205A Homework #1, due Tuesday 6 September.

1. [Bill. 2.4] Let \( F_n \) be classes of subsets of \( S \). Suppose each \( F_n \) is a field, and \( F_n \subset F_{n+1} \) for \( n = 1, 2, \ldots \). Define \( F = \bigcup_{n=1}^{\infty} F_n \). Show that \( F \) is a field. Give an example to show that, if each \( F_n \) is a \( \sigma \)-field, then \( F \) need not be a \( \sigma \)-field.

2. [Bill. 2.5(b)] Given a non-empty collection \( \mathcal{A} \) of sets, we defined \( F(\mathcal{A}) \) as the intersection of all fields containing \( \mathcal{A} \). Show that \( F(\mathcal{A}) \) is the class of sets of the form \( \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} A_{ij} \), where for each \( i \) and \( j \) either \( A_{i,j} \in \mathcal{A} \) or \( A_{ij}^c \in \mathcal{A} \), and where the \( m \) sets \( \bigcap_{j=1}^{n_i} A_{ij}, \ 1 \leq i \leq m \) are disjoint.

3. [Bill. 2.8] Suppose \( B \in \sigma(\mathcal{A}) \), for some collection \( \mathcal{A} \) of subsets. Show there exists a countable subcollection \( \mathcal{A}_B \) of \( \mathcal{A} \) such that \( B \in \sigma(\mathcal{A}_B) \).

4. Show that the Borel \( \sigma \)-field on \( \mathbb{R}^d \) is the smallest \( \sigma \)-field that makes all continuous functions \( f : \mathbb{R}^d \to \mathbb{R} \) measurable.

5. [Durr. 1.3.5] A function \( f : \mathbb{R}^d \to \mathbb{R} \) is lower semicontinuous (l.s.c.) if \( \liminf_{y \to x} f(y) \geq f(x) \) for all \( x \). A function is upper semicontinuous (u.s.c.) if \( \limsup_{y \to x} f(y) \leq f(x) \) for all \( x \). Show that, if \( f \) is l.s.c. or u.s.c., then \( f \) is measurable.
1. [similar Bill. 2.15] Let $\mathcal{B}$ be the Borel subsets of $\mathbb{R}$. For $B \in \mathcal{B}$ define

$$
\mu(B) =
\begin{cases}
1 & \text{if } (0, \varepsilon) \subset B \text{ for some } \varepsilon > 0 \\
0 & \text{if not}
\end{cases}
$$

(a) Show that $\mu$ is not finitely additive on $\mathcal{B}$.
(b) Show that $\mu$ is finitely additive but not countably additive on the field $\mathcal{B}_0$ of finite disjoint unions of intervals $(a, b]$.

2. Show that, in the definition of “a probability measure $\mu$ on a measurable space $(S, \mathcal{S})$”, we may replace “countably additive” by “finitely additive, and satisfies

$$
\text{if } A_n \downarrow \phi \text{ then } \mu(A_n) \to 0.
$$

3. [similar Durr. A.1.1] Give an example of a measurable space $(S, \mathcal{S})$, a collection $\mathcal{A}$ and probability measures $\mu$ and $\nu$ such that

(i) $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$
(ii) $S = \sigma(\mathcal{A})$
(iii) $\mu \neq \nu$.

Note: this can be done with $S = \{1, 2, 3, 4\}$

4. [similar Durr. Lemma A.2.1] Let $\mu$ be a probability measure on $(S, \mathcal{S})$, where $\mathcal{S} = \sigma(\mathcal{F})$ for a field $\mathcal{F}$. Show that for each $B \in \mathcal{S}$ and $\varepsilon > 0$ there exists $A \in \mathcal{F}$ such that $\mu(B \Delta A) < \varepsilon$.

5. Let $g : [0, 1] \to \mathbb{R}$ be integrable w.r.t. Lebesgue measure. Let $\varepsilon > 0$. Show that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$
\int |f(x) - g(x)| \, dx \leq \varepsilon.
$$
1. Use the monotone convergence theorem to prove the following.
   (i) If $X_n \geq 0$, $X_n \downarrow X$ a.s. and $EX_n < \infty$ for some $n$ then $EX_n \rightarrow EX$.
   (ii) If $E|X| < \infty$ then $E|X|1_{(|X|>n)} \rightarrow 0$ as $n \rightarrow \infty$.
   (iii) If $E|X_1| < \infty$ and $X_n \uparrow X$ a.s. then either $EX_n \uparrow EX \leq \infty$ or else $EX_n \uparrow \infty$ and $E|X| = \infty$.
   (iv) If $X$ takes values in the non-negative integers then
   $$EX = \sum_{n=1}^{\infty} P(X \geq n).$$

2. (i) For a counting r.v. $X = \sum_{i=1}^{n} 1_{A_i}$, give a formula for the variance of $X$ in terms of the probabilities $P(A_i)$ and $P(A_i \cap A_j)$, $i \neq j$.
   (ii) If $k$ balls are put at random into $n$ boxes, what is the variance of $X = \text{number of empty boxes}$?

