205B homework, week 2; due Thursday February 5

Durrett Chapter 5 Exercises 1.6, 1.7, 1.8, 2.3, 2.5, 2.10, 2.11
Durrett Chapter 5 Exercises 4.3, 4.4, 4.8, 4.10

1. Give an example to show that, if $X_n$ is a Markov chain, then $f(X_n)$ need not be a Markov chain.

2. Let $A$ and $B$ be disjoint subsets of a finite state space $S$. Let $f(i) = P_i(\tau_A < \tau_B)$.
   (a) Write down equations satisfied by $(f(i) : i \in S)$.
   (b) Under what conditions is $(f(i))$ the unique solution of these equations? [Hint: consider the decomposition into strongly connected components]

3. Show that for $K < \infty$ and $a > 0$ there is a constant $C(K,a) < \infty$ such that: in every finite irreducible chain

   $$\max_{i,j} E_i T_j \leq C(K,a)$$

   where $K =$ number of states and $a = \min\{P(i, j) : P(i, j) > 0\}$.

4. Let $X_n$ be an irreducible chain with transition matrix $P$. Let $Y_n$ be the jump chain $Y_n = X(T_n)$ where $T_0 = 0$ and

   $$T_{n+1} = \min\{m > T_n : X_m \neq X(T_n)\}.$$

   (a) Show that $Y_n$ is Markov, and write its transition matrix $Q$ in terms of $P$.
   (b) Show that $(Y_n)$ is recurrent iff $(X_n)$ is recurrent.
   (c) Assuming recurrence, find the relation between the $P$-invariant measure and the $Q$-invariant measure.
   (d) Deduce that, on an infinite state space, it is possible for $(Y_n)$ to be positive-recurrent while $(X_n)$ is not.

5. Let $X_n$ be an finite irreducible chain with transition matrix $P$. Fix a subset $A$ of $S$. Define a transition matrix $Q$ on $A$ by

   $$q(i, j) = p(i, j)/\sum_{k \in A} p(i, k).$$

   Suppose $Q$ is irreducible. In the case where $P$ is reversible, find a simple explicit formula for the stationary distribution $\pi^*$ of $Q$ in terms of $P$ and its stationary distribution $\pi$. Give an example to show that the formula may not hold in the non-reversible case.
Durrett Chapter 5 Exercises 5.7, 5.8, 5.11

1. Let \((X_n)\) be an irreducible Markov chain on \(S\) with transition matrix \((p(x,y))\). Let \(B\) be a finite subset of \(S\) such that the chain a.s. visits \(B\) infinitely often. Let \((Z_m)\) be the chain watched only on \(B\). Then \(Z\) is irreducible, and so has stationary distribution \(\hat{\pi}\), say. Define

\[
\mu(x,y) = E_x \sum_{n=0}^{\infty} 1_{(X_n=y,T_B>n)}, \quad x \in B, \ y \in S.
\]

\[
\pi(y) = \sum_{x \in B} \hat{\pi}(x) \mu(x,y).
\]

Show that \(\pi\) is invariant, in the sense

\[
\pi(y) = \sum_{z \in S} \pi(z)p(z,y) \leq \infty, \ y \in S.
\]

2. A population consists of \(X_n\) individuals at times \(n = 0, 1, 2, \ldots\). Between time \(n\) and time \(n+1\) each of these individuals dies with probability \(p\) independently of the others; and the population at time \(n+1\) consists of the survivors together with an independent random (Poisson \((\lambda)\)) number of immigrants.

Let \(X_0\) have arbitrary distribution. What happens to the distribution of \(X_n\) as \(n \to \infty\)? [Hint: consider first the case where \(X_0\) has Poisson \((\lambda_0)\) distribution]
These are miscellaneous questions on Markov chains, not necessarily closely connected to this week’s class material.

1. Let $X_n$ be the Markov chain on states $0, 1, \ldots, K$ with transition matrix

   \[ p(i, i + 1) = \frac{2}{3} \quad \text{and} \quad p(i, i - 1) = \frac{1}{3}; \quad 1 \leq i \leq K - 1 \]

   \[ p(0, 0) = p(K, K) = 1 \]

   and initial state $i_0 \neq 0, K$. Let $X^*_n$ be the process $X_n$ conditioned on the event \{ $X_m = K$ ultimately \}.  
   (a) Prove carefully that $X^*_n$ is Markov.  
   (b) Find its transition matrix.  
   (c) Find the distribution of $\min_{n \geq 0} X^*_n$.

