}^d$ and the labeled-bond space $(\mathbb{R}^d \wedge \mathbb{R}^d) \times \mathbb{R}$, restricted so that if $\langle x, y \rangle$ is a bond, then $x$ and $y$ are sites, and with distribution invariant under simultaneous translation of sites and labeled bonds. Here $\mathbb{R}^d \wedge \mathbb{R}^d$ denotes the set of pairs $\langle x, y \rangle$, $x, y \in \mathbb{R}^d$, with $\langle x, y \rangle$ and $\langle y, x \rangle$ identified. Such a graph can include sites of infinite degree, but a.s. has only finitely many sites in each bounded region; the asymptotic density of sites may be infinite. Multiple bonds between a fixed pair of sites are not allowed, but one can obtain results for graphs with such multiple bonds by deleting all bonds between each fixed pair $x$ and $y$ except the one with the smallest label. Bonds of form $\langle x, x \rangle$, called *loop bonds*, are allowed. We let $V$ denote the set of sites, $\mathscr{B}$ the set of bonds and $f$ the labeling function, so that $X$ can be identified with the triple $(V, \mathscr{B}, f)$. Alternately, $\mathbb{R}^d$ may be replaced throughout by a lattice $L$, with translation allowed by elements of $L$ only; without further mention, we use this formulation when appropriate, as in the lattice/uniform model.

To ensure that the graphs $G = (V, \mathscr{B}, f)$ we deal with have a well-defined unique MST or MSF, we will make two assumptions throughout: (1) all bond labels are distinct and (2) for every component $C$ and every finite proper subset $A$ of $V \cap C$, there is a unique $f$-minimizing bond among all bonds that connect sites in $A$ to sites in $V \setminus A$. Assuming (1), a sufficient condition for (2) is that every site has finite degree in $G_{\leq r}$ for every $r$; this is satisfied a.s. in the lattice/uniform model and in the stationary/Euclidean model provided the stationary point process of sites is locally finite. We call $G$ *locally finite* if $V \cap R$ is finite for all bounded regions $R$ and call a labeled graph that satisfies (1) and (2) *ambiguity-free*.

Given a graph $G = (V, \mathscr{B}, f)$ and a subgraph $H$, define

$$\partial H := \{ b = \langle x, y \rangle \in \mathscr{B} \colon x \in H,\ y \notin H \}.$$

In a mild abuse of terminology we will call $f(b)$ the *length* of $b$ and refer to bonds as *shorter, longest*, and so forth; this should not cause confusion because we never make use of Euclidean length of bonds except when it coincides with the labeling by $f$.

The MST in finite graphs can be constructed by an inductive "invasion" procedure known as Prim's algorithm [16]; the same procedure was used by Aldous and Steele [1] to define their MSF for certain infinite graphs. Specifically, let $(V, \mathscr{B}, f)$ be a labeled graph and let $v \in V$ be a site. Let $J_0(v) := \{v\}$ and, given $J_n(v)$, let $J_{n+1}(v)$ be $J_n(v)$ together with the shortest (that is, $f$-minimizing) bond $B_{n+1}$ in $\partial J_n(v)$, including the endpoint of $B_{n+1}$ in $V \setminus J_n(v)$. Let $J_\infty(v) := \bigcup_{n \geq 0} J_n(v)$. We say a bond $b$ is *invaded from* $v$ if $b \in J_\infty(v)$.

For $\gamma$ a self-avoiding path in a graph and $u, v$ sites in $\gamma$, let $\gamma_{uv}$ denote the segment of $\gamma$ from $u$ to $v$. We say a self-avoiding path $\gamma$ in a labeled graph $(V, \mathscr{B}, f)$ is *locally $f$-minimax* if for every pair $u, v$ of sites in $\gamma$ and every path $\alpha$ from $u$ to $v$,

$$\max\{ f(b) \colon b \in \gamma_{uv} \} \leq \max\{ f(b) \colon b \in \alpha \}.$$

To define our MSF, we will need the next proposition. All the equivalences are well known in the case of finite graphs; see [15]. For infinite graphs, we essentially need only verify that existing ideas for finite graphs can establish equivalence among (2.1)–(2.5) without using equivalence to (2.6) or finiteness of the graph. The proofs of this and all results in this section appear in Section 3.

PROPOSITION 2.1. *Let $G = (V, \mathscr{B}, f)$ be a (finite or infinite) ambiguity-free labeled graph and let $b = \langle x, y \rangle \in \mathscr{B}$ with $x \neq y$. The following are equivalent*:

(2.1) *There exists a set $A$ of sites such that $b$ is the shortest bond from $A$ to $V \setminus A$.*

(2.2) *There exists no path from $x$ to $y$ with all bonds strictly shorter than $b$.*

(2.3) *$b$ is a bond in some locally $f$-minimax path.*

*The following are equivalent*:

(2.4) $b$ *is invaded either from* $x$ *or from* $y$.

(2.5) $x$ *and* $y$ *are in distinct strict* $f(b)$*-clusters and these clusters are not both infinite.*

*If strict simultaneous uniqueness holds for* $G$, *then* (2.1)–(2.5) *are all equivalent. If* $G$ *is finite*, *then* (2.1)–(2.5) *are each equivalent to*

(2.6) $b$ *is a bond of the MST of* $G$.

Again, *shorter* means having a smaller label. Equivalence of (2.4) and (2.6) for finite graphs is the basis of Prim's algorithm [15] for constructing the MST and shows that the set of invaded bonds does not depend on the starting site in finite graphs. We call (2.2) the *creek-crossing criterion*, by analogy to a hiker trying to cross a creek by stepping from stone to stone, avoiding getting wet by never taking an available step if a path of all strictly shorter steps exists.

Aldous and Steele [1] defined a spanning forest consisting of all bonds satisfying (2.4) and proved that (2.4) and (2.5) are equivalent. We prefer to use the creek-crossing criterion as our definition, that is, we define the *minimal spanning forest* (or MSF) of an ambiguity-free labeled graph $G = (V, \mathscr{B}, f)$ to be the graph with site set $V$ and bond set

$$\{b = \langle x, y\rangle \in \mathscr{B}\text{: there is no path from } x \text{ to } y \text{ consisting entirely of bonds } e \text{ with } f(e) < f(b)\}.$$

Because simultaneous uniqueness plays a major role in this work, we will give a sufficient condition for it from [2]. For this we need the notion of *positive finite energy*: for a full definition in our context, see [2]; the idea appears in [6]. Loosely, positive finite energy means that conditioning on the graph outside a finite box, together with certain partial information about bonds crossing the box boundary, a.s. yields a nonzero probability that all sites within that box are connected at a given level $r$, at least for $r$ large enough that percolation occurs at level $r$.

