1. [Bill. 2.4] Let $F_n$ be classes of subsets of $S$. Suppose each $F_n$ is a field, and $F_n \subset F_{n+1}$ for $n = 1, 2, \ldots$. Define $F = \bigcup_{n=1}^{\infty} F_n$. Show that $F$ is a field. Give an example to show that, if each $F_n$ is a $\sigma$-field, then $F$ need not be a $\sigma$-field.

2. [Bill. 2.5(b)] Given a non-empty collection $\mathcal{A}$ of sets, we defined $F(\mathcal{A})$ as the intersection of all fields containing $\mathcal{A}$. Show that $F(\mathcal{A})$ is the class of sets of the form $\bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij}$, where for each $i$ and $j$ either $A_{i,j} \in \mathcal{A}$ or $A_{i,j}^c \in \mathcal{A}$, and where the $m$ sets $\bigcap_{j=1}^{n_i} A_{ij}$, $1 \leq i \leq m$ are disjoint.

3. [Bill. 2.8] Suppose $B \in \sigma(\mathcal{A})$, for some collection $\mathcal{A}$ of subsets. Show there exists a countable subcollection $\mathcal{A}_B$ of $\mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$.

4. Show that the Borel $\sigma$-field on $\mathbb{R}^d$ is the smallest $\sigma$-field that makes all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ measurable.

5. [Durr. 1.3.5] A function $f : \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous (l.s.c.) if $\liminf_{y \to x} f(y) \geq f(x)$ for all $x$. A function is upper semicontinuous (u.s.c.) if $\limsup_{y \to x} f(y) \leq f(x)$ for all $x$. Show that, if $f$ is l.s.c. or u.s.c., then $f$ is measurable.
205A Homework #2, due Tuesday 13 September.

1. [similar Bill. 2.15] Let \( \mathcal{B} \) be the Borel subsets of \( \mathbb{R} \). For \( B \in \mathcal{B} \) define

\[
\mu(B) = \begin{cases} 
1 & \text{if } (0, \varepsilon) \subset B \text{ for some } \varepsilon > 0 \\
0 & \text{if not}
\end{cases}
\]

(a) Show that \( \mu \) is not finitely additive on \( \mathcal{B} \).

(b) Show that \( \mu \) is finitely additive but not countably additive on the field \( \mathcal{B}_0 \) of finite disjoint unions of intervals \((a,b]\).

2. Show that, in the definition of “a probability measure \( \mu \) on a measurable space \((\mathcal{S},\mathcal{S})\)”, we may replace “countably additive” by “finitely additive, and satisfies

\[
\text{if } A_n \downarrow \phi \text{ then } \mu(A_n) \to 0.
\]

3. [similar Durr. A.1.1] Give an example of a measurable space \((\mathcal{S},\mathcal{S})\), a collection \( \mathcal{A} \) and probability measures \( \mu \) and \( \nu \) such that

(i) \( \mu(A) = \nu(A) \) for all \( A \in \mathcal{A} \)

(ii) \( \mathcal{S} = \sigma(\mathcal{A}) \)

(iii) \( \mu \neq \nu \).

Note: this can be done with \( S = \{1,2,3,4\} \)

4. [similar Durr. Lemma A.2.1] Let \( \mu \) be a probability measure on \((\mathcal{S},\mathcal{S})\), where \( \mathcal{S} = \sigma(\mathcal{F}) \) for a field \( \mathcal{F} \). Show that for each \( B \in \mathcal{S} \) and \( \varepsilon > 0 \) there exists \( A \in \mathcal{F} \) such that \( \mu(B \Delta A) < \varepsilon \).

5. Let \( g : [0,1] \to \mathbb{R} \) be integrable w.r.t. Lebesgue measure. Let \( \varepsilon > 0 \). Show that there exists a continuous function \( f : [0,1] \to \mathbb{R} \) such that \( \int |f(x) - g(x)| \, dx \leq \varepsilon \).
205A Homework #3, due Tuesday 20 September.

1. Use the monotone convergence theorem to prove the following.
   (i) If $X_n \geq 0$, $X_n \downarrow X$ a.s. and $EX_n < \infty$ for some $n$ then $EX_n \to EX$.
   (ii) If $E|X| < \infty$ then $E|X| 1_{(|X|>n)} \to 0$ as $n \to \infty$.
   (iii) If $E|X_1| < \infty$ and $X_n \uparrow X$ a.s. then either $EX_n \uparrow EX < \infty$ or else $EX_n \uparrow \infty$ and $E|X| = \infty$.
   (iv) If $X$ takes values in the non-negative integers then
   \[ EX = \sum_{n=1}^{\infty} P(X \geq n). \]

2. (i) For a counting r.v. $X = \sum_{i=1}^{n} 1_{A_i}$, give a formula for the variance of $X$ in terms of the probabilities $P(A_i)$ and $P(A_i \cap A_j)$, $i \neq j$.
   (ii) If $k$ balls are put at random into $n$ boxes, what is the variance of $X =$ number of empty boxes?

