xxx Chap4-9.tex

1 Notes to JF: "Poincare method" section of new Chapter

What exists so far, if I haven't forgotten anything, is

Chapter 4 section 4.3 (10/11/94 version)

Use in 'low-temperature Metropolis" (Chap 12 sec 6.1) (12/22/94 version)

Use in "subperiodic trees" (saw.tex section 1.1)

The last two were intended as examples to be studied later in the book, in detail by various methods; but it's OK with me if you prefer to put them in the new chapter.

I suggest that in writing the new chapter section you start with your own thoughts, rather than by editing the existing Chapter 4 sec 4.3. But here are two thoughts on section 4.3 that occurred to me while teaching.

1. Example 35 shows that Corollary 34 does not hold for vertex-transitive graphs. But a small modification of the proof of Cor 34 (bound flow thru edge by flow thru vertex) shows than on a vertex-transitive graph

$$\tau_2 \leq r\ ED^2; \quad r = \text{degree}.$$

2. Heuristically one expects that any "reasonable" graph has $\tau_2 = O(\Delta^2)$, for $\Delta =$ diameter. The argument for Cor 34 seems to imply one can't hope to do better than this using the (naive, i.e. without choosing weights in Cauchy-S) Poincare method. The method is perhaps most relevant (xxx check with examples!!!!) where one believes that in fact $\tau_2 = \Theta(\Delta^2)$ and one seeks to prove a bound as close to this as one can. The (new) example below illustrates this point.

3. We should fix a name for this method. Persi (and you?) call it *Poincare*; I've been calling it *distinguished paths*, and Alistair Sinclair calls it *multicommodity flow*. I don't really object to *Poincare*, except for preferring descriptive names in principle, so I would be happiest calling it *the (adjective) flows method* if we could find a suitable adjective: "locally bounded" or "diffuse" make sense but are a bit bland.

1.1 A chain on cladograms

xxx JF: I just made this up. Look complicated, but actually worked quite well in class.

An n-leaf cladogram is a tree with n leaves labeled $\{1, 2, \ldots, n\}$, an unlabeled root of degree 1 and internal branchpoints of degree 3, where we do not distinguish between left and right branches. See figure 1. We define a Markov chain on the set I_n of such trees as follows.

> Pick a uniform random leaf i and remove leaf i (in more detail: remove the edge from i to its branchpoint b, and remove b to collapse two edges into one). Then pick a uniform random edge e of the remaining tree, create a new branchpoint b' in the interior of e, and reattach leaf i via a new edge to b'.

Figure 1 illustrates a typical transition: from the first tree, leaf 3 is removed and reattached to the edge above the branchpoint of $\{11, 8\}$.

This chain has symmetric transition matrix, so is reversible with uniform stationary distribution π. It is known (see Remarks later) that $\tau_2 = \Omega(n^2)$ and there is reason to conjecture that in fact $\tau_2 = \Theta(n^2)$. Since the diameter of the transition graph is $O(n)$ (this can be deduced from our construction below), Corollary 34 (yyy 10/11/94 version of Chapter 4) would imply $\tau_2 = O(n^2)$ if the chain were edge-transitive. In fact the chain is not even vertex-transitive (essentially, because different trees have different "shapes"), but (xxx tie up with comment **2** above: this is the type of example where we expect Poincare to do well) we shall use the distinguished paths method to prove

$$\tau_2 = O(n^{7/2}). \tag{1}$$

Our argument has "slack" at two places (which we will point out), and it would be interesting to seek to improve the analysis.

We first need some elementary observations concerning I_n. Each tree $\mathbf{x} \in I_n$ has exactly $2n - 1$ edges. A new tree in I_n can be obtained from an old tree in I_{n-1} by attaching leaf n to any edge of the old tree, and different choices of (old tree, edge) lead to different new trees. So $|I_n| = |I_{n-1}| \times (2n - 3)$ and inductively

$$|I_n| = (2n - 3)(2n - 5)(2n - 7) \cdot \cdot 3 \cdot 1 := c_n \text{ say.}$$

This argument is tantamount to the observation that a uniform random tree in I_n can be constructed in the natural sequential way, as follows.