Letting $\Lambda_t$ denote $[-t, t]^d$, let us define the *site density* of a stationary random graph $X = (V, \mathscr{B}, f)$ to be

$$\lim_{t\to\infty} |V \cap \Lambda_t|/|\Lambda_t|,$$

where $[\cdot]$ denotes cardinality for finite sets and volume for regions in $\mathbb{R}^d$. Stationarity ensures that this limit exists a.s.

PROPOSITION 2.2 ([2], Theorem 1.8 and Remark 1.10). *Suppose* $X$ *is a stationary random labeled graph in* $\mathbb{R}^d$, *with positive finite energy and finite site density a.s. Then both simultaneous uniqueness and strict simultaneous uniqueness hold for* $X$, *a.s.*

The lattice/uniform and Poisson/Euclidean models clearly have positive finite energy and finite site density, so Proposition 2.2 shows that for these models, our definition (2.2) is equivalent to the definition (2.4) used by Aldous and Steele [1].

The distinction between strict and nonstrict simultaneous uniqueness is important in our results. Therefore, we will explicate the distinction by way of the following result, though Proposition 2.2 makes it nonessential to our main results.

LEMMA 2.3. (i) *Suppose $G$ is an infinite labeled graph with all labels distinct. If strict simultaneous uniqueness holds for $G$, then so does simultaneous uniqueness.*

(ii) *Suppose $X$ is a stationary random labeled graph in $\mathbb{R}^d$ that a.s. has finite site density and all labels distinct. Then with probability* 1, *simultaneous uniqueness holds for $X$ if and only if strict simultaneous uniqueness holds for $X$.*

The assumption of all labels distinct cannot be eliminated in Lemma 2.3. It is easily verified that in the stationary random labeled graph of Example 1.9 of [2], strict simultaneous uniqueness holds a.s. but not simultaneous uniqueness. See also Example 5.2 below.

It is immediate from the creek-crossing criterion (2.2) that the longest bond in any cycle in a labeled graph $G$ is not in the MSF. Thus the MSF is acyclic. Criterion (2.1) ensures that in an infinite ambiguity-free connected labeled graph, every component of the MSF is infinite. In particular, there are no one-point components, meaning the MSF spans the site set $V$. This proves the following, which justifies the name "minimal spanning forest."

LEMMA 2.4. *In an ambiguity-free labeled graph $G = (V, \mathscr{B}, f)$ with all components infinite, the MSF is a forest that spans $V$ and consists of infinite trees.*

For an infinite lattice, a completely different way of obtaining a random spanning forest is considered in [14].

Let $C_\infty(G, r)$ denote the union of all infinite $r$-clusters in the graph $G$ and let $C_\infty^{\mathrm{st}}(G, r)$ be the union of all infinite strict $r$-clusters. Here is our main result.

THEOREM 2.5. *Suppose $X$ is a stationary random labeled graph in $\mathbb{R}^d$ that a.s. is ambiguity-free, has finite site density and has all components infinite. Then with probability* 1:

(i) *The MSF contains no zero-ended trees or trees with three or more ends.*

(ii) *If simultaneous uniqueness holds for $X$, then the MSF includes at most one two-ended tree; all other trees are one-ended. If a two-ended tree $T$ exists, then there is percolation at level $r_c(X)$ in the corresponding percolation model,*

*$T$ contains all sites of $C_\infty^{st}(X, r_c(X))$ and the trunk of $T$ is contained in $C_\infty^{\mathrm{st}}(X, r_c(X))$.*

If $X$ is ergodic, then there is a nonrandom almost-sure value of the critical point $r_c(X)$, which we denote $r_c$.

Consider a planar graph $G$, without loop bonds, embedded in $\mathbb{R}^2$. The bonds, viewed as curves in $\mathbb{R}^2$, divide the plane into *faces*, which are connected components of the complement of the graph. We call $G$ *latticelike* if (i) $G$ is locally finite, (ii) every site in $G$ has finite degree, (iii) $G$ is connected, (iv) $G$ is planar, (v) $G$ has no loop bonds and (vi) every face is bounded. For a lattice-like graph, a *dual graph*, also planar, is obtained by selecting an arbitrary "face site" in each face and then, for each bond $b$ that forms part of the boundary between two distinct faces, putting a dual bond $b^*$ between the face sites in these faces. For $G$ labeled, a dual bond $b^*$ is said to be *strictly occupied* at level $r$ if and only if $f(b) \geq r$.

THEOREM 2.6. *Suppose $G$ is an ambiguity-free lattice-like labeled graph in $\mathbb{R}^2$. If the MSF of $G$ consists of more than one tree, then there is percolation of strictly occupied dual bonds at level $r_c(G)$ in the corresponding percolation model.*

EXAMPLE 2.7. For the lattice/uniform model on the hypercubic lattice, it is known for dimensions $d = 2$ [11] and for sufficiently large $d$ [9] that there is no percolation at the critical point in the corresponding percolation model; the best result at present is that $d \geq 19$ is "sufficiently large" [10]. Therefore, all trees in the MSF are one-ended. For $d = 2$, where the lattice and its dual are isomorphic, there is also no percolation in the dual graph at level $r_c$ [11], so the MSF consists of a single one-ended tree, as was proved in [4].

EXAMPLE 2.8. For the two-dimensional stationary/Euclidean model, in place of the complete graph on the site set $V$, one may consider the Voronoi graph, defined as follows. Let $d(\cdot,\cdot)$ denote Euclidean distance. Given $V$, divide the plane into the polygonal cells

$$Q(v) := \{y \in \mathbb{R}^2 : d(y, v) \leq d(y, x) \text{ for all } x \in V\}, \qquad v \in V.$$

The *Voronoi graph* has site set $V$ and bond set

$$\{\langle u, v\rangle : u, v \in V, Q(u) \text{ and } Q(v) \text{ have an edge in common}\},$$

labeled by Euclidean length. It is easily checked that only Voronoi bonds can satisfy (2.1) or (2.2), so the MSF is the same for the Voronoi graph as for the complete graph. This is well known for finite graphs; see [15]. The Voronoi graph is a.s. lattice-like, so Theorem 2.6 can be applied. Percolation at level $r$ in the graph dual to the Voronoi graph is equivalent to percolation of vacant space in the corresponding blob model at level $r$ (radius $r/2$). In the Poisson case, it is proved in [3] that there is no percolation of vacant or occupied space at level $r_c$. Therefore, the MSF consists of a single one-ended tree, as conjectured by Aldous and Steele [1].