2. Let $S$ be a finite set. Let $p(i, j)$ be an irreducible Markov transition matrix on $P$, with stationary distribution $\pi$. Consider a cat-and-mouse game, as follows. A state $i$ is chosen at random according to $\pi$; the cat and mouse are both placed at $i$, but before the cat can do anything the mouse jumps to another state chosen according to $p(i, \cdot)$. Thereafter, the mouse doesn’t move. The cat now searches for the mouse by moving at random according to the “time-reversed” Markov chain, i.e. the chain with transition matrix

   \[ q(i, j) = \frac{\pi(j)p(j, i)}{\pi(i)}. \]

   Find a simple formula for the expected number of steps taken by the cat until it finds the mouse.
3. Let $P(i,j)$ be a Markov transition matrix on $\{0, 1, 2, \ldots\}$. Give a simple necessary and sufficient condition, in terms of $P$, for the following assertion to be true.

For any pair $i_0 < j_0$ it is possible to construct $(X_n, Y_n; n \geq 0)$ such that

1. $X$ is the $(i_0, P)$-chain
2. $Y$ is the $(j_0, P)$-chain
3. $X_n \leq Y_n$ for all $n$.

4. Let $(X_n)$ be irreducible positive-recurrent with stationary distribution $\pi$. Fix a subset $B$ of $S$. Let

$$ T_B = \min\{n \geq 1 : X_n \in B\}, $$

$$ A_{kn} = \{X_m \in B^c \text{ for all } k \leq m \leq n\}. $$

(a) Show that for the stationary chain, $P(A_{kn})$ depends only on $n - k$, and deduce that for the stationary chain

$$ P(X_0 \in B, T_B \geq n) = P(T_B = n). $$

(b) Use (a) to give a new proof that $E_i T_i = 1/\pi(i)$.

(c) Use (a) to prove

$$ E_i(T_i)^2 = 2E_iT_i(\sum_j (E_j T_i/E_j T_j) - 1). $$

5. Let $(X_n; n \geq 0)$ be a finite-state irreducible Markov chain. Write $\pi$ for the stationary distribution and

$$ T_j = \min\{n \geq 0 : X_n = j\} $$

for the first hitting time.

(a) Prove that $\sum_j \pi_j E_i T_j$ does not depend on $i$.

(b) Give an example to show that $\sum_i \pi_i E_i T_j$ may depend on $j$. 

5
1. Let \((X_n)\) be an irreducible Markov chain on states \(I = \{0, 1, 2, \ldots\}\). Let \(g : I \to \mathbb{R}\) be such that

(a) \(E_i g(X_1) \geq g(i)\) for all \(i\), with strict inequality for some \(i\).

(b) \(\sup_i E_i |g(X_1) - g(i)| < \infty\).

Prove that \((X_n)\) is not positive-recurrent. Give an example to show it may be null-recurrent.

2. Let \((X_n : n \geq 0)\) be a non-homogeneous Markov chain on states \(\{1, 2, \ldots, K\}\). Let \(T\) be its tail \(\sigma\)-field. Prove that there exists a partition \((B_1, \ldots, B_m)\), \(m \leq K\) of \(\Omega\) such that \(T = \sigma(B_1, \ldots, B_m)\) up to null sets.

[Hint. Consider \(E(Z|X_n)\) for tail-measurable \(Z\).]

3. Let \(P(i, j)\) be a Markov transition matrix on \(\{0, 1, 2, \ldots, K\}\) such that 0 and \(K\) are absorbing, \(\{1, 2, \ldots, K-1\}\) forms a strongly connected component and

\[ \sum_j j P(i, j) = i \quad \text{for each } 0 \leq i \leq K. \]

Fix \(B \geq 2\). Define a Markov process \((X_n; n \geq 0)\) on state-space \(\{1, 2, \ldots, K\}^B\) as follows. A state \((x(1), \ldots, x(B))\) represents the positions of \(B\) particles, particle \(b\) being in position \(x(b)\). Initially all particles are at position \(i_0\), for some \(1 \leq i_0 \leq K - 1\). A step of the process \(X\) is as follows. Pick one of the particles uniformly at random, and let it perform a move according to \(P(\cdot, \cdot)\). If the move takes the particle to a position which is not \(0\), that concludes the step of \(X\). Otherwise the particle tries to move to \(0\), in which case it is immediately replaced at the position of another particle, picked uniformly at random from the other \(B-1\) particles. Call this latter move a 0-jump.

Ultimately the process will reach the absorbing state with all particles in position \(K\). Let \(N\) be the random total number of 0-jumps made. Prove

\[ E \left( \frac{B-1}{B} \right)^N = \frac{i_0}{K}. \]

What can you deduce about \(EN\)?

