1 STAT 205B homework solutions; week 9

Problem 21.
Note that $g(X_n)$ is a submartingale, and $g(j) < E_j g(X_1)$ for some $j$. Let $T = \min\{n \geq 1|X_n = j\}$. Then $g(j) = E_j g(X_T) < E_j g(X_1)$, so $ET = \infty$, which would contradict the optional stopping theorem if $ET < \infty$ held (along with condition (b)). Thus, $ET = \infty$, that is, the chain is not positive recurrent.

An example which is null-recurrent is the simple symmetric random walk on the non-negative integers with $P_{0 \rightarrow 1} = 1$ and with $g(i) = i$ ($j = 0$ in this case.)

Problem 22.
Let $F_n = \sigma(X_0, X_1, \ldots, X_n)$, $F_\infty = \sigma(X_0, X_1, \ldots)$. Then, using the Markov property

$$E[Z|X_n] = E[Z|X_0, X_1, \ldots, X_n] = E[Z|F_n] \rightarrow E[Z|F_\infty] = Z$$

(as $n \rightarrow \infty$, almost surely) since $Z$ is $\mathcal{T}$-measurable and $\mathcal{T} \subset F_\infty$. Now observe that since $X_n$ can take only at most $K$ different values, the range of $E[Z|X_n]$ also has cardinality at most $K$, for each $n$. Hence, it is true for the limit $Z$ as well. That is, any tail-measurable random variable takes at most $K$ different values, and this implies that $\mathcal{T}$ is generated by a partition of at most $K$ disjoint sets.

Problem 23.
(i) \[ E[\lambda^{-(n+1)} f(X_{n+1})|X_n = i] = \lambda^{-(n+1)} \sum_j p_{ij} f(j) = \lambda^{-n} f(i) \]
by the assumption, for every state $i$.

(ii) It suffices to show that there is at least one state $i$ for which

$$E[\lambda^{-\tau_b}|X_0 = i] = \infty.$$ 

Assume to the contrary that all these conditional expectations are finite. Using the optional stopping theorem for the martingale in (i), starting from state $i$, with stopping time $\tau_b$ ($E_i[\lambda^{-\tau_b}] < \infty$ by the assumption and the martingale differences are bounded because the chain is finite, irreducible) yields

$$E_i[\lambda^{-\tau_b} f(b)] = f(i). \quad (1)$$

Now, if $f(b) = 0$, then $f(i) = 0$ for every $i$ which contradicts the assumption that $f$ is non-constant. Otherwise, we can assume that $f(b) > 0$. Then (1) implies that $f(i) > 0$ for every $i$. Let $i^* = \text{argmin} f(i)$. Then

$$f(i^*) \leq \sum_j p_{i^*j} f(j) = \lambda f(i^*)$$

leads to contradiction, since $\lambda < 1$. 

Problem 24.

If $A_n$ is the average position of the $B$ particles after $n$ steps, then it is easy to see that

$$E[A_{n+1}|X_n] = A_n 1\{\text{there is no 0-jump at step } n+1\} + \frac{B}{B-1} A_n 1\{\text{there is a 0-jump at step } n+1\}$$

$$= A_n \left(1 + \frac{1}{B-1} 1\{\text{there is a 0-jump at step } n+1\}\right).$$

This easily implies that $(\frac{B}{B-1})^{-N_n} A_n$ is a martingale, where $N_n$ is the number of 0-jumps made up to step $n$. Using the optional stopping theorem with the stopping time when all the particles reach state $K$ we get

$$E \left(\frac{B}{B-1}\right)^{-N} K = i_0.$$

To get a bound for $EN$, observe that

$$\frac{i_0}{K} = E \left(1 - \frac{1}{B}\right)^N \geq E \left(1 - \frac{N}{B}\right) = 1 - \frac{EN}{B},$$

which implies

$$EN \geq B(1 - i_0/K).$$