## 3. Proofs of the main results.

PROOF OF PROPOSITION 2.1. We will show (2.1) $\Leftrightarrow$ (2.2), (2.2) $\Leftrightarrow$ (2.3) and, under strict simultaneous uniqueness, (2.2) $\Leftrightarrow$ (2.5). The equivalence (2.4) $\Leftrightarrow$ (2.5) is proved in [1]. The equivalence (2.1) $\Leftrightarrow$ (2.6) for finite graphs can be found in [15].

Suppose first that (2.1) holds and $x \in A$, $y \in V \setminus A$. If $\gamma$ is a path from $x$ to $y$, then $\gamma$ includes a bond $e$ from $A$ to $V \setminus A$. By (2.1), $f(e) \geq f(b)$. Since $\gamma$ is arbitrary, (2.2) follows. Conversely suppose (2.2) holds and let $A$ be the strict $f(b)$-cluster containing $x$. By (2.2), $y \in V \setminus A$ and by definition of "strict $f(b)$-cluster," there is no bond shorter than $b$ from $A$ to $V \setminus A$. Thus (2.1) holds and we have (2.1) $\Leftrightarrow$ (2.2).

Next suppose (2.3) holds. Suppose $\alpha$ is a path from $x$ to $y$. Then by definition of "locally $f$-minimax," $\max\{f(e): e \in \alpha\} \geq f(b)$. Since $\alpha$ is arbitrary, (2.2) holds. Conversely, (2.2) says that $b$ by itself constitutes a locally $f$-minimax path. Thus (2.3) $\Leftrightarrow$ (2.2).

Now (2.2) says that $x$ and $y$ are in distinct strict $f(b)$-clusters. Under strict simultaneous uniqueness, these clusters are necessarily not both infinite, so (2.2) and (2.5) are equivalent. □

For a graph $G$ in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, let $\theta_x G$ denote the translation of $G$ by $-x$; thus for every site $v$ of $G$, $\theta_v G$ has a site at 0.

The proof of Theorem 2.5 is based principally on the idea that certain possible structures in labeled graphs are prohibited a.s. by stationarity and finite site density. We begin with four propositions on this theme. The first says, loosely, that anything that happens only finitely many times per infinite cluster actually never happens.

PROPOSITION 3.1 ([2]). *Suppose $X = (V, \mathscr{B}, f)$ is a stationary random labeled graph in $\mathbb{R}^d$ with finite site density, and $A$ is a set of labeled graphs $G$ in which the origin is a site in an infinite component of $G$. Suppose that with probability* 1, *there are only finitely many sites $v$ in each infinite component of $X$ for which $\theta_v X \in A$. Then*

$$P[\,\theta_v X \in A \text{ for some } v \in V\,] = 0.$$

PROOF OF LEMMA 2.3. (i) Suppose $G$ includes two disjoint infinite $r$-clusters for some $r$. Since there is at most one bond in $G$ with label $r$, each of these two $r$-clusters contains an infinite strict $r$-cluster, so strict simultaneous uniqueness fails.

(ii) We may assume the random graph is ergodic, so $r_c(X) = r_c$ a.s.

Suppose that in the graph $X$, simultaneous uniqueness holds. We may assume all labels are distinct. For each $r > r_c$, $M(r) := \bigcup_{s<r} C_\infty(X, s)$ is an infinite strict $r$-cluster; we call any other infinite strict $r$-cluster *extraneous*.

For $r > r_c$ we call an infinite strict $r$-cluster *hollow* if it contains no infinite $s$-cluster, for every $s < r$. Thus every extraneous infinite strict cluster is hollow. For each site $v$ of $X$ let $r_\infty(v) := \inf\{r > r_c: v \in C_\infty(X, r)\}$. If $v$ is a site of an extraneous infinite strict $r$-cluster for some $r > r_c$, then $v \notin C_\infty(X, s)$ for any $s < r$, so $r_\infty(v) = r$; in particular, there is at most one such $r$ for each site $v$ of $X$. Therefore, distinct extraneous infinite strict clusters are disjoint, even if the corresponding values of $r$ are distinct.

If $C$ is an extraneous infinite strict $r$-cluster for some $r > r_c$, then since $C_\infty(X, r)$ is connected, there must be a bond $b = \langle x, y \rangle \in \partial C$ with $x \in C$ and $f(b) = r$. Since labels are distinct, there is at most one such choice of $b$, $x$ and $y$; we call $b$ the *attachment bond* and $x$ the *attachment site* of $C$. To detach all extraneous clusters from each other, we wish to replace the attachment bond $\langle x, y \rangle$ with the loop bond $\langle x, x \rangle$. We call $\langle x, x \rangle$ the *altered attachment bond* of $C$ and give it the same label $r$ as the attachment bond. Let $Y$ be the stationary random labeled graph whose components are the extraneous infinite strict clusters in $X$ together with their altered attachment bonds. In each component of $Y$, the altered attachment bond is the unique longest bond. Let $A$ be the set of labeled graphs in which 0 is an endpoint of the unique longest bond in its connected component. If $v$ is a site of $Y$, then $\theta_v Y \in A$ if and only if $v$ is the altered attachment site of some extraneous infinite strict cluster in $X$. Hence $\theta_v Y \in A$ for exactly one site $v$ in each infinite cluster in $Y$. By Proposition 3.1, we may assume there are no sites $v$ with $\theta_v Y \in A$; this means $Y$ is empty, so there are no extraneous infinite strict clusters in $X$.

It remains to establish uniqueness of the strict $r_c$-cluster. Another application of Proposition 3.1 and the distinct-labels property shows that there are a.s. no bonds $b$ in $X$ with $f(b) = r_c$. However, this means every infinite strict $r_c$-cluster is also an infinite $r_c$-cluster, and there is at most one of the latter. □

If $F$ is a finite set of sites in a single component of a graph $G$, we write $C(G, F)$ for this component and let $G \setminus F$ denote the subgraph of $G$ obtained by deleting $F$ and all bonds emanating from $F$. We write $C(G, v)$ for $C(G, \{v\})$. We call such a finite $F$ a *core* if there are infinitely many finite components in $G \setminus F$ that are contained in $C(G, F)$; that is, removing $F$ splits off infinitely many new finite clusters. An example is furnished by the "infinite-spoked bicycle wheel," in which a countable number of "rim sites" are located on some circle, and there are two other "hub-end" sites not on this circle; there is a bond between each hub-end site and each rim site. The two hub-end sites then form a core. Neither hub-end site is a core by itself.