3. (i) Suppose $EX = 0$ and $\text{var}(X) = \sigma^2 < \infty$. Prove
   $$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}, \quad a > 0.$$  
   (ii) Suppose $X \geq 0$ and $EX^2 < \infty$. Prove
   $$P(X > 0) \geq \frac{(EX)^2}{EX^2}.$$  

4. Chebyshev’s other inequality.
   Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and increasing functions. Prove that, for any r.v. $X$,
   $$E(f(X)g(X)) \geq (Ef(X))(Eg(X)).$$
[In other words, $f(X)$ and $g(X)$ are positively correlated. This is intuitively obvious, but a little tricky to prove. Hint: consider an independent copy $Y$ of $X$. For this and the next question you may need the product rule for expectations of independent r.v.s]

5. Let $X$ have Poisson($\lambda$) distribution and let $Y$ have Poisson($2\lambda$) distribution.
   (i) Prove $P(X \geq Y) \leq \exp(-(3 - \sqrt{8})\lambda)$ if $X$ and $Y$ are independent.
   (ii) Find constants $A < \infty$, $c > 0$, not depending on $\lambda$, such that, without assuming independence, $P(X \geq Y) \leq A \exp(-c\lambda)$. 


1. Monte Carlo integration [cf. Durr. 2.2.3] Let \( f : [0,1] \to \mathbb{R} \) be such that \( \int_0^1 f^2(x) \, dx < \infty \). Let \((U_i)\) be i.i.d. Uniform(0,1). Let

\[
D_n := n^{-1} \sum_{i=1}^{n} f(U_i) - \int_0^1 f(x) \, dx.
\]

(i) Use Chebyshev’s inequality to bound \( P(|D_n| > \varepsilon) \).
(ii) Show this bound remains true if the \((U_i)\) are only pairwise independent.

2. Let \( X \geq 0 \) and \( Y \geq 0 \) be independent r.v.’s with densities \( f \) and \( g \). Calculate the densities of \( XY \) and of \( X/Y \).

Note: this is just to remind you of “undergraduate” results.

3. [Durr. 2.2.2.] Let \((X_i)\) be r.v.’s with \( EX_i = 0 \) and \( EX_i X_j \leq r(j-i), 1 \leq i \leq j < \infty \), where \( r(n) \) is a deterministic sequence with \( r(n) \to 0 \) as \( n \to \infty \). Prove that \( n^{-1} \sum_{i=1}^{n} X_i \to 0 \) in probability.

4. [Durr. 2.3.11] Suppose events \( A_n \) satisfy \( P(A_n) \to 0 \) and

\[
\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty.
\]

Prove that \( P(A_n \text{ occurs infinitely often}) = 0 \).

5. (a) Let \( Z \) have standard Normal distribution. Show

\[
P(Z > z) \sim z^{-1}(2\pi)^{-1/2} \exp(-z^2/2) \text{ as } z \to \infty.
\]

(b) Let \((Z_1, Z_2, \ldots)\) be independent with standard Normal distribution. Find constants \( c_n \to \infty \) such that

\[
\limsup_n Z_n/c_n = 1 \text{ a.s.}
\]
205A Homework #5, due Tuesday 4 October.

1. Let $(X_n)$ be i.i.d. with $E|X_1| < \infty$. Let $M_n = \max(X_1, \ldots, X_n)$. Prove that $n^{-1}M_n \to 0$ a.s.

2. [Durr. 2.3.2] Let $0 \leq X_1 \leq X_2 \leq \ldots$ be r.v.'s such that $EX_n \sim an^\alpha$ and $\text{var}(X_n) \leq Bn^\beta$, where $0 < a, B < \infty$ and $0 < \beta < 2\alpha < \infty$. Prove that $n^{-\alpha}X_n \to a$ a.s.

