Durrett Chapter 3 Exercises

2.2: Convergence of Maxima.
2.6 The Levy metric.
2.13 Converging together lemma.

1. Show that, given $X$, we can find random variables $X_n$ such that each $X_n$ takes only finitely many values but $X_n \overset{d}{\to} X$.

2. Suppose $X$ has a continuous density $f$. Let $\delta_n \downarrow 0$. Suppose $X_n$ takes only values which are integer multiples of $\delta_n$. And suppose that

$$ k_n \delta_n \to x \implies \frac{P(X_n = k_n \delta_n)}{\delta_n} \to f(x). $$

Show that $X_n \overset{d}{\to} X$.

3. For real $x > 0$ let $\text{dec}(x)$ be the decimal part of $x$, and let $\text{sig}(x)$ be the first significant digit of $x$. Thus

$$ \text{dec}(83.472) = 0.472, \quad \text{sig}(83.472) = 8. $$

Let $X_n > 1$ be r.v.’s such that $\text{dec}(\log_{10} X_n) \overset{d}{\to} U$, where $U$ is uniform on $(0, 1)$. Prove that $\text{sig}(X_n) \overset{d}{\to} D$ where $D$ has a certain distribution (which you should specify) on $\{1, 2, \ldots, 9\}$.

Remark. **Benford’s law** asserts that, in real-world data where entries differ by several orders of magnitude, the observed distribution of first significant digit is approximately $D$. 


205B Homework #2, due Tuesday February 7

Durrett, Chapter 3, section 3, Exercises 1, 3, 5, 6, 7, 13, 15, 17, 18, 19, 22, 23.

[Note most of these have short proofs – nothing complicated is needed.]
4. Let \((X_j, j \geq 1)\) be independent and take values \((-2j^2, -j, 0, j, 2j^2)\) with

\[
P(X_j = -2j^2) = P(X_j = 2j^2) = \frac{1}{12j^2}
\]

\[
P(X_j = -j) = P(X_j = j) = \frac{1}{12}.
\]

Show that the Lindeberg condition is not satisfied. Show that nonetheless there exist constants \(b_n\) (which you should specify) such that

\[
\frac{1}{b_n} \sum_{j=1}^{n} X_j \overset{d}{\rightarrow} \text{Normal}(0, 1).
\]
Durrett Chapter 3 Exercises 9.5, 3.27, 3.28.

5. Let $U$ and $V$ be independent, uniform $[0,1]$. Define $Y_n = nU - \lfloor nU \rfloor$. Show that $$(U,Y_n) \overset{d}{\to} (U,V).$$

6. Let $Q = Q(x,A)$ be a kernel from $\mathbb{R}$ to $\mathbb{R}$. For each $1 \leq n \leq \infty$ let $(X_n,Y_n)$ be such that $Q$ is the conditional probability kernel for $Y_n$ given $X_n$. Suppose $X_n \overset{d}{\to} X_\infty$. Give an example to show that $$\quad (X_n,Y_n) \overset{d}{\to} (X_\infty,Y_\infty) \quad (\ast)$$ is not necessarily true. Show that either of the following two extra assumptions is sufficient to imply $(\ast)$.
   (a) For each $1 \leq n \leq \infty$ there is a density $f_n$ for $X_n$, and $\int |f_n(x) - f_\infty(x)|dx \to 0$.
   (b) $Q(x_m,\cdot) \to Q(x,\cdot)$ weakly whenever $x_m \to x$.

7. Give an example of a distribution $(Y_1,Y_2)$ such that $Y_1$ and $Y_2$ are Normal($0,1$) but the joint distribution is not bivariate Gaussian.

8. Let $(X_1,X_2)$ be bivariate Gaussian with $\mathbb{E}X_1^2 = \mathbb{E}X_2^2 = 1$ and $\mathbb{E}X_1X_2 = \rho$. Calculate $\mathbb{P}(X_1 > 0, X_2 > 0)$.
   [Hint: for independent standard Normals $(Z_1,Z_2)$ the density is rotationally invariant]

9. For integers $x \geq 0, k \geq 1$ write $[x]_k = x(x-1)\ldots(x-k+1)$. Prove that, for positive integer-valued r.v.'s $X(n)$,
   
   if $\mathbb{E}[X(n)] \to \lambda^k$ as $n \to \infty$, for each $k \geq 1$
   
   then $X(n) \overset{d}{\to} \text{Poisson}(\lambda)$. 


Durrett Chapter 6 Exercises 2.7, 2.8, 2.9, 3.10, 3.11, 3.12

10. Give an example to show that, if $X_n$ is a Markov chain and $f$ a function defined on its state space, then $f(X_n)$ need not be a Markov chain.