PROPOSITION 3.2 ([2]). *Suppose $X$ is a stationary random labeled graph in $\mathbb{R}^d$ with finite site density. Then with probability* 1, *$X$ contains no core.*

The easy proof of the following lemma is contained in the proof of Lemma 2.3 of [2].

LEMMA 3.3. *Suppose $G$ is an infinite connected graph that contains no core, and $v$ is a site of $G$. Then $G$ contains an infinite self-avoiding path starting at $v$.*

PROPOSITION 3.4. *Suppose $T$ is an infinite tree that has at most finitely many topological ends and contains no core. Then all sites of $T$ have finite degree.*

PROOF. Suppose $T$ is an infinite tree that has $k$ topological ends ($0 \le k < \infty$), and $v$ is a site of infinite degree in $T$. Then $T \setminus \{v\}$ has infinitely many components, at most $k$ of which contain infinite self-avoiding paths. By Lemma 3.3, all other components of $T \setminus \{v\}$ are finite. However, this means $\{v\}$ is a core. $\square$

For $G = (V, \mathscr{B}, f)$ a graph and $W \subset \mathbb{R}^d$, define the *density* of $W$ in $G$ to be

$$\rho(W, G) := \lim_{t \to \infty} |W \cap V \cap \Lambda_t| / |\Lambda_t|$$

whenever this limit exists. For $A$ a set of graphs in $\mathbb{R}^d$ in which the origin is a site, define $V_A := \{v \in V\colon \theta_v G \in A\}$. Define the *density* of $A$, or of $V_A$, in $G$ to be $\rho(V_A, G)$. If $A$ is measurable and $X$ is stationary, then $\rho(V_A, X)$ exists a.s. and

$$\text{if } \rho(V_A, X) = 0 \text{ a.s., then } V_A = \phi \text{ a.s.} \tag{3.1}$$

Further, $\rho(V_\bullet, X)$ is a.s. a measure—the Palm measure associated with the ergodic component of $X$. $\rho(V, X)$ is just the site density of $X$. Such facts are well known in the context of stationary point processes; see, for example, [12]. For a discussion in the context of stationary random graphs, see [2].

A *branch point* in a component $C$ of a graph is a site $v$ such that at least three components of $C \setminus \{v\}$ are infinite. The proof of the following proposition is an adaptation of the proof of Theorem 1 of [6].

PROPOSITION 3.5. *Suppose $X = (V, \mathscr{B}, f)$ is a stationary random graph in $\mathbb{R}^d$ with finite site density. Then $X$ contains no branch points, a.s.*

PROOF. Let $A$ be the collection of graphs in which the origin is a branch point, so $V_A$ is the set of branch points in $X$. For $t > 0$, $k \ge 1$ and $v$ a branch point in a component $C$ of $X$, let $C_v^{(i)}(t, k)$, $i = 1, \dots, n_v(t, k)$, be a listing of those components of $C \setminus \{v\}$ that have at least $k$ sites in the translate $v + \Lambda_t$. From the definition of branch point, $n_v(t, k) \ge 3$ if $t$ is sufficiently large; let $A(t, k)$ be the set of graphs in which 0 is a branch point with $n_0(t, k) \ge 3$. Then for $s > 0$, the hypotheses of Lemma 2 of [6] are satisfied for $V \cap \Lambda_{t+s}$ (in place of $S$), $V_{A(t,k)} \cap \Lambda_s$ (in place of $R$) and $C_v^{(i)}(t, k)$ (in place of $C_v^{(i)}$), yielding

$$|V_{A(t,k)} \cap \Lambda_s| \le k^{-1} |V \cap \Lambda_{t+s}|.$$

Dividing by $|\Lambda_s|$ and letting $s \to \infty$, we see that

(3.2) $$\rho(V_{A(t,k)}, X) \le k^{-1}\rho(V, X) \quad \text{a.s.}$$

Now $A(t,k)$ increases to $A$ as $t \to \infty$, and $\rho(V_\bullet, X)$ is a.s. a measure, so letting $t \to \infty$ in (3.2), we obtain $\rho(V_A, X) \le k^{-1}\rho(V, X)$ a.s. Since $k$ is arbitrary, this shows $\rho(V_A, X) = 0$, so by (3.1) $V_A = \phi$ a.s. □

The following lemma is related to the criterion (2.3) for membership in the MSF.

LEMMA 3.6. *Let $G$ be an ambiguity-free labeled graph with MSF $F$. Then every self-avoiding path in $F$ is locally $f$-minimax.*

PROOF. Let $\gamma$ be a self-avoiding path in $F$, let $u, v$ be sites in $\gamma$ and let $\alpha$ be another path from $u$ to $v$ in $G$. Following $\gamma_{uv}$ from $u$ to $v$, then following $\alpha$ backward from $v$ to $u$ produces a circuit. If the longest bond $b$ in this circuit appears in only one of $\gamma_{uv}$ and $\alpha$, then either $b$ is a loop bond or $b$ fails the creek-crossing criterion (2.2), so $b$ is not a bond of $F$. Therefore, $b$ is a bond of $\alpha$ and the lemma follows. □

LEMMA 3.7. *Suppose $T$ is a component of the MSF of an ambiguity-free infinite labeled graph $G$ and $b = \langle x, y\rangle \in \partial T$, with $x \in T$. Then $f(b) \ge r_c(G)$. If all sites of $T$ have finite degree in $T$, then there is an infinite self-avoiding path $\gamma$ in $T$ starting at $x$ such that $f(e) < f(b)$ for all bonds $e$ in $\gamma$.*

PROOF. Let $b_0 = b$, $x_0 = x$ and $y_0 = y$, and let $F$ be the MSF of $G$. Then $b_0 \notin F$, so there exists a path $\gamma_0$ in $G$ from $x_0$ to $y_0$ consisting entirely of bonds $e$ with $f(e) < f(b_0)$. Let $b_1 = \langle x_1, y_1\rangle$ be the first bond in $\gamma_0$ that is in $\partial T$, with $x_1 \in T$; such a bond necessarily exists since $x_0 \in T$ and $y_0 \notin T$. By Lemma 3.6, the unique locally $f$-minimax path from $x_0$ to $x_1$ lies in $T$ and also consists of bonds $e$ with $f(e) < f(b_0)$, so we may assume the section of $\gamma_0$ from $x_0$ to $x_1$ is precisely this $f$-minimax path; in particular this means this section of $\gamma_0$ lies in $T$. Similarly, there exists a path $\gamma_1$ in $G$ from $x_1$ to $y_1$ consisting entirely of bonds $e$ with $f(e) < f(b_1)$, and a first bond $b_2 = \langle x_2, y_2\rangle$ in $\gamma_1$ that is in $\partial T$, with the section of $\gamma_1$ from $x_1$ to $x_2$ lying in $T$; inductively, this process can be continued indefinitely. Let $\gamma$ be the path in $T$ that follows $\gamma_0$ from $x_0$ to $x_1$, then $\gamma_1$ from $x_1$ to $x_2$ and so on. Let $S := \gamma \cup \{b_1, b_2, \dots\}$. Now the bonds $b_i$ are distinct, since $f(b_0) > f(b_1) > \cdots$, so $S$ is infinite and connected, and all bonds $e$ in $S$ have $f(e) < f(b)$. Therefore, $f(b) \ge r_c(G)$.