Lemma 1 *Start with the unique tree in I_2, and for $k = 2, 3, \ldots, n-1$ attach leaf $k+1$ to the middle of a uniform random edge of the previous tree. The final random cladogram is distributed uniformly on I_n. This remains true if we add leaves in some arbitrary order $\pi(1), \ldots, \pi(n)$.*

We now specify the distinguished paths. Fix an initial tree $\mathbf{x}^0 \in I_n$. Create a uniform random permutation $\pi(1), \pi(2), \ldots, \pi(n)$ of leaves. Remove leaf $\pi(1)$ and reattach to a random edge of the remaining tree. Then remove $\pi(2)$ and reattach to the edge incident to $\pi(1)$. This creates a subtree $\mathcal{S}_2$ which is a cladogram on the two leaves $\{\pi(1), \pi(2)\}$. Inductively, for $k = 2, 3, \ldots, n-1$ remove leaf $\pi(k+1)$ and reattach to a uniform random edge of $\mathcal{S}_k$ to create the subtree $\mathcal{S}_{k+1}$ on leaves $\pi(1), \ldots, \pi(k+1)$. We end with a random tree $\mathcal{S}_n$. Note that throughout the construction we want each $\mathcal{S}_k$ to be a cladogram, so $\mathcal{S}_k$ is the spanning subtree on leaves $\pi(1), \ldots, \pi(k)$ of the current tree, together with the edge upwards from the top branchpoint of that subtree. Write $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n = \mathcal{S}_n$ for the sequence of n-leaf cladograms in this construction.

It follows from the lemma that $\mathcal{S}_n$ is uniform on I_n. Thus by taking the initial tree $\mathbf{x}^0$ to be uniform random, we have defined a flow from π to π. Write $f(\mathbf{e})$ for the flow through a typical edge $\mathbf{e}$ of the transition graph in this scheme. Since the stationary flow through $\mathbf{e}$ equals $\frac{1}{c_n} \frac{1}{n} \frac{1}{2n-3}$ and each path has length n, by (xxx the fundamental inequality: Theorem 32)

