1 STAT 205B homework solutions; week 5

Problem 6.2.7 (Durrett).
Observe that
\[
P(X_{n+1} = x_{n+1}|X_k = x_k, 1 \leq k \leq n) = P(\xi_{n+1} \notin \{\xi_1, \ldots, \xi_n\}|X_k = x_k, 1 \leq k \leq n)
\]
\[
= \frac{N - x_n}{N} \quad \text{by independence}
\]
For \(x_n = x_{n+1}\) we get \(P(X_{n+1} = x_n|X_k = x_k, 1 \leq k \leq n) = \frac{x_n}{N}\) and for all other cases 0.
We see that \(P(X_{n+1} = x_n + 1|\sigma(X_1, \ldots, X_n))\) is \(\sigma(X_n)\)-measurable and thus \(P(X_{n+1} = x_n + 1|\sigma(X_1, \ldots, X_n)) = P(X_{n+1} = x_n + 1|\sigma(X_n))\), so \(X\) is a Markov chain.

Problem 6.2.8 (Durrett).
Consider the sequence \((X_1, X_2, X_3) = (1, 1, 1)\). This can happen in two ways: \((S_1, S_2, S_3) = (1, 0, -1)\), or \((S_1, S_2, S_3) = (1, 0, 1)\). Thus, \(P(X_4 = 2|X_1 = 1, X_2 = 1, X_3 = 1) = .25\). Now consider the sequence \((X_1, X_2, X_3) = (0, 0, 1)\). This can only happen if \((S_1, S_2, S_3) = (-1, 0, 1)\).
Thus, \(P(X_4 = 2|X_1 = 0, X_2 = 0, X_3 = 1) = .5\). Thus, \(X\) is not a Markov chain.

Problem 6.2.9 (Durrett).
Let \(p(s_n) = \frac{E^{\theta s_n + 1 + 2}}{E^{\theta s_n + 1}(1 - \theta)^{n - s_n}} = \frac{s_n + n + 2}{2(n+2)}\). We will show
\(P(X_{n+1} = 1|S_1, \ldots, S_n) = p(S_n)\). Then all the assertions will follow.
\[
\int_{[S_1 = s_1, \ldots, S_n = s_n]} p(S_n) dP = p(s_n)P(S_1 = s_1, \ldots, S_n = s_n)
\]
\[
= p(s_n)EP(U_k \leq \theta \text{ for } k \in A, U_k > \theta \text{ for } k \in B|\theta)
\]
where \(A = \{k : s_k - s_{k-1} = 1\}, B = \{k : s_k - s_{k-1} = -1\}\)
\[
= p(s_n)E\theta|A|(1 - \theta)^{|B|} \quad \text{by independence}
\]
\[
= p(s_n)E\theta^{s_n}(1 - \theta)^{n-s_n} = E\theta^{s_n + 2}(1 - \theta)^{n-s_n} = E\theta^{s_n + 1}(1 - \theta)^{|B|}
\]
\[
= EP(U_k \leq \theta \text{ for } k \in A \cup \{n + 1\}, U_k > \theta \text{ for } k \in B|\theta)
\]
\[
= P(S_1 = s_1, \ldots, S_n = s_n, X_{n+1} = 1) = \int_{[S_1 = s_1, \ldots, S_n = s_n]} 1_{[X_{n+1} = 1]} dP
\]

Problem 6.3.10 (Durrett).
(i) \(x \notin A\) gives \(\tau_A \geq 1\) and so
\[
E_x(\tau_A|\mathcal{F}_1) = E_{X_1}(\tau_A + 1)
\]
as \(\tau_A \geq 1\) gives \(\tau_A = 1 + \tau_A \circ \theta_1\). Thus
\[
g(x) = E_{x\tau_A} = E_x(E_x(\tau_A|\mathcal{F}_1)) = 1 + E_x(E_{X_1}(\tau_A)) = 1 + E_xg(X_1) = 1 + \sum_y p(x, y)g(y).
\]
(ii) 
\[
E[g(X((n+1) \land \tau_A)) + ((n+1) \land \tau_A)|\mathcal{F}_n] = 1\{\tau_A \leq n\}(g(X(\tau_A)) + \tau_A) + 1\{\tau_A > n\}(g(X(n)) - 1 + n + 1) \\
= g(X(n \land \tau_A)) + (n \land \tau_A)
\]

using (*)