If all sites in $T$ have finite degree, then since all $b_i$ are distinct, there must be infinitely many sites $x_i$ in $\gamma$. Therefore, $\gamma$ contains an infinite self-avoiding path in $T$ starting at $x$, consisting of bonds $e$ with $f(e) < f(b)$. □

Suppose $T$ is a two-ended tree. Then for each site $v$ in $T$ there is a unique trunk site, denoted $z(v)$, such that every infinite path in $T$ starting from $v$ first meets the trunk at $z(v)$.

PROOF OF THEOREM 2.5. Let $F$ denote the MSF of $X$. We may assume $X$ is ergodic, so $r_c(X) = r_c$ a.s.

(i) The absence of zero-ended trees follows from Proposition 3.2, applied to $F$, and Lemma 3.3. The absence of trees with three or more ends follows from Proposition 3.5 applied to $F$.

(ii) Suppose $T$ is a two-ended tree of the MSF with trunk $\gamma$. Let the bonds of $\gamma$ be labeled as $\{\dots, e_{-1}, e_0, e_1, \dots\}$ in the order in which they appear when following $\gamma$ in an arbitrarily chosen direction, and let

$$l^+(T) = \limsup_{n\to+\infty} f(e_n), \qquad l^-(T) = \limsup_{n\to-\infty} f(e_n).$$

By Proposition 3.1, for each rational $q$ there is a.s. no first or last bond $e_n$ with $f(e_n) > q$, so we must have $l^+(T) = l^-(T)$; thus we denote the common value by $l(T)$. If $f(e_n) > l(T)$ for some $n$, then there is an $f$-maximizing bond in $\gamma$. However, by Proposition 3.1 again, using the fact that all labels are distinct, there is a.s. no $f$-maximizing bond in $\gamma$ and no bond with $f(e_n) = l(T)$. Therefore, $f(e_n) < l(T)$ for all $n$; thus $\gamma$ is part of a infinite strict $l(T)$-cluster in $X$ and $l(T) \geq r_c$.

By strict simultaneous uniqueness, the infinite strict $l(T)$-cluster in $X$ is unique; we claim that $T$ contains all sites of this cluster. If $T = F$, there is nothing to prove, so suppose $T \neq F$ and let $b \in \partial T$. By Propositions 3.2 (applied to $F$) and 3.4, all sites of $T$ have finite degree in $T$. Therefore, by Lemma 3.7 there is an infinite self-avoiding path in $T$ consisting of bonds $e$ with $f(e) < f(b)$. An infinite self-avoiding path in $T$ must include $\{e_n: n \geq m\}$ or $\{e_n: n \leq m\}$ for some $m$, so it follows that $l(T) \leq f(b)$. Since $b \in \partial T$ is arbitrary, the infinite strict $l(T)$-cluster cannot cross $\partial T$ and the claim follows.

Let us show that $l(T) = r_c$. Suppose not and fix $l(T) > r > r_c$. Since $T$ contains all sites of $C_\infty^{\mathrm{st}}(X, l(T))$, $T$ also contains all sites of $C_\infty^{\mathrm{st}}(X, r)$. By Propositions 3.2, 3.4 and 3.5, $z^{-1}(v)$ is finite for each trunk site $v$, so there must exist $x, y \in C_\infty^{\mathrm{st}}(X, r)$ with $z(x) \neq z(y)$. By uniqueness, $C_\infty^{\mathrm{st}}(X, r)$ is connected, so for any such $x, y$ there is a path $\beta$ in $C_\infty^{\mathrm{st}}(X, r)$ from $x$ to $y$. There is also a self-avoiding path $\alpha$ in $T$ that goes from $x$ to $z(x)$, then via $\gamma$ to $z(y)$, then to $y$. By Lemma 3.6, $\alpha$ is $f$-minimax. Since all bonds in $\beta$ have label less than $r$, the same must be true for all bonds in $\alpha$, so $\alpha$ is contained in $C_\infty^{\mathrm{st}}(X, r)$. It follows that $\gamma \cap C_\infty^{\mathrm{st}}(X, r)$ is a nonempty connected subset of $\gamma$. By Proposition 3.1, this connected subset a.s. has no first or last bond, so must be all of $\gamma$. However, this would mean $l(T) \leq r$, contrary to our assumption. Hence no such $r$ exists, that is, $l(T) = r_c$, so $T$ contains all sites of an infinite strict $r_c$-cluster. Since there is at most one such cluster, the theorem follows. □

PROOF OF THEOREM 2.6. Suppose $T$ is a tree of the MSF $F$ of $G$, with $T \neq F$. Let $\partial^* T$ be the set of dual bonds $\{b^*: b \in \partial T\}$. Since $G$ is lattice-like, $\partial T$ is infinite and every component of $\partial^* T$ is infinite. By Lemma 3.7,

$f(b) \geq r_c(G)$ for all $b \in \partial T$, so every component of $\partial^* T$ is part of an infinite cluster of occupied dual bonds at level $r_c(G)$. □

**4. Invasion percolation and optimal paths to infinity.** The following is immediate from Proposition 2.1, Lemma 3.6 and Theorem 2.5.

PROPOSITION 4.1. *For $X$ as in Theorem 2.5 with strict simultaneous uniqueness holding a.s., with probability 1 for each site $v$ there exist either one or two self-avoiding locally $f$-minimax paths in $X$ from $v$ to infinity. If there is no percolation at the critical point in the corresponding percolation model, there is a unique such path for each $v$.*

For $X$ as in Theorem 2.5, let $\gamma_{x\infty}$ denote the locally $f$-minimax path in $X$ from $x$ to infinity when there is only one, and the union of both when there are two. In the lattice/uniform model, the $f$-minimax property makes $\gamma_{0\infty}$, or the tree containing 0, candidates to be called an incipient infinite cluster, but we have not investigated how their properties relate to those of other candidates that have been put forth in the literature; see [7], Section 7.4. In the Poisson/Euclidean model, the same holds with 0 replaced by the closest site to 0.