3. (i) Suppose $EX = 0$ and $\text{var}(X) = \sigma^2 < \infty$. Prove
   \[ P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \ a > 0. \]
   (ii) Suppose $X \geq 0$ and $EX^2 < \infty$. Prove
   \[ P(X > 0) \geq \frac{(EX)^2}{EX^2}. \]

4. Chebyshev’s other inequality.
   Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be bounded and increasing functions.
   Prove that, for any r.v. $X$,
   \[ E(f(X)g(X)) \geq (Ef(X))(Eg(X)). \]
   [In other words, $f(X)$ and $g(X)$ are positively correlated. This is intuitively obvious, but a little tricky to prove. Hint: consider an independent copy $Y$ of $X$. For this and the next question you may need the product rule for expectations of independent r.v.s]

5. Let $X$ have Poisson($\lambda$) distribution and let $Y$ have Poisson($2\lambda$) distribution.
   (i) Prove $P(X \geq Y) \leq \exp(-(3 - \sqrt{8})\lambda)$ if $X$ and $Y$ are independent.
   (ii) Find constants $A < \infty$, $c > 0$, not depending on $\lambda$, such that, without assuming independence, $P(X \geq Y) \leq A \exp(-c\lambda)$. 

3
1. **Monte Carlo integration** [cf. Durr. 2.2.3] Let \( f : [0, 1] \to \mathbb{R} \) be such that \( \int_0^1 f^2(x) \, dx < \infty \). Let \((U_i)\) be i.i.d. Uniform(0,1). Let

\[
D_n := n^{-1} \sum_{i=1}^{n} f(U_i) - \int_0^1 f(x) \, dx.
\]

(i) Use Chebyshev’s inequality to bound \( P(|D_n| > \varepsilon) \).
(ii) Show this bound remains true if the \((U_i)\) are only pairwise independent.

2. Let \( X \geq 0 \) and \( Y \geq 0 \) be independent r.v.’s with densities \( f \) and \( g \).

Calculate the densities of \( XY \) and of \( X/Y \).

*Note:* this is just to remind you of “undergraduate” results.

3. [Durr. 2.2.2.] Let \((X_i)\) be r.v.’s with \( EX_i = 0 \) and \( EX_iX_j \leq r(j-i), 1 \leq i \leq j < \infty \), where \( r(n) \) is a deterministic sequence with \( r(n) \to 0 \) as \( n \to \infty \).

Prove that \( n^{-1} \sum_{i=1}^{n} X_i \to 0 \) in probability.

4. [Durr. 2.3.11] Suppose events \( A_n \) satisfy \( P(A_n) \to 0 \) and

\[
\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty.
\]

Prove that

\[
P(A_n \text{ occurs infinitely often}) = 0.
\]

5. (a) Let \( Z \) have standard Normal distribution. Show

\[
P(Z > z) \sim z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \text{ as } z \to \infty.
\]

(b) Let \((Z_1, Z_2, \ldots)\) be independent with standard Normal distribution. Find constants \( c_n \to \infty \) such that

\[
\limsup_{n} Z_n/c_n = 1 \text{ a.s.}
\]
205A Homework #5, due Tuesday 4 October.

1. Let \((X_n)\) be i.i.d. with \(E|X_1| < \infty\). Let \(M_n = \max(X_1, \ldots, X_n)\). Prove that \(n^{-1}M_n \to 0\) a.s.

2. [Durr. 2.3.2] Let \(0 \leq X_1 \leq X_2 \leq \ldots\) be r.v.'s such that \(EX_n \sim an^{\alpha}\) and \(\text{var}(X_n) \leq Bn^{\beta}\), where \(0 < a, B < \infty\) and \(0 < \beta < 2\alpha < \infty\). Prove that \(n^{-\alpha}X_n \to a\) a.s.

3. Prove that the following are equivalent.
   (i) \(X_n \to X\) in probability.
   (ii) There exist \(\varepsilon_n \downarrow 0\) such that \(P(|X_n - X| > \varepsilon_n) \leq \varepsilon_n\).
   (iii) \(E \min(|X_n - X|, 1) \to 0\).


