1 STAT 205B homework solutions; week 3

Problem 3.4.4 (Durrett).

\[ 2(\sqrt{S_n} - \sqrt{n}) = \left( \frac{S_n - n}{\sqrt{n}} \right) \times \left( \frac{2}{\sqrt{S_n/n} + 1} \right) \]

The factor on the right converges to 1 a.s. by SLLN, and the convergence together lemma for products (proved just like the one for sums) implies the result.

Problem 3.4.6 (Durrett).

Follow hint, and assume \( a_n \to \infty \).

\[ P(|X_n - Y_n| \geq \delta) \leq P(|1 - N_n/a_n| \geq \varepsilon) + 2\varepsilon/\delta^2, \]

by Kolmogorov’s inequality, so \( X_n - Y_n \overset{d}{\to} 0 \) and the convergence together lemma implies the result.

Problem 3.4.7 (Durrett).

WLOG \( \mu = 1 \). \( (N_t + 1)/t \overset{a.s.}{\to} 1 \) by Theorem 3.4.1 in Durrett, so also \( N_t/t \overset{a.s.}{\to} 1 \). Now

\[ \frac{(N_t + 1) - S_{N_t+1}}{\sigma \sqrt{t}} - \frac{1}{\sigma \sqrt{t}} \leq \frac{N_t - t}{\sigma \sqrt{t}} \leq \frac{N_t - S_{N_t}}{\sigma \sqrt{t}} \]

The first term on the left (so the whole left) and the whole right converge to \( N(0, 1) \) in distribution by the random index CLT applied to \( \sum (1 - Y_i) \) and \( N_t + 1 \), respectively \( N_t \). Now in general if \( X_n \leq Y_n \leq Z_n \) and \( X_n, Z_n \overset{d}{\to} X \), then \( Y_n \overset{d}{\to} X \), as it is easily seen by taking \( P(\cdot \leq a) \) of all three sequences. The problem that \( t \) is not discrete is solved by taking any subsequence going to \( \infty \). Also, note \( Y_{N_t} \) does not in general have the same distribution as \( Y_1 \).

Problem 3.4.9 (Durrett).

\( E(X_m) = 0 \) for all \( m \), hence

\[ Var(S_n)/n = \frac{1}{n} \sum_{m=1}^{n} Var(X_m) = \frac{1}{n} \sum_{m=1}^{n} (m^{-2}m^2 + (1 - m^{-2})) \to 2. \]

On the other hand, for the truncated variables \( Y_m = \text{sgn}(X_m) \) (Bernoulli(1/2) variables) the CLT clearly holds, so \( \sum_{m=1}^{n} Y_m / \sqrt{n} \overset{d}{\to} \chi \). Moreover,

\[ \frac{\sum_{m=1}^{n} X_m - Y_{m a.s.}}{\sqrt{n}} \overset{a.s.}{\to} 0, \]

since \( P(X_m \neq Y_m) = m^{-2} \) which has a convergent sum in \( m \), thus, the Borel-Cantelli lemma implies that \( P(X_m \neq Y_m \text{ infinitely often}) = 0 \). This implies \( \sum_{m=1}^{n} X_m / \sqrt{n} \overset{d}{\to} \chi \).
Problem 3.4.11 (Durrett). This follows from the version of Lindeberg-Feller CLT with the Lyapunov condition: with $s_n^2 = \sum_{m=1}^{n} \text{Var}(X_m) = n$,

$$
\sum_{m=1}^{n} \frac{E|X_m|^{2+\delta}}{s_n^{2+\delta}} \leq \frac{nC'}{n^{(2+\delta)/2}} = \frac{C'}{n^{\delta/2}} \to 0.
$$

Problem 4.

Let $Y_j = X_j 1(X_j \leq j)$, and $Z_j = X_j - Y_j$. Let $b_n = n^{3/2}/18$. It is easy to check that the array $Y_{n,j} = b_n^{-1} Y_j$ satisfies Lindeberg’s condition, and that if $S_n = \sum_{j=1}^{n} Y_{n,j}$, then $\text{Var} S_n \to 1$, so $S_n \overset{d}{\to} N(0,1)$. Also, by Borel-Cantelli, $Z_j = 0$ for all but finitely many times a.s. Thus $S_n' = b_n^{-1} \sum_{j=1}^{n} Z_j \overset{a.s.}{\to} 0$, and so also in distribution. By the converging together lemma,

$$
\frac{1}{b_n} \sum_{j=1}^{n} X_j = S_n + S_n' \overset{d}{\to} N(0,1).
$$

Note that the variances do not converge to the variance of the limit, which would follow if the Lindeberg condition were satisfied.