$$\tau_2 \leq n \sup_{\mathbf{e}} n(2n-3)c_n \ f(\mathbf{e}). \tag{2}$$

To bound $f(\mathbf{e})$ we first write

$$f(\mathbf{e}) = \sum_{k=0}^{n-1} f_k(\mathbf{e})$$

where $f_k(\mathbf{e})$ is the contribution to flow from cases where traversal of edge $\mathbf{e}$ is caused by moving of leaf $\pi(k+1)$. Fix $\mathbf{e}$; we illustrate with the edge from the first tree to the second tree in figure 1. First observe that $f_k(\mathbf{e})$ is non-zero only for certain values of k. In figure 1, $\mathbf{e}$ could be traversed at the first step ($k = 0$), or at the third step ($k = 2$) if the subtree $\mathcal{S}_2$ were the subtree on $\{11, 8\}$. But this $\mathbf{e}$ cannot be traversed with $k = 1$, because then 3 must be reattached adjacent to some leaf $\pi(1)$. This $\mathbf{e}$ could also be traversed with $k = 4$ or 5 but for no further values of k. Fix a value of k for which the traversal (from some unspecified initial $\mathbf{x}^0$) is possible. At the end of the argument we will sum over all k without seeking to exploit the "possibility" constraint – this is the first point where there is slack in the argument. We illustrate by considering the edge $\mathbf{e} = (\mathbf{x}^k, \mathbf{x}^{k+1}$ in figure 1 and taking $k = 4$. So $\mathcal{S}_k$ is the subtree shown on the right of figure 2. The initial tree $\mathbf{x}^0$ must have the leaves (excluding $\{6, 1, 11, 8\}$) in the same relative positions as in $\mathbf{x}^k$. That is, $\mathbf{x}^0$ must be like the tree on the left of figure 2 with leaves $\{6, 1, 11, 8\}$ added in arbitrary positions. The chance that a uniform random tree in I_n is of this form equals $1/c_{n-k}$; this is the first term in inequality (3) below which summarizes the argument. Given $\mathbf{x}^0$ is of this form, it is next necessary (in order that $\mathbf{x}^k$ be as illustrated) that $\{\pi(1), \pi(2), \pi(3), \pi(4)\}$ be the (unordered) set $\{6, 1, 11, 8\}$, and this has chance $1/\binom{n}{k}$. Next, when $\pi(1)$ is reattached to form $\mathbf{x}^1$ it must be reattached to a "permissible" edge, i.e. an edge which can lead eventually to $\mathbf{x}^k$. At first sight it might appear that there is only one such edge, but figure 3 illustrates a "bad" initial configuration in which there are four possible such edges. We shall just bound the probability of reattachment to a permissible edge by 1; this is the second point of slack in the analysis. Next, in building the subtree $\mathcal{S}_k$ we must build the correct subtree (illustrated on the right of figure 2), and this have chance $1/c_k$. Multiplying the terms so far gives an upper bound on the chance that a particular tree $\mathbf{x}^k$ arises after step k. In order than the next step traverses the edge $\mathbf{e} = (\mathbf{x}^k, \mathbf{x}^{k+1})$ it is necessary that $\pi(k+1)$ is a particular leaf (leaf 3, in figure 1) and that it be reattached to the correct edge of $\mathcal{S}_k$; these chances are $1/(n-k)$ and $1/(2k-1)$. In the case $k = 0$ the second chance is $1/(2n-3)$; we leave this variation implicit in the argument below.

Collecting these estimates

$$
\begin{array}{rcll}
f_k(\mathbf{e}) \leq & 1/c_{n-k} & \text{(possible initial } \mathbf{x}^0) & \\
\times & 1/\binom{n}{k} & \text{(choose } \{\pi(1), \ldots, \pi(k)\}) & \\
\times & 1 & (\pi(1) \text{ reattached to feasible edge}) & \\
\times & 1/c_k & (\pi(1), \ldots, \pi(k) \text{ build correct subtree to create } \mathbf{x}^k) & \\
\times & 1/(n-k) & (\pi(k+1) \text{ is correct leaf}) & \\
\times & 1/(2k-1) & (\pi(k+1) \text{ reattached to correct edge}). & (3)
\end{array}
$$

Rearranging,

$$
n(2n-3)c_n f(\mathbf{e}) \leq \frac{n(2n-3)}{(n-k)(2k-1)} \ \frac{c_n}{c_{n-k} c_k \binom{n}{k}}.
$$

We can rewrite c_n as $\frac{(2n)!}{(2n-1)2^n n!}$, and then a routine argument from Stirling's formula shows the right side is asymptotic to

$$2\pi^{1/2}(k+1)^{1/2}(n-k)^{1/2}n^{1/2} \quad \text{as } k \to \infty,\ n-k \to \infty$$

and has the same order of magnitude bound if $n \to \infty$ with k or $n-k$ bounded. So

$$\sup_{\mathbf{e}} n(3n-1)c_n f_k(\mathbf{e}) = O(n^{3/2}) \text{ uniformly in } k = 0, 1, \ldots, n-1.$$

Since $f(\mathbf{e}) = \sum_{k=0}^{m-1} f_k(\mathbf{e})$, summing over k and applying (2) gives the desired bound (1).

Remarks. (a) For a typical edge $\mathbf{e}$ (rather that the worst-case edge) one can remove the slack in the argument above to show that the right side of (2) is $O(n^2)$. So it is plausible that more elaborate analysis might be able to establish $\tau_2 = O(n^2)$ using different bounds for the exceptional edges.

(b) Using coupling, Aldous [**?**] shows $\tau_1 = O(n^3)$. The coupling construction uses similar ideas but its analysis is more complicated. By using a suitable test function in the extremal characterization, it is not hard to show [**?**] $\tau_2 = \Omega(n^2)$. The conjecture $\tau_2 = \Theta(n^2)$ parallels a conjecture that we can apply weak convergence methodology here (xxx point to discussion to be written in Chapter 13).