Invasion percolation, introduced in the mathematical literature in [5], is defined as follows in an ambiguity-free labeled graph. Let $I_0(x) := \{x\}$ for some site $x$. Given $I_n(x)$, let $\Delta_n(x)$ be the set of bonds not in $I_n(x)$ but with at least one endpoint in $I_n(x)$, let $E_{n+1}(x)$ be the $f$-minimizing bond in $\Delta_n(x)$, let $I_{n+1}(x) = I_n(x) \cup \{E_{n+1}(x)\}$ and let $I_\infty(x) := \bigcup_{n=1}^{\infty} I_n(x)$. (Here bonds are viewed as containing the sites that are their endpoints.) Note that this differs from Prim's algorithm, the invasion procedure described in Section 2. More precisely, we call $E_{n+1}(x)$ a *backfill bond* (with respect to $x$) if it has both endpoints in $I_n(x)$, and a *breakout bond* if it has only one endpoint in $I_n(x)$. The breakout bonds in $I_\infty(x)$ are precisely the bonds invaded in Prim's algorithm, that is, the bonds in $J_\infty(x)$.

THEOREM 4.2. *Let $X$ be as in Theorem 2.5, with strict simultaneous uniqueness holding a.s. and let $F$ denote the MSF of $X$. Then with probability one*:

(i) *An invaded bond is a breakout bond if and only if it is a bond of $F$.*
(ii) *For every $x$, $I_\infty(x)$ contains $\gamma_{x\infty}$.*
(iii) *The symmetric difference $I_\infty(x) \vartriangle I_\infty(y)$ is finite if and only if $x$ and $y$ are in the same tree of $F$.*

For the square lattice, (iii) reproduces Theorem A.1 of [5], where, by Example 2.7, $F$ consists of a single tree. Nonrigorous arguments in [13] suggest that for the integer lattice in dimension greater than 8 there is a positive probability for $x \neq y$ that $I_\infty(x)$ and $I_\infty(y)$ are disjoint, much as the paths of independent random walks started at $x$ and $y$ can be disjoint. By (iii), this would imply that the MSF is not connected in high dimensions.

For $A \subset G$, let $\overline{A} = \{b \in G$: $b$ has an endpoint in $A\}$.

PROOF OF THEOREM 4.2. We may assume that $X$ is ergodic and, by Propositions 3.2 and 3.4, that all sites have finite degree in $F$.

If $b = \langle u, v \rangle$ is a breakout bond, it follows from the criterion (2.1) that $b$ is a bond of $F$. Conversely, if $b$ is a backfill bond, then there is a path of breakout bonds from $u$ to $v$. Since this path lies in $F$ and $F$ is acyclic, $b$ is not a bond of $F$. This proves (i).

If the tree $T$ of $F$ that contains $x$ is one-ended and $b$ is a bond of $\gamma_{x\infty}$, then only finitely many bonds of $F$ can be reached from $x$ via paths in $F$ without passing through $b$. Hence by (i) only finitely many breakout bonds can be invaded without invading $b$. Therefore, $b \in I_\infty(x)$ and (ii) follows for one-ended $T$.

By Theorem 2.5 and Proposition 3.4, to prove (ii) in general it remains to consider two-ended $T$, with all sites of finite degree in $T$. The trunk consists of two disjoint paths, say $\alpha$ and $\beta$, from $z(x)$ to infinity. Let $a_0 = z(x)$ and let $a_1, a_2, \dots$ be the bonds of $\alpha$ and $b_1, b_2, \dots$ the bonds of $\beta$, each listed starting from $z(x)$. From (i), $I_\infty(x)$ must contain at least one of these paths, say $\beta$, and $I_n(x) \cap (\alpha \cup \beta)$ is a single interval of $\alpha \cup \beta$ for each $n$. Suppose $a_0, \dots, a_{k-1}$ are in $I_\infty(x)$, with $a_{k-1}$ invaded at some time $\tau \geq 0$ and $I_\tau(x) \cap (\alpha \cup \beta) = \{a_{k-1}, \dots, a_1, a_0, b_1, \dots, b_j\}$ for some $j$. By Theorem 2.5, $f(a_k) < r_c$, so there is a.s. no infinite $f(a_k)$-cluster; in particular, $f(b_m) > f(a_k)$ for some $m > j$. However, this means $a_k$ must be invaded before $b_m$. Thus by induction, $a_k \cap I_\infty(x)$ for all $k$. Hence $I_\infty(x)$ contains $\alpha \cup \beta$ and (ii) follows when $T$ is a two-ended tree as well.

Turning to (iii), if $x$ and $y$ are in different trees of $F$, then it follows from (i) that $I_\infty(x) \cap I_\infty(y) = \phi$. Thus suppose $x$ and $y$ are in the same tree $T$ of $F$. We consider two cases.

*Case* 1: *$T$ is one-ended*. There exists a site $s_{xy}$, where $\gamma_{x\infty}$ first meets $\gamma_{y\infty}$, starting from $x$ or $y$. If $s_{xy}$ is neither $x$ nor $y$, then $T \setminus \{s_{xy}\}$ includes finite components $T_x$ and $T_y$ containing $x$ and $y$, respectively. Let $T_x$ (or $T_y$) be $\phi$ if $x$ (or $y$) is $s_{xy}$. Let $\beta_n(x)$ be the $n$th bond invaded, starting from $x$, which is not in $\overline{T}_x \cup \overline{T}_y$. Then $\beta_n(x)$ is the $f$-minimizing bond in

$$\Big\{b \in G: b \notin \overline{T}_x \cup \overline{T}_y \cup \{\beta_1(x),, \dots, \beta_{n-1}(x)\},$$
$$b \text{ has an endpoint in } \{s_{xy}, \beta_1(x), \dots, \beta_{n-1}(x)\}\Big\}.$$

Therefore, if $\beta_i(x) = \beta_i(y)$ for all $1 \leq i \leq n-1$, then $\beta_n(x) = \beta_n(y)$. It follows that $\beta_n(x) = \beta_n(y)$ for all $n$ and

$$I_\infty(x) \vartriangle I_\infty(y) \subset \overline{T}_x \cup \overline{T}_y. \tag{4.1}$$

Note that, although $T_x$ is finite, $\partial T_x$ and thus $\overline{T}_x$ may be infinite if $T_x$ has sites of infinite degree.