(iii) First, observe that \(P_x(\tau_A < \infty) > 0\) gives the existence of an \(n(x)\) with \(p^{n(x)}(x, A) > 0\), so as \(S - A\) is finite we get for some \(N < \infty, \epsilon > 0\) that

\[
E_x\tau_A = \sum_{k=0}^{\infty} P_x(\tau_A > k) \text{ as } \tau_A \in \{0, 1, 2, \ldots\} \\
\leq \sum_{k=0}^{\infty} NP_x(\tau_A > kN) \text{ as } P_x(\tau_A > m) \leq P_x(\tau_A > n) \text{ if } m \geq n \\
\leq N \sum_{k=0}^{\infty} (1 - \epsilon)^k < \infty \text{ as } \epsilon > 0.
\]

That is, \(E_x\tau_A\) is a bounded function.

The conditional expectation of the above martingale given \(X(0)\) is (using the \(n = 0\) case)
\(g(X(0))\). On the other hand, since \(S - A\) is finite, and \(g\) is 0 on \(A\), \(g\) is a bounded function, and convergence of the martingale to \(g(X(\tau_A)) + \tau_A = \tau_A\) gives that \(E[\tau_A | X(0)] = g(X(0))\).

**Problem 6.3.11 (Durrett).**

Observe \(p((V, H), (H, W)) = p((V, T), (T, W)) = 1/2\) for \(V, W \in \{H, T\}, p(\cdot, \cdot) = 0\) else. Let \(A = (H, H)\) and check that all conditions in 2.11. are satisfied. So \(E_xN_1\) is the only solution of

\[
g(T, H) = 1 + 1/2(g(H, H) + g(H, T)) = 1 + 1/2g(H, T)
\]

max

\[
g(T, T) = 1 + 1/2(g(T, T) + g(T, H)) \\
g(H, T) = 1 + 1/2(g(T, T) + g(T, H))
\]

with \(g(H, H) = 0\).

Find \(g(H, T) = g(T, T) = 2 + g(T, H) = 6\.

So \(EN_1 = 4\).

**Problem 6.3.12 (Durrett).**

Let \(A = \{0, N\}\) and use the result of 6.3.10. It is easy to check that \(g(x) = x(N - x)\) solves the system of equations (*), and thus, \(g(x) = E_x\tau_A\).

**Problem 10.**

Let \(X\) be a simple symmetric random walk on \(\mathbb{Z}\) with \(X_0 = 0\) and let \(f(x) = x\) if \(x\) is odd, \(f(x) = 0\) if \(x\) is even. Then \(P(f(X_3) = 3|f(X_2) = 0) = P(f(X_3) = 3) = \frac{1}{8}\) while \(P(f(X_3) = 3|f(X_2) = 0, f(X_1) = -1) = 0\). Thus, \((f(X_n))\) is not a Markov chain.
Problem 11.

Let \( n_0(i, j) = \min \{ n : P^n(i, j) > 0 \} \). Then \( n_0(i, j) < K \) because we can always avoid a cycle. Now

\[
\max_{i \neq j} P_i(T_j > K - 1) \leq \max_{i \neq j} P_i(T_j > n_0(i, j)) = \max_{i \neq j} (1 - p^{n_0(i,j)}(i,j))
\]

\[
\leq \max_{i \neq j} (1 - a^{n_0(i,j)}) \leq 1 - a^{K-1}
\]

Let \( P_{(-j)} = ((p(i, k) : i, k \neq j)) \). Then

\[
\| P_{(-j)}^{K-1} \| = \max_{i \neq j} \sum_{l \neq j} P_{(-j)}^{K-1}(i, l) = \max_{i \neq j} P_i(T_j > K - 1) \leq 1 - a^{K-1}
\]

and \( \| P_{(-j)}^m \| \leq (1 - a^{K-1})^m, \ m = 0, 1, 2, ... \)

So

\[
E_i T_j = \sum_{n=0}^\infty P_i(T_j > n) = \sum_{n=0}^\infty \sum_{n=m(K-1)}^{(m+1)(K-1)-1} P_i(T_j > n)
\]

\[
\leq \sum_{m=0}^\infty (K - 1)P_i(T_j > m(K - 1)) \leq \sum_{m=0}^\infty (K - 1) \| P_{(-j)}^{m(K-1)} \|
\]

\[
\leq (K - 1) \sum_{m=0}^\infty (1 - a^{K-1})^m = \frac{K - 1}{a^{K-1}}
\]

Further, \( E_j T_j \leq 1 + \max_i E_i T_j \)

So we can take \( C(K, a) = 1 + \frac{K-1}{a^{K-1}} \).