5. Prove the deterministic lemma we used in the proof of the Glivenko-Cantelli Theorem.

   **Lemma.** If \(F_1, F_2, \ldots, F\) are distribution functions and
   (i) \(F_n(x) \to F(x)\) for each rational \(x\)
   (ii) \(F_n(x) \to F(x)\) and \(F_n(x-) \to F(x-)\) for each atom \(x\) of \(F\)
   then \(\sup_x |F_n(x) - F(x)| \to 0\).
1. [Durr. 2.5.9] Let \((X_i)\) be independent, \(S_n = \sum_{i=1}^{n} X_i\), \(S^*_n = \max_{i \leq n} |S_i|\). Prove that
\[
P(S^*_n > 2a) \leq \frac{P(|S_n| > a)}{\min_{j \leq n} P(|S_n - S_j| \leq a)}, \quad a > 0.
\]
[Hint. If \(|S_j| > 2a\) and \(|S_n - S_j| \leq a\) then \(|S_n| > a\).]

2. [Durr. 2.5.10 and 11] In the setting of the previous question, prove
(i) if \(\lim_{n \to \infty} S_n\) exists in probability then the limit exists a.s.
(ii) if the \((X_i)\) are identically distributed and if \(n^{-1} S_n \to 0\) in probability then \(n^{-1} \max_{m \leq n} S_m \to 0\) in probability.

3. [cf. Durr 2.2.8] Let \((X_i)\) be i.i.d. taking values in \([-1, 1, 3, 7, 15, \ldots]\), such that
\[
P(X_1 = 2^k - 1) = \frac{1}{k(k+1)2^k}, \quad k \geq 1
\]
(which implicitly specifies \(P(X_1 = -1)\)).
(a) Show \(EX_1 = 0\).
(b) Show that for all \(\alpha < 1\),
\[
P\left(S_n < -\frac{\alpha n}{\log_2 n}\right) \to 1.
\]

Comment. This is sometimes described as “an unfair, fair game”. It shows that the conclusions of the SLLN and the “recurrence of sums” theorem can’t be strengthened much.
205A Homework #7, due Tuesday 18 October.

1. Suppose $S$ and $T$ are stopping times. Are the following necessarily stopping times? Give proof or counter-example.
   (a) $\min(S,T)$
   (b) $\max(S,T)$
   (c) $S + T$.

2. Let $(X_i)$ be i.i.d. with $EX_i^2 < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Let $T$ be a bounded stopping time. Is it true in general that
   $$\text{var}(S_T) = (\text{var}(X_1))(ET)?$$
   If not, is it true in the special case $EX_1 = 0$?

3. Let $(X_i)$ be a sequence of random variables, and let $\mathcal{T}$ be its tail $\sigma$-field. Let $S_n = \sum_{i=1}^n X_i$. Let $b_n \uparrow \infty$ be constants. Which of the following events must be in $\mathcal{T}$? Give proof or counter-example.
   (i) $\{X_n \to 0\}$
   (ii) $\{S_n \text{ converges}\}$
   (iii) $\{X_n > b_n \text{ infinitely often}\}$
   (iv) $\{S_n > b_n \text{ infinitely often}\}$
   (v) $\left\{\frac{\sqrt{\sum_{i=1}^n X_i^2}}{S_n} \to 0\right\}$.

4. Let $S_n = \sum_{i=1}^n X_i$, where $(X_i)$ are i.i.d. with exponential(1) distribution. Use the large deviation theorem to get explicit limits for $n^{-1} \log P(n^{-1}S_n \geq a)$, $a > 1$ and $n^{-1} \log P(n^{-1}S_n \leq a)$, $a < 1$.

5. Oriented first passage percolation. Consider the lattice quadrant $\{(i,j) : i, j \geq 0\}$ with directed edges $(i, j) \to (i+1, j)$ and $(i, j) \to (i, j+1)$. Associate to each edge $e$ an exponential(1) r.v. $X_e$, independent for different edges. For each directed path $\pi$ of length $d$ started at $(0,0)$, let $S_\pi = \sum_{e \in \text{path } \pi} X_e$. Let $H_d$ be the minimum of $S_\pi$ over all such paths $\pi$ of length $d$. It can be shown that $d^{-1}H_d \to c$ a.s., for some constant $c$. Give explicit upper and lower bounds on $c$.
   [Hint: use result of previous question for lower bound.]
205A Homework #8, due Tuesday 1 November.