Fix $b = \langle u, v\rangle \in \partial T_x$, with $u \in T_x$. Suppose $b \in I_\infty(x)$; we claim that $b \in I_\infty(y)$. By (i), since $b \notin F$, $b$ must be a backfill bond. Therefore, there is a path $\lambda$ from $u$ to $v$ consisting entirely of breakout bonds (with respect to $x$); by (i), $\lambda$ is a path in $T$. It follows from the creek-crossing criterion (2.2) that $b$ is the longest bond in $\lambda \cup \{b\}$. Further, since $v \notin T_x$, $\lambda$ must pass through $s_{xy}$, and therefore $b$ is invaded after $s_{xy}$. Letting $\beta_0(x) := \{s_{xy}\}$, we let $n \geq 0$ be such that $b$ is invaded (starting from $x$) after $\beta_n(x)$ is invaded but before $\beta_{n+1}(x)$ is invaded. Then

$$f(\beta_{n+1}(x)) > f(b) > f(e) \quad \text{for all bonds } e \text{ in } \lambda. \tag{4.2}$$

Now starting from $y$, $s_{xy}$ is invaded before $\beta_{n+1}(y) = \beta_{n+1}(x)$ is invaded. Since $s_{xy}$ is a site of $\lambda$, it follows from (4.2) that starting from $y$, all of $\lambda \cup \{b\}$ will be invaded before $\beta_{n+1}(y)$, and our claim that $b \in I_\infty(y)$ follows.

Conversely, for the same $b \in \partial T_x$, if $b \in I_\infty(y)$, then virtually the same proof shows $b \in I_\infty(x)$. Therefore, we have

$$\big(I_\infty(x) \triangle I_\infty(y)\big) \cap \big(\partial T_x \cup \partial T_y\big) = \phi.$$

With (4.1) this establishes (iii) for one-ended trees.

*Case* 2: *$T$ is a two-ended tree with trunk $\gamma$.* Let $A_v$ denote the (finite) component of a site $v$ in $T \setminus \{z(v)\}$, together will all bonds of $X$ which have both endpoints in this component (so $A_v = \phi$ if $v \in \gamma$). We claim that

$$C_\infty^{\mathrm{st}}(X, r_c) \subset I_\infty(v) \subset C_\infty^{\mathrm{st}}(X, r_c) \cup A_v. \tag{4.3}$$

Let $\sigma$ be the least $n$ such that $I_n(v)$ contains $z(v)$; such an $n$ always exists by (i). Since, by Theorem 2.5 $z(v) \in C_\infty^{\mathrm{st}}(X, r_c)$, at all future times $i \geq \sigma$, $\Delta_i(v)$ includes a bond of $C_\infty^{\mathrm{st}}(X, r_c)$. Thus every bond $b$ outside $A_v$ invaded after time $\sigma$ has $f(b) < r_c$, so in fact all such bonds are in $C_\infty^{\mathrm{st}}(X, r_c)$, and the second inclusion in (4.3) follows. For the first inclusion, suppose there exists a bond $b \in C_\infty^{\mathrm{st}}(X, r_c) \setminus I_\infty(v)$. Since by uniqueness $C_\infty^{\mathrm{st}}(X, r_c)$ is connected, we can find such a $b$ with at least one endpoint in $I_\infty(v)$, so $b \in \Delta_n(v)$ for some $n$. However, then $f(e) < f(b)$ for all $e \in I_\infty(v) \setminus I_n(v)$. The graph consisting of such $e$ has only finitely many connected components so it includes an infinite $f(b)$-cluster. Since $f(b) < r_c$, there is no such infinite cluster a.s. It follows that $C_\infty^{\mathrm{st}}(X, r_c) \setminus I_\infty(v) = \phi$ a.s. This proves (4.3), which in turn shows $I_\infty(x) \triangle I_\infty(y)$ is finite. □

COROLLARY 4.3. *In the lattice/uniform model, for every $r > r_c$ and every site $x$, at most finitely many bonds of $\gamma_{x\infty}$ are outside the infinite r-cluster of the corresponding percolation model, a.s.*

PROOF. The analogous fact for $I_\infty(x)$ was proved in [5], so the result follows from Theorem 4.1(ii). In [5], $r_c$ is replaced by a percolation threshhold possibly different from $r_c$, but it was proved in [8] that the two thresholds are the same. □

**5. Stability of the MSF under local changes.** We consider next the extent to which local changes in a graph can produce global changes in the

MSF. Specifically, for an ambiguity-free labeled graph $G$ in $\mathbb{R}^d$, let us define the function $\delta_G$ on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\delta_G(x, y) := \begin{cases} 1, & \text{if } x \text{ and } y \text{ are sites in the same tree of the MSF of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Local changes in $G$ may cause bonds to be added to or deleted from the MSF, and as a result the function $\delta_G$ may change.

To begin, we define some modifications of the graph $G = (V, \mathscr{B}, f)$. For $\Gamma \subset \mathbb{R}^d$ let

$$D_\Gamma(G) := \{x \in V \cap \Gamma : \langle x, y \rangle \in \mathscr{B} \text{ for some } y \in \Gamma^c\}.$$

Define the *restriction* $G|_\Gamma$ of $G$ to $\Gamma$ by

$$\mathscr{B}|_\Gamma := = \{\langle x, y \rangle \in \mathscr{B} : x, y \in \Gamma\},$$
$$G|_\Gamma := (V \cap \Gamma, \mathscr{B}|_\Gamma, f).$$

Here, in a slight abuse of notation, $f$ is actually the restriction of $f$ to $\mathscr{B}|_\Gamma$. To add in bonds crossing $\partial \Gamma$, let $G_\Gamma^+$ denote the restriction of $G$ to $\Gamma \cup D_{\Gamma^c}(G)$. We say two labeled graphs $G_1$ and $G_2$ *agree outside* $\Gamma$ if $G_1|_{\Gamma^c}^+ = G_2|_{\Gamma^c}^+$.

THEOREM 5.1. (i) *Suppose $G_1$ and $G_2$ are ambiguity-free locally finite labeled graphs in $\mathbb{R}^d$ that agree outside $\Lambda_t$ for some $t > 0$, and let $F_i$ be the MSF of $G_i$. Then the symmetric difference $F_1 \triangle F_2$ is a finite set of bonds.*

(ii) *Let $X$ be as in Theorem 2.5, with strict simultaneous uniqueness holding a.s. There exists a set $A$ of labeled graphs such that* (a) $P(X \in A) = 1$ *and* (b) *if $t > 0$ and $G_i = (V_i, \mathscr{B}_i, f_i)$ are labeled graphs in $A$ that agree outside $\Lambda_t$, then there exists $s > t$ such that $\delta_{G_1}(x, y) = \delta_{G_2}(x, y)$ for all $x, y \in \Lambda_s^c$.*

Roughly speaking, (ii) says that changes in a finite box can only shift a finite number of sites to different trees of the MSF. Finite changes cannot, for example, split a two-ended tree into two one-ended trees or glue two one-ended trees into a two-ended tree, except possibly by creating a labeled graph of a type that a.s. does not occur, that is, one not in A.