[Hint: Let \(A_n\) be the average position of the \(B\) particles after \(n\) steps. Find a martingale related to \(A_n\).]
4. Let \((X_n, n \geq 0)\) be a finite-state irreducible Markov chain with transition matrix \(P\). Let \(f\) be a non-constant real-valued function and \(0 < \lambda < 1\) be such that \(\sum_j p_{ij} f(j) = \lambda f(i) \ \forall i\).

(i) Show that \(\lambda^{-n} f(X_n)\) is a martingale.

(ii) Let \(\tau_b\) be the first hitting time on a state \(b\). Show that

\[
\sup \{ \theta : E(\theta^{\tau_b} | X_0 = i) < \infty \ \forall i \} \leq 1/\lambda.
\]
205B homework, week 7; due Thursday March 12

Durrett section 4.3 Exercises 1, 2, 3, 4, 9, 10, 11
1. Prove the following slight extension of Azuma’s inequality.

Let \((M_n)\) be a martingales such that \(|M_n - M_{n-1}| \leq K_n\) for constants \(K_n\). Then for \(x > 0\)

\[
P(|M_n - M_0| \geq x) \leq 2 \exp\left(-\frac{1}{2}x^2/\sum_{i=1}^{n} K_i^2\right).
\]

2. Suppose you have \(n\) items, the \(i\)'th item having size \(V_i > 0\) and reward \(0 \leq W_i \leq B\) for constant \(B\), where the r.v.’s \((V_i, W_i)\) are all independent. You have a box of fixed size \(c\). You pack items into the box (subject to the constraint that the sum of sizes of packed objects is at most \(c\)), choosing the subset of items which maximizes the total reward (that is \(\sum_i W_i\), summed over the packed items). Write \(X\) for this maximum total reward. Show

\[
P(|X - EX| \geq x) \leq 2 \exp(-x^2/(2nB^2)), \quad x > 0.
\]

3. Let \(S_n = \sum_{i=1}^{n} X_i\), where \((X_i)\) are i.i.d. with exponential(1) distribution. Use the large deviation theorem to get explicit limits for

\[
n^{-1} \log P(n^{-1}S_n \geq a), \quad a > 1 \quad \text{and} \quad n^{-1} \log P(n^{-1}S_n \leq a), \quad a < 1.
\]

4. Oriented first passage percolation. Consider the lattice quadrant \(\{(i, j) : i, j \geq 0\}\) with directed edges \((i, j) \rightarrow (i+1, j)\) and \((i, j) \rightarrow (i, j+1)\). Associate to each edge \(e\) an exponential(1) r.v. \(X_e\), independent for different edges. For each directed path \(\pi\) of length \(d\) started at \((0,0)\), let \(S_\pi = \sum_{\text{edges } e \text{ in path } \pi} X_e\). Let \(H_d\) be the minimum of \(S_\pi\) over all such paths \(\pi\) of length \(d\). It can be shown that \(d^{-1}H_d \rightarrow c\) a.s., for some constant \(c\). Give explicit upper and lower bounds on \(c\).

[Hint: use result of previous question for lower bound.]
205B Homework, Week 9; due Thursday Apr 2

Durrett Chapter 6 problems 1.7, 3.1, 3.4, 3.5.
Durrett Chapter 6 problems 7.1, 7.2, 7.4.

1. (This is a more careful statement of Durrett Chapter 6 Exercise 6.2.) Let \((X_1, X_2, \ldots; Y_1, Y_2, \ldots)\) be i.i.d. taking values 0 or 1 with probability 1/2 each. Define \(L_{0,n}\) as in Example 6.4, so that \(n^{-1}L_{0,n} \to \gamma\) a.s. for some constant \(\gamma\).

   (a) Compute \(E L_{0,2}\), and deduce a lower bound on \(\gamma\).

   (b) For \(a < 1\) compute the expected number of common subsequence pairs \(i_1 < i_2 < \ldots < i_k \leq n; j_1 < j_2 < \ldots < j_k \leq n\) where \(k = \lfloor an \rfloor\). Deduce that \(\gamma \leq a_0\) for a certain \(a_0\). What is the approximate numerical value of \(a_0\)?
205B Homework, Week 11-12; due Thursday Apr 23

Durrett Chapter 7 problems 1.3, 2.4, 3.3, 3.6, 3.7, 5.3
205B Homework, Week 13-14; due Thursday May 7

Durrett Chapter 7 problems 5.4, 6.1, 6.3, 7.1