[Theorem 7 and Corollary 8 refer to the notes linked from the “week 8” row of the schedule.]

1. Suppose probability measures satisfy \( \pi \ll \nu \ll \mu \). Show that

\[
\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \times \frac{d\nu}{d\mu}.
\]

2. In the setting of Theorem 7 [hard part], where \( S_2 \) is nice, show that \( Q \) is unique in the following sense. If \( Q^* \) is another conditional probability kernel for \( \mu \), then

\[
\mu_1 \{ x : Q^*(x, B) = Q(x, B) \text{ for all } B \in S_2 \} = 1.
\]

3. Let \( F \) be a distribution function. Let \( c > 0 \). Find a simple formula for

\[
\int_{-\infty}^{\infty} (F(x + c) - F(x)) \, dx.
\]

4. In the proof of Corollary 8 we used the inverse distribution function

\[
f(x, u) = \inf \{ y : u \leq Q(x, (-\infty, y)) \}
\]

associated with the kernel \( Q \). Show that \( f \) is product measurable.

5. Given a triple \((X_1, X_2, X_3)\), we can define 3 p.m.’s \( \mu_{12}, \mu_{13}, \mu_{23} \) on \( \mathbb{R}^2 \) by

\[
\mu_{ij} \text{ is the distribution of } (X_i, X_j). \quad (1)
\]

These p.m.’s satisfy a consistency condition:

the marginal distribution \( \mu_{1} \) obtained from \( \mu_{12} \) must coincide

with the marginal obtained from \( \mu_{13} \), and similarly for \( \mu_{2} \) and \( \mu_{3} \). \quad (2)

Give an example to show that the converse is false. That is, give an example of \( \mu_{12}, \mu_{13}, \mu_{23} \) satisfying (2) but for which there does not exist a triple \((X_1, X_2, X_3)\) satisfying (2).
1. Let $X,Y$ be random variables, and suppose $Y$ is measurable with respect to some sub-$\sigma$-field $\mathcal{G}$. Let $\mu(\omega, \cdot)$ be a regular conditional distribution for $X$ given $\mathcal{G}$. Prove that, for bounded measurable $h$,

$$E(h(X,Y)|\mathcal{G})(\omega) = \int h(x,Y(\omega))\mu(\omega, dx) \text{ a.s.}$$

2. For $i = 1, 2$ let $X_i$ be a r.v. defined on $(\Omega, \mathcal{F}, P)$ taking values in $(S_i, S_i)$. Let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$. Prove that assertions (a),(b) and (c) below are equivalent. When these assertions hold, we say call $X_1$ and $X_2$ are conditionally independent given $\mathcal{G}$.

(a) $P(X_1 \in A_1, X_2 \in A_2|\mathcal{G}) = P(X_1 \in A_1|\mathcal{G})P(X_2 \in A_2|\mathcal{G})$ for all $A_i \in S_i$.

(b) $E(h_1(X_1)h_2(X_2)|\mathcal{G}) = E(h_1(X_1)|\mathcal{G})E(h_2(X_2)|\mathcal{G})$ for all bounded measurable $h_i : S_i \to \mathbb{R}$.

(c) $E(h_1(X_1)|\mathcal{G}, X_2) = E(h_1(X_1)|\mathcal{G})$ for all bounded measurable $h_1 : S_1 \to \mathbb{R}$.

3. Suppose $X$ and $Y$ are conditionally independent given $Z$. Suppose $X$ and $Z$ are conditionally independent given $\mathcal{F}$, where $\mathcal{F} \subseteq \sigma(Z)$. Prove that $X$ and $Y$ are conditionally independent given $\mathcal{F}$.

4. Let $(X_n)$ and $(Y_n)$ be submartingales w.r.t. $(\mathcal{F}_n)$. Show that $(X_n + Y_n)$ and that $(\max(X_n, Y_n))$ are also submartingales w.r.t. $(\mathcal{F}_n)$.