11. Show that for $K < \infty$ and $a > 0$ there is a constant $C(K, a) < \infty$ such that: in every finite irreducible chain

$$\max_{i,j} E_i T_j \leq C(K, a)$$

where $K =$ number of states and $a = \min\{p(i,j) : p(i,j) > 0\}$.
Let $X_n$ be an irreducible chain with transition matrix $P$. Let $Y_n$ be the 
\textit{jump chain} $Y_n = X(T_n)$ where $T_0 = 0$ and 
\[ T_{n+1} = \min\{m > T_n : X_m \neq X(T_n)\}. \]

(a) Show that $Y_n$ is Markov, and write its transition matrix $Q$ in terms of 
$P$.
(b) Show that $(Y_n)$ is recurrent iff $(X_n)$ is recurrent.
(c) Assuming recurrence, find the relation between the $P$-invariant measure 
and the $Q$-invariant measure.
(d) Deduce that, on an infinite state space, it is possible for $(Y_n)$ to be 
positive-recurrent while $(X_n)$ is not.

Let $X_n$ be an finite irreducible chain with transition matrix $P$. Fix a 
subset $A$ of $S$. Define a transition matrix $Q$ on $A$ by 
\[ q(i, j) = p(i, j)/\sum_{k \in A} p(i, k). \]

Suppose $Q$ is irreducible. In the case where $P$ is \textit{reversible}, find a simple explicit formula for the stationary distribution $\pi^*$ of $Q$ in terms of $P$ and its stationary distribution $\pi$. Give an example to show that the formula may not hold in the non-reversible case.
205B homework, #7; due Tuesday March 14

Durrett Chapter 6 Exercises 6.6, 6.7

14. Let \((X_n)\) be an irreducible Markov chain on \(S\) with transition matrix \((p(x, y))\). Let \(B\) be a finite subset of \(S\) such that the chain a.s. visits \(B\) infinitely often. Let \((Z_m)\) be the chain watched only on \(B\). Then \(Z\) is irreducible, and so has stationary distribution \(\hat{\pi}\), say. Define

\[
\mu(x, y) = \mathbb{E}_x \sum_{n=0}^{\infty} 1_{\{X_n=y, T_B>n\}}, \quad x \in B, \ y \in S.
\]

\[
\pi(y) = \sum_{x \in B} \hat{\pi}(x) \mu(x, y).
\]

Show that \(\pi\) is invariant, in the sense

\[
\pi(y) = \sum_{z \in S} \pi(z) p(z, y) \leq \infty, \ y \in S.
\]

15. A population consists of \(X_n\) individuals at times \(n = 0, 1, 2, \ldots\). Between time \(n\) and time \(n + 1\) each of these individuals dies with probability \(p\) independently of the others; and the population at time \(n + 1\) consists of the survivors together with an independent random (Poisson \((\lambda)\)) number of immigrants.

Let \(X_0\) have arbitrary distribution. What happens to the distribution of \(X_n\) as \(n \to \infty\)? [Hint: consider first the case where \(X_0\) has Poisson \((\lambda_0)\) distribution]
16. Let $X_n$ be the Markov chain on states $0, 1, \ldots, K$ with transition matrix

$$p(i, i + 1) = \frac{2}{3} \text{ and } p(i, i - 1) = \frac{1}{3}; \ 1 \leq i \leq K - 1$$

and initial state $i_0 \neq 0, K$. Let $X_n^*$ be the process $X_n$ conditioned on the event $\{X_m = K \text{ ultimately}\}$.

(a) Prove carefully that $X_n^*$ is Markov.

(b) Find its transition matrix.

(c) Find the distribution of $\min_{n \geq 0} X_n^*$.

17. Let $S$ be a finite set. Let $p(i, j)$ be an irreducible Markov transition matrix on $S$, with stationary distribution $\pi$. Consider a cat-and-mouse game, as follows. A state $i$ is chosen at random according to $\pi$; the cat and mouse are both placed at $i$, but before the cat can do anything the mouse jumps to another state chosen according to $p(i, \cdot)$. Thereafter, the mouse doesn’t move. The cat now searches for the mouse by moving at random according to the “time-reversed” Markov chain, i.e. the chain with transition matrix

$$q(i, j) = \pi(j)p(j, i)/\pi(i).$$

Find a simple formula for the expected number of steps taken by the cat until it finds the mouse.

[more questions on next page]
18. Let \( P = p(i, j) \) be a Markov transition matrix on \( \{0, 1, 2, \ldots\} \). Give a simple necessary and sufficient condition, in terms of \( P \), for the following assertion to be true.

For any pair \( i_0 < j_0 \) it is possible to construct \( (X_n, Y_n; n \geq 0) \) such that

1. \( X \) is the \((i_0, P)\)-chain
2. \( Y \) is the \((j_0, P)\)-chain
3. \( X_n \leq Y_n \) for all \( n \).