PROOF OF THEOREM 5.1. (i) For $u, v$ distinct sites in $D_{\Lambda_t}(G_i)$, define

$$r_{uv} := \inf\{r > 0 : u \text{ is connected to } v \text{ at level } r \text{ in } G_i|_{\Lambda_t^c}^+\}.$$

Note $D_{\Lambda_t}(G_i)$ and $G_i|_{\Lambda_t^c}^+$ do not depend on $i$. Suppose $b = \langle x, y \rangle$ is a bond of $G_i|_{\Lambda_t^c}^+$ that is in $F_1$ but not $F_2$. From the creek-crossing criterion (2.2), there is a path in $G_2$ from $x$ to $y$ consisting of bonds $e$ with $f(e) < f(b)$, but no such path exists in $G_1$ and hence none exists in $G_i|_{\Lambda_t^c}^+$. Therefore, this path in $G_2$ must have a bond in $G_2|_{\Lambda_t}$, so for some pair $u, v$ of distinct sites in $D_{\Lambda_t}(G_i)$, there exist disjoint paths in $G_i|_{\Lambda_t^c}^+$ from $x$ to $u$ and from $y$ to $v$ at some level less than $f(b)$. Thus $r_{uv} \le f(b)$. However, there can be no path from $u$ to $v$ at a level less than $f(b)$ in $G_i|_{\Lambda_t^c}^+$, for otherwise we would also have such a path from $x$ to $y$. Thus $r_{uv} = f(b)$; since labels are distinct there is at most one such $b$ for each pair $u, v$. Since $G_1$ and $G_2$ are locally finite, (i) follows.

(ii) We may assume $X$ is ergodic, so by Theorem 2.5 there exists an $n \in \{0, 1\}$ such that the MSF of $X$ has exactly $n$ two-ended trees, a.s. Let $A$ be the set of graphs for which the MSF has exactly $n$ two-ended trees and satisfies the description in Theorem 2.5(ii). If $n = 0$, then for each bond $b$ in $F_1 \setminus F_2$, removing $b$ from $F_1$ splits the tree of $F_1$ containing $b$ into a finite and an infinite piece, and similarly for $b \in F_2 \setminus F_1$. If $x$ and $y$ are sites of $X$ and neither $x$ nor $y$ is in one of these finite pieces, then $\delta_{G_1}(x, y) = \delta_{G_2}(x, y)$. Thus for $n = 0$, (b) follows from (i). If $n = 1$, it follows from (i) that to prove (b) we need only eliminate the possibility that for some site $z$ in a one-ended tree $T$ of $F_1$, the path $\gamma_{z\infty}$ in $T$ is contained in the trunk of the two-ended tree of $F_2$. However, this would imply that all bonds $e$ in $\gamma_{z\infty}$ have $f(e) \le r_c$, so $T$ would meet the infinite strict $r_c$-cluster, so in $G_1$ the infinite strict $r_c$-cluster would not be contained in the two-ended tree, meaning $G_1 \notin A$. □

Without strict simultaneous uniqueness, Theorem 5.1(ii) is false. The MSF of $X$ may then include multiple two-ended trees. Through changes in a finite box, it may be possible to in effect cut in half the trunks of two such trees and then glue the four resulting rays back together in a different pairing, as the following example, similar to Example 1.9 of [2], shows.

EXAMPLE 5.2. Consider a random labeled graph $X$ in $\mathbb{R}^2$ with site set $\mathbb{Z}^2$. Vertical bonds are nearest neighbor; horizontal bonds are long range. Let us call the subgraph in the vertical line at $m$, *column* $m$. Let $\{U_m: m \in \mathbb{Z}\}$ be iid uniform $[0, 1]$ random variables and let $\{T_j: j \in \mathbb{Z}\}$ be the arrival times of a stationary renewal process in $\mathbb{Z}$, with interarrival times not identically 1. Let us refer to $[T_j, T_{j+1}) \cap \mathbb{Z}$ as *block* $j$. Conditionally on $\{U_m: m \in \mathbb{Z}\}$ and $\{T_j: j \in \mathbb{Z}\}$, we label bonds as follows: For $m$ in block $j$, vertical bonds $\langle (m, k), (m, k + 1)\rangle$ in column $m$ get labels $W_{mk}$ iid uniform in $[0, U_j)$; for $m$ in block $i$ and $n$ in block $j$, each horizontal bond $\langle (m, k), (n, k)\rangle$ independently gets a label uniform in $(\max(U_i, U_j), 1]$. The discussion in Example 1.9 of [2] establishes that $X$ does not satisfy simultaneous uniqueness, strict or not, though for fixed $r > 0$ there is a.s. a unique infinite cluster at level $r$. Uniqueness fails precisely at the levels $U_j$. It is easily verified that the MSF of $X$ consists precisely of all vertical bonds, so is composed of infinitely many two-ended trees. We now introduce local modifications to $X$. We can construct a *crossover* in a square $[m, m + 1] \times [k, k + 1]$ by deleting the vertical bonds $\langle (m, k), (m, k + 1)\rangle$, $(m + 1, k + 1)\rangle$ and adding diagonal bonds $\langle (m, k), (m + 1, k + 1)\rangle$ with label $W_{mk}$ and $\langle (m, k + 1), (m + 1, k)\rangle$ with label $W_{m+1,k}$. Let $\{Z_{mk}: m, k \in \mathbb{Z}\}$ be idd taking values 0 and 1 with probability 1/2 each. Starting with the graph $X$, in each square $[m, m + 1] \times [k, k + 1]$ we construct a crossover if $m$ and $m + 1$ are in the same block, and $Z_{m-1,k} = 0$, $Z_{mk} = 1$ and $Z_{m+1,k} = 0$. We denote the resulting graph $Y$. Thus no two horizontally adjacent cubes both have crossovers. The subgraph of $Y$ consisting of precisely the vertical and diagonal bonds is made up of infinitely many distinct infinite lines; let us call these lines *strands*. Note that a horizontal bond emanating from a given strand is longer than any of the bonds of that strand.

Using the creek-crossing criterion (2.2) it follows easily that the MSF of Y a.s. consists precisely of the infinite collection of strands. Comparing $Y$ itself to $Y$ with a single added crossover, we see that Theorem 5.1 (ii)(b) does not hold. □

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