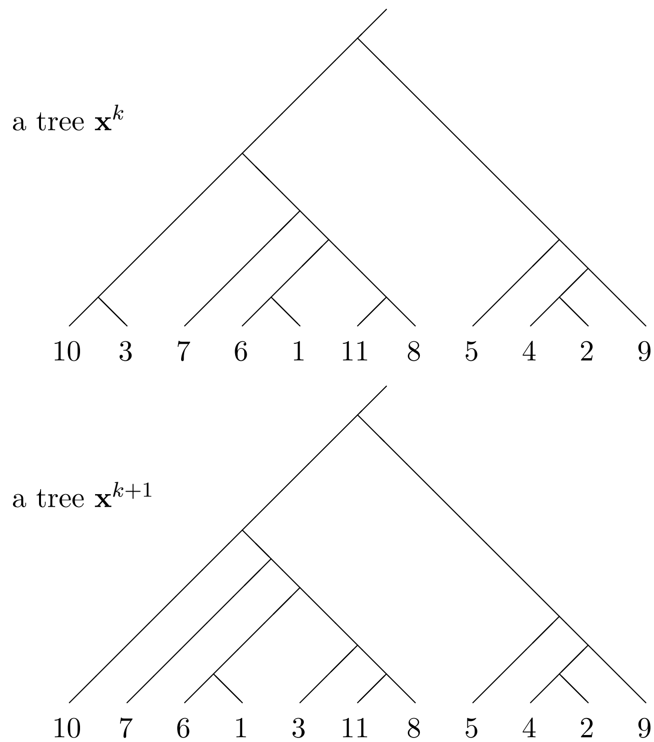

Figure 1. An edge $\mathbf{e} = (\mathbf{x}^k, \mathbf{x}^{k+1})$ in the transition graph, for $n = 11$.

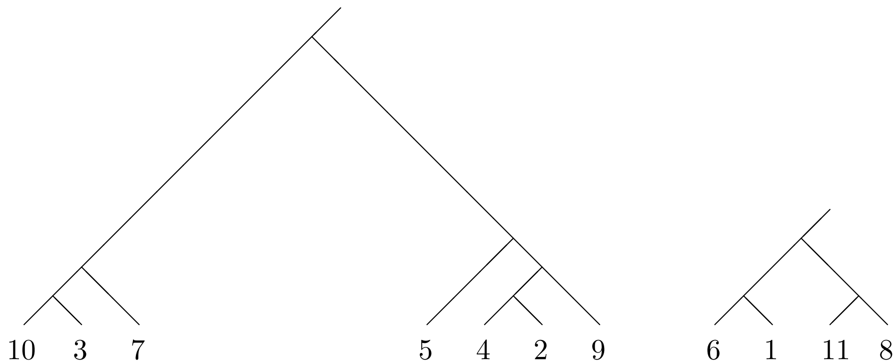

Figure 2. In order that the transition at step $k = 5$ be the transition in figure 1, the subtree $\mathcal{S}_4$ must be the right tree and the initial tree $\mathbf{x}^0$ must be compatible with the left tree.

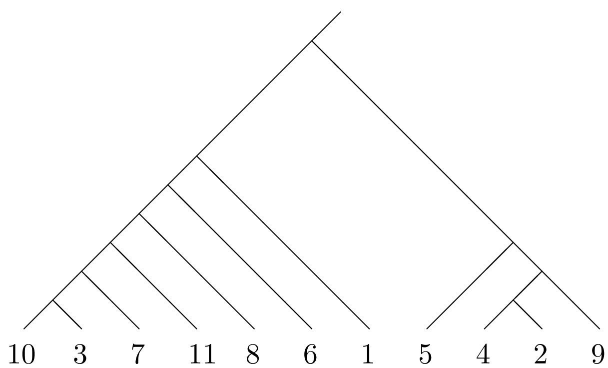

Figure 3. A bad possible initial tree $\mathbf{x}^0$.