19. Let \( (X_n) \) be irreducible positive-recurrent with stationary distribution \( \pi \). Fix a subset \( B \) of \( S \). Let

\[
T_B = \min\{n \geq 1 : X_n \in B\}.
\]
\[
A_{kn} = \{X_m \in B^c \text{ for all } k \leq m \leq n\}.
\]

(a) Show that for the stationary chain, \( \mathbb{P}(A_{kn}) \) depends only on \( n - k \), and deduce that for the stationary chain

\[
\mathbb{P}(X_0 \in B, T_B \geq n) = \mathbb{P}(T_B = n).
\]

(b) Use (a) to give a new proof that \( \mathbb{E}_i T_i = 1/\pi(i) \).

(c) Use (a) to prove

\[
\mathbb{E}_i (T_i^2) = 2\mathbb{E}_i T_i (\sum_j (\mathbb{E}_j T_i / \mathbb{E}_j T_j) - 1).
\]

20. Let \( (X_n; n \geq 0) \) be a finite-state irreducible Markov chain. Write \( \pi \) for the stationary distribution and

\[
T_j = \min\{n \geq 0 : X_n = j\}
\]

for the first hitting time.

(a) Prove that \( \sum_j \pi_j \mathbb{E}_i T_j \) does not depend on \( i \).

(b) Give an example to show that \( \sum_i \pi_i \mathbb{E}_i T_j \) may depend on \( j \).
21. Let \((X_n)\) be an irreducible Markov chain on states \(I = \{0, 1, 2, \ldots\}\). Let 
\(g : I \rightarrow \mathbb{R}\) be such that 
(a) \(\mathbb{E}_i g(X_1) \geq g(i)\) for all \(i\), with strict inequality for some \(i\). 
(b) \(\sup_i \mathbb{E}_i |g(X_1) - g(i)| < \infty\). 
Prove that \((X_n)\) is not positive-recurrent. Give an example to show it may be null-recurrent.

22. Let \((X_n : n \geq 0)\) be a non-homogeneous Markov chain on states \(\{1, 2, \ldots, K\}\). Let \(\mathcal{T}\) be its tail \(\sigma\)-field. Prove that there exists a partition \((B_1, \ldots, B_m)\), \(m \leq K\) of \(\Omega\) such that \(\mathcal{T} = \sigma(B_1, \ldots, B_m)\) up to null sets. 
[Hint. Consider \(\mathbb{E}(Z|X_n)\) for tail-measurable \(Z\).]

23. Let \((X_n, n \geq 0)\) be a finite-state irreducible Markov chain with transition matrix \(P\). Let \(f\) be a non-constant real-valued function and \(0 < \lambda < 1\) be such that \(\sum_j p_{ij} f(j) = \lambda f(i) \ \forall i\). 
(i) Show that \(\lambda^{-n} f(X_n)\) is a martingale. 
(ii) Let \(\tau_b\) be the first hitting time on a state \(b\). Show that 
\[\sup \{\theta : \mathbb{E}(\theta^{\tau_b}|X_0 = i) < \infty \ \forall i\} \leq 1/\lambda.\]

[more questions on next page]
24. Let \( p(i, j) \) be a Markov transition matrix on \( \{0, 1, 2, \ldots, K\} \) such that 0 and \( K \) are absorbing, \( \{1, 2, \ldots, K-1\} \) forms a strongly connected component and
\[
\sum_j j p(i, j) = i \quad \text{for each} \quad 0 \leq i \leq K.
\]

Fix \( B \geq 2 \). Define a Markov process \((X_n; n \geq 0)\) on state-space \( \{1, 2, \ldots, K\}^B \) as follows. A state \((x(1), \ldots, x(B))\) represents the positions of \( B \) particles, particle \( b \) being in position \( x(b) \). Initially all particles are at position \( i_0 \), for some \( 1 \leq i_0 \leq K-1 \). A step of the process \( X \) is as follows. Pick one of the particles uniformly at random, and let it perform a move according to \( p(\cdot, \cdot) \). If the move takes the particle to a position which is not 0, that concludes the step of \( X \). Otherwise the particle tries to move to 0, in which case it is immediately replaced at the position of another particle, picked uniformly at random from the other \( B-1 \) particles. Call this latter move a 0-jump.

Ultimately the process will reach the absorbing state with all particles in position \( K \). Let \( N \) be the random total number of 0-jumps made. Prove

\[
\mathbb{E} \left( \left( \frac{B-1}{B} \right)^N \right) = \frac{i_0}{K}.
\]

What can you deduce about \( \mathbb{E}N \)?

[Hint: Let \( A_n \) be the average position of the \( B \) particles after \( n \) steps. Find a martingale related to \( A_n \).]
205B Homework #10; due Tuesday April 11

Durrett Chapter 7 Exercises 1.6, 3.1, 3.3, 3.4
205B Homework #11; due Tuesday April 18

Durrett Chapter 7 Exercises 4.1, 5.1, 5.2, 5.4.
TBA