3. Prove that the following are equivalent.
   (i) $X_n \to X$ in probability.
   (ii) There exist $\varepsilon_n \downarrow 0$ such that $P(|X_n - X| > \varepsilon_n) \leq \varepsilon_n$.
   (iii) $E \min(|X_n - X|, 1) \to 0$.


5. Prove the deterministic lemma we used in the proof of the Glivenko-Cantelli Theorem.

   Lemma. If $F_1, F_2, \ldots, F$ are distribution functions and
   (i) $F_n(x) \to F(x)$ for each rational $x$
   (ii) $F_n(x) \to F(x)$ and $F_n(x^-) \to F(x^-)$ for each atom $x$ of $F$
   then $\sup_x |F_n(x) - F(x)| \to 0$. 

1. [Durr. 2.5.9] Let $(X_i)$ be independent, $S_n = \sum_{i=1}^{n} X_i$, $S_n^* = \max_{i \leq n} |S_i|$. Prove that

$$P(S_n^* > 2a) \leq \frac{P(|S_n| > a)}{\min_{j \leq n} P(|S_n - S_j| \leq a)} , \quad a > 0.$$ 

[Hint. If $|S_j| > 2a$ and $|S_n - S_j| \leq a$ then $|S_n| > a$.]

2. [Durr. 2.5.10 and 11] In the setting of the previous question, prove (i) if $\lim_{n \to \infty} S_n$ exists in probability then the limit exists a.s. (ii) if the $(X_i)$ are identically distributed and if $n^{-1}S_n \to 0$ in probability then $n^{-1}\max_{m \leq n} S_m \to 0$ in probability.

3. [cf. Durr 2.2.8] Let $(X_i)$ be i.i.d. taking values in \{-1, 1, 3, 7, 15, \ldots\}, such that

$$P(X_1 = 2^k - 1) = \frac{1}{k(k+1)2^k}, \quad k \geq 1$$

(which implicitly specifies $P(X_1 = -1)$).

(a) Show $EX_1 = 0$.

(b) Show that for all $\alpha < 1$,

$$P\left(S_n < -\frac{\alpha n}{\log_2 n}\right) \to 1.$$ 

Comment. This is sometimes described as “an unfair, fair game”. It shows that the conclusions of the SLLN and the “recurrence of sums” theorem can’t be strengthened much.
1. Suppose $S$ and $T$ are stopping times. Are the following necessarily stopping times? Give proof or counter-example.
   (a) $\min(S,T)$
   (b) $\max(S,T)$
   (c) $S + T$.

2. Let $(X_i)$ be i.i.d. with $EX_i^2 < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Let $T$ be a bounded stopping time. Is it true in general that
   $$\text{var}(S_T) = (\text{var}(X_1))(ET)?$$
   If not, is it true in the special case $EX_1 = 0$?

3. Let $(X_i)$ be a sequence of random variables, and let $\mathcal{T}$ be its tail $\sigma$-field. Let $S_n = \sum_{i=1}^{n} X_i$. Let $b_n \uparrow \infty$ be constants. Which of the following events must be in $\mathcal{T}$? Give proof or counter-example.
   (i) $\{X_n \to 0\}$
   (ii) $\{S_n \text{ converges}\}$
   (iii) $\{X_n > b_n \text{ infinitely often}\}$
   (iv) $\{S_n > b_n \text{ infinitely often}\}$
   (v) $\left\{\frac{\sqrt{\sum_{i=1}^{n} X_i^2}}{S_n} \to 0\right\}$.

4. Let $S_n = \sum_{i=1}^{n} X_i$, where $(X_i)$ are i.i.d. with exponential(1) distribution. Use the large deviation theorem to get explicit limits for
   $$n^{-1} \log P(n^{-1} S_n \geq a), \ a > 1 \text{ and } n^{-1} \log P(n^{-1} S_n \leq a), \ a < 1.$$}

5. Oriented first passage percolation. Consider the lattice quadrant $\{(i,j): i,j \geq 0\}$ with directed edges $(i,j) \to (i+1,j)$ and $(i,j) \to (i,j+1)$. Associate to each edge $e$ an exponential(1) r.v. $X_e$, independent for different edges. For each directed path $\pi$ of length $d$ started at $(0,0)$, let $S_\pi = \sum$ edges $e$ in path $X_e$. Let $H_d$ be the minimum of $S_\pi$ over all such paths $\pi$ of length $d$. It can be shown that $d^{-1} H_d \to c$ a.s., for some constant $c$. Give explicit upper and lower bounds on $c$.
   [Hint: use result of previous question for lower bound.]
205A Homework #8, due Tuesday 1 November.