5. Give an example where

$(X_n)$ is a submartingale w.r.t. $(\mathcal{F}_n)$
$(Y_n)$ is a submartingale w.r.t. $(\mathcal{G}_n)$
$(X_n + Y_n)$ is not a submartingale w.r.t. any filtration.
1. Let $S_n = \sum_{i=1}^{n} \xi_i$, where the $(\xi_i)$ are independent, $E\xi_i = 0$ and $\text{var} \xi_i < \infty$. Let $s_n^2 = \sum_{i=1}^{n} \text{var} \xi_i$. So we know that $(S_n^2 - s_n^2)$ is a martingale. Suppose also that $|\xi_i| \leq K$ for some constant $K$. Show that

$$P\left( \max_{m \leq n} |S_m| < x \right) \leq s_n^{-2}(K + x)^2, \quad x > 0.$$

2. Let $(X_n)$ be a martingale with $X_0 = 0$ and $EX_n^2 < \infty$. Using the fact that $(X_n + c)^2$ is a submartingale, show that

$$P\left( \max_{m \leq n} X_m \geq x \right) \leq \frac{EX_n^2}{x^2 + EX_n^2}, \quad x > 0.$$

3. Let $(X_n)$ and $(Y_n)$ be martingales with $E(X_n^2 + Y_n^2) < \infty$. Show that

$$EX_nY_n - EX_0Y_0 = \sum_{m=1}^{n} E(X_m - X_{m-1})(Y_m - Y_{m-1}).$$

4. Let $(X_n, F_n), n \geq 0$ be a positive submartingale with $X_0 = 0$. Let $V_n$ be random variables such that

(i) $V_n \in F_{n-1}, \ n \geq 1$

(ii) $B \geq V_1 \geq V_2 \geq \ldots \geq 0$, for some constant $B$.

Prove that for $\lambda > 0$

$$P(\max_{1 \leq j \leq n} V_j X_j > \lambda) \leq \lambda^{-1} \sum_{j=1}^{n} E[V_j (X_j - X_{j-1})].$$

5. Prove Dubins’ inequality. If $(X_n)$ is a positive martingale then the number $U$ of upcrossings of $[a, b]$ satisfies

$$P(U \geq k) \leq (a/b)^k E\min(X_0/a, 1).$$

[if you follow sketch in Durrett then prove the quoted exercise]
205A Homework #11, due Tuesday 22 November.

In each question, there is some fixed filtration \((\mathcal{F}_n)\) with respect to which martingales are defined.

1. Let \((X_n)\) be a submartingale such that \(\sup_n X_n < \infty\) a.s. and \(\mathbb{E}[\sup_n (X_n - X_{n-1})^+] < \infty\). Show that \(X_n\) converges a.s.

2. For a sequence \((A_n)\) of events, show that
\[
\sum_{n=2}^\infty P(A_n|\cap_{m=1}^{n-1} A_m^c) = \infty \implies P(\cup_{m=1}^\infty A_m) = 1.
\]

3. Let \((X_n)\) be a martingale and write \(\Delta_n = X_n - X_{n-1}\), Suppose that \(b_m \uparrow \infty\) and \(\sum_{m=1}^\infty b_m^{-2} E\Delta_m^2 < \infty\). Prove that \(X_n/b_n \to 0\) a.s.

4. Let \((X_n)\) be a martingale with \(\sup_n \mathbb{E}|X_n| < \infty\). Show that there is a representation \(X_n = Y_n - Z_n\) where \((Y_n)\) and \((Z_n)\) are non-negative martingales such that \(\sup_n \mathbb{E} Y_n < \infty\) and \(\sup_n \mathbb{E} Z_n < \infty\).

5. Let \((X_n)\) be adapted to \((\mathcal{F}_n)\) with \(0 \leq X_n \leq 1\). Let \(\alpha, \beta > 0\) with \(\alpha + \beta = 1\). Suppose \(X_0 = x_0\) and
\[
P(X_{n+1} = \alpha + \beta X_n|\mathcal{F}_n) = X_n, \quad P(X_{n+1} = \beta X_n|\mathcal{F}_n) = 1 - X_n.
\]
Show that \(X_n \to X_\infty\) a.s., where \(P(X_\infty = 1) = x_0\) and \(P(X_\infty = 0) = 1 - x_0\).

6. Suppose \(\mathcal{F}_n \uparrow \mathcal{F}_\infty\) and \(Y_n \to Y_\infty\) in \(L^1\). Show that \(\mathbb{E}(Y_n|\mathcal{F}_n) \to \mathbb{E}(Y_\infty|\mathcal{F}_\infty)\) in \(L^1\).

7. Let \(S_n\) be the total assets of an insurance company at the end of year \(n\). Suppose that in year \(n\) the company receives premiums of \(c\) and pays claims totaling \(\xi_n\), where \(\xi_n\) are independent with \(\text{Normal}(\mu, \sigma^2)\) distribution, where \(0 < \mu < c\). The company is ruined if its assets fall to 0 or below. Show
\[
P(\text{ruin}) \leq \exp\left(-2(c - \mu)S_0/\sigma^2\right).
\]