[Theorem 7 and Corollary 8 refer to the notes linked from the “week 8” row of the schedule.]

1. Suppose probability measures satisfy $\pi \ll \nu \ll \mu$. Show that

$$\frac{d\pi}{d\mu} = \frac{d\pi}{d\nu} \times \frac{d\nu}{d\mu}.$$ 

2. In the setting of Theorem 7 [hard part], where $S_2$ is nice, show that $Q$ is unique in the following sense. If $Q^*$ is another conditional probability kernel for $\mu$, then

$$\mu_1 \{ x : Q^*(x, B) = Q(x, B) \text{ for all } B \in S_2 \} = 1.$$ 

3. Let $F$ be a distribution function. Let $c > 0$. Find a simple formula for

$$\int_{-\infty}^{\infty} (F(x + c) - F(x)) \, dx.$$ 

4. In the proof of Corollary 8 we used the inverse distribution function

$$f(x, u) = \inf \{ y : u \leq Q(x, (\infty, y]) \}$$

associated with the kernel $Q$. Show that $f$ is product measurable.

5. Given a triple $(X_1, X_2, X_3)$, we can define 3 p.m.’s $\mu_{12}, \mu_{13}, \mu_{23}$ on $\mathbb{R}^2$ by

$$\mu_{ij} \text{ is the distribution of } (X_i, X_j). \quad (1)$$

These p.m.’s satisfy a consistency condition:

the marginal distribution $\mu_1$ obtained from $\mu_{12}$ must coincide
with the marginal obtained from $\mu_{13}$, and similarly for $\mu_2$ and $\mu_3$. \quad (2)

Give an example to show that the converse is false. That is, give an example of $\mu_{12}, \mu_{13}, \mu_{23}$ satisfying (2) but for which there does not exist a triple $(X_1, X_2, X_3)$ satisfying (1).
1. Let $X, Y$ be random variables, and suppose $Y$ is measurable with respect to some sub-$\sigma$-field $\mathcal{G}$. Let $\mu(\omega, \cdot)$ be a regular conditional distribution for $X$ given $\mathcal{G}$. Prove that, for bounded measurable $h$,

$$E(h(X,Y)|\mathcal{G})(\omega) = \int h(x,Y(\omega)) \mu(\omega, dx) \text{ a.s.}$$

2. For $i = 1, 2$ let $X_i$ be a r.v. defined on $(\Omega, \mathcal{F}, P)$ taking values in $(S_i, S_i)$. Let $\mathcal{G}$ be a sub-$\sigma$-field of $\mathcal{F}$. Prove that assertions (a),(b) and (c) below are equivalent. When these assertions hold, we say call $X_1$ and $X_2$ are conditionally independent given $\mathcal{G}$.

(a) $P(X_1 \in A_1, X_2 \in A_2|\mathcal{G}) = P(X_1 \in A_1|\mathcal{G})P(X_2 \in A_2|\mathcal{G})$ for all $A_i \in S_i$.

(b) $E(h_1(X_1)h_2(X_2)|\mathcal{G}) = E(h_1(X_1)|\mathcal{G})E(h_2(X_2)|\mathcal{G})$ for all bounded measurable $h_i : S_i \rightarrow \mathbb{R}$.

(c) $E(h_1(X_1)|\mathcal{G}, X_2) = E(h_1(X_1)|\mathcal{G})$ for all bounded measurable $h_1 : S_1 \rightarrow \mathbb{R}$.

3. Suppose $X$ and $Y$ are conditionally independent given $Z$. Suppose $X$ and $Z$ are conditionally independent given $\mathcal{F}$, where $\mathcal{F} \subseteq \sigma(Z)$. Prove that $X$ and $Y$ are conditionally independent given $\mathcal{F}$.

4. Let $(X_n)$ and $(Y_n)$ be submartingales w.r.t. $(\mathcal{F}_n)$. Show that $(X_n + Y_n)$ and that $(\max(X_n, Y_n))$ are also submartingales w.r.t. $(\mathcal{F}_n)$.

5. Give an example where

$(X_n)$ is a submartingale w.r.t. $(\mathcal{F}_n)$

$(Y_n)$ is a submartingale w.r.t. $(\mathcal{G}_n)$

$(X_n + Y_n)$ is not a submartingale w.r.t. any filtration.
1. Let $S_n = \sum_{i=1}^{n} \xi_i$, where the $(\xi_i)$ are independent, $E\xi_i = 0$ and var $\xi_i < \infty$. Let $s_n^2 = \sum_{i=1}^{n} \text{var} \xi_i$. So we know that $(S_n^2 - s_n^2)$ is a martingale. Suppose also that $|\xi_i| \leq K$ for some constant $K$. Show that

$$P\left( \max_{m \leq n} |S_m| < x \right) \leq s_n^{-2}(K + x)^2, \quad x > 0.$$

2. Let $(X_n)$ be a martingale with $X_0 = 0$ and $EX_n^2 < \infty$. Using the fact that $(X_n + c)^2$ is a submartingale, show that

$$P\left( \max_{m \leq n} X_m \geq x \right) \leq \frac{EX_n^2}{x^2 + EX_n^2}, \quad x > 0.$$

3. Let $(X_n)$ and $(Y_n)$ be martingales w.r.t. the same filtration with $E(X_n^2 + Y_n^2) < \infty$. Show that

$$EX_nY_n - EX_0Y_0 = \sum_{m=1}^{n} E(X_m - X_{m-1})(Y_m - Y_{m-1}).$$

4. Let $(X_n, F_n), n \geq 0$ be a positive submartingale with $X_0 = 0$. Let $V_n$ be random variables such that

(i) $V_n \in F_{n-1}, \ n \geq 1$
(ii) $B \geq V_1 \geq V_2 \geq \ldots \geq 0$, for some constant $B$.

Prove that for $\lambda > 0$

$$P\left( \max_{1 \leq j \leq n} V_j X_j > \lambda \right) \leq \lambda^{-1} \sum_{j=1}^{n} E[V_j (X_j - X_{j-1})].$$

5. Prove Dubins’ inequality. If $(X_n)$ is a positive martingale then the number $U$ of upcrossings of $[a, b]$ satisfies

$$P(U \geq k) \leq (a/b)^k E \min(X_0/a, 1).$$

[if you follow sketch in Durrett then prove the quoted exercise]
205A Homework #11, due Tuesday 22 November.

In each question, there is some fixed filtration $\mathcal{F}_n$ with respect to which martingales are defined.

1. Let $(X_n)$ be a submartingale such that $\sup_n X_n < \infty$ a.s. and $E \sup_n (X_n - X_{n-1})^+ < \infty$. Show that $X_n$ converges a.s.

2. For a sequence $(A_n)$ of events, show that
$$
\sum_{n=2}^{\infty} P(A_n | \cap_{m=1}^{n-1} A_m^c) = \infty \implies P(\cup_{m=1}^{\infty} A_m) = 1.
$$

3. Let $(X_n)$ be a martingale and write $\Delta_n = X_n - X_{n-1}$, Suppose that $b_m \uparrow \infty$ and $\sum_{m=1}^{\infty} b_m^{-2} E \Delta_m^2 < \infty$. Prove that $X_n / b_n \to 0$ a.s.

4. Let $(X_n)$ be a martingale with $\sup_n E|X_n| < \infty$. Show that there is a representation $X_n = Y_n - Z_n$ where $(Y_n)$ and $(Z_n)$ are non-negative martingales such that $\sup_n EY_n < \infty$ and $\sup_n EZ_n < \infty$.

5. Let $(X_n)$ be adapted to $(\mathcal{F}_n)$ with $0 \leq X_n \leq 1$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Suppose $X_0 = x_0$ and
$$
P(X_{n+1} = \alpha + \beta X_n | \mathcal{F}_n) = X_n, \quad P(X_{n+1} = \beta X_n | \mathcal{F}_n) = 1 - X_n.
$$
Show that $X_n \to X_\infty$ a.s., where $P(X_\infty = 1) = x_0$ and $P(X_\infty = 0) = 1 - x_0$.

6. Suppose $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y_\infty$ in $L^1$. Show that $E(Y_n | \mathcal{F}_n) \to E(Y_\infty | \mathcal{F}_\infty)$ in $L^1$.

7. Let $S_n$ be the total assets of an insurance company at the end of year $n$. Suppose that in year $n$ the company receives premiums of $c$ and pays claims totaling $\xi_n$, where $\xi_n$ are independent with Normal$(\mu, \sigma^2)$ distribution, where $0 < \mu < c$. The company is ruined if its assets fall to 0 or below. Show
$$
P(\text{ruin}) \leq \exp(-2(c - \mu)S_0 / \sigma^2).
$$