5.8 Exercises

where \(|A|\) denotes the number of elements in \(A\).
(b) Obtain a lower bound for the mean number of flips required until all \(2^k\) patterns of length \(k\) have appeared when a fair coin is repeatedly flipped.

7. Consider a Markov chain whose state space is the set of non-negative integers. Suppose its transition probabilities are given by

\[ P_{0,i} = p_i, \quad i \geq 0, \quad P_{i,i-1} = 1, \quad i > 0 \]

where \(\sum_i i p_i < \infty\). Find the limiting probabilities for this Markov chain.

8. Consider a Markov chain with states 0, 1, \ldots, \(N\) and transition probabilities

\[ P_{0N} = 1, \quad P_{ij} = 1/i, \quad i > 0, \quad j < i \]

That is, from state 0 the chain always goes to state \(N\), and from state \(i > 0\) it is equally likely to go to any lower numbered state. Find the limiting probabilities of this chain.

9. Consider a Markov chain with states 0, 1, \ldots, \(N\) and transition probabilities

\[ P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, \ldots, N-1 \]

\[ P_{0,0} = P_{N,N} = 1 \]

Suppose that \(X_0 = i\), where \(0 < i < N\). Argue that, with probability 1, the Markov chain eventually enters either state 0 or \(N\). Derive the probability it enters state \(N\) before state 0. This is called the gambler's ruin probability.

10. If \(X_n\) is a stationary ergodic Markov chain, show that \(X_1, X_2, \ldots\) is an ergodic sequence.

11. Suppose \(X_1, X_2, \ldots\) are iid integer valued random variables with \(M_n = \max_{1 \leq i \leq n} X_i\). Is \(M_n\) necessarily a Markov chain? If yes, give its transition probabilities; if no, construct a counterexample.
12. Suppose $X_n$ is a finite state stationary Markov chain, and let $T = \min\{n > 0 : X_n = X_0\}$. Compute $E[T]$.

13. Hastings-Metropolis algorithm. Given an irreducible Markov chain with transition probabilities $P_{ij}$ and any positive probability vector $\{\pi_i\}$ for these states, show that the Markov chain with transition probabilities $Q_{ij} = \min(P_{ij}, \pi_j P_{ji}/\pi_i)$ if $i \neq j$ and $Q_{ii} = 1 - \sum_{j \neq i} Q_{ij}$, is time reversible and has stationary distribution $\{\pi_i\}$.

14. Consider a time reversible Markov chain with transition probabilities $P_{ij}$ and stationary probabilities $\pi_i$. If $A$ is a set of states of this Markov chain, then we define the $A$-truncated chain as being a Markov chain whose set of states is $A$ and whose transition probabilities $P_{ij}^A$, $i, j \in A$, are given by

$$P_{ij}^A = \begin{cases} P_{ij} & \text{if } j \neq i \\ P_{ii} + \sum_{k \notin A} P_{ik} & \text{if } j = i \end{cases}$$

If this truncated chain is irreducible, show that it is time reversible, with stationary probabilities

$$\pi_i^A = \frac{\pi_i}{\sum_{j \in A} \pi_j}, \quad i \in A$$

15. A collection of $M$ balls are distributed among $m$ urns. At each stage one of the balls is randomly selected, taken from whatever urn it is in and then randomly placed in one of the other $m - 1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, n_2, \ldots, n_m)$ where $n_i$ is the number of balls in urn $i$. Show that this Markov chain is time reversible and find its stationary probabilities.

16. Let $Q$ be an irreducible symmetric transition probability matrix on the states $1, \ldots, n$. That is,

$$Q_{ij} = Q_{ji}, \quad i, j = 1, \ldots, n$$

Let $b_i, i = 1, \ldots, n$ be specified positive numbers, and consider
16. Let \( Z_0, Z_1, Z_2, \ldots \) be a realization of a finite-state ergodic chain. If we sample every \( k \)th epoch, then show (a) that the sampled chain \( Z_0, Z_k, Z_{2k}, \ldots \) is ergodic, (b) that it possesses the same equilibrium distribution as the original chain, and (c) that it is reversible if the original chain is. Thus, based on the ergodic theorem, we can estimate theoretical means by sample averages using only every \( k \)th epoch of the original chain.

17. Take three numbers \( x_1, x_2, \) and \( x_3 \) and form the successive running averages \( x_n = (x_{n-3} + x_{n-2} + x_{n-1})/3 \) starting with \( x_4 \). Prove that

\[
\lim_{n \to \infty} x_n = \frac{x_1 + 2x_2 + 3x_3}{6}
\]

18. Consider a random walk on the integers \( \{0, 1, \ldots, n\} \). States 0 and \( n \) are absorbing in the sense that \( p_{00} = p_{nn} = 1 \). If \( i \) is a transient state, then the transition probabilities \( p_{i,i+1} = p \) and \( p_{i,i-1} = q \), where \( p + q = 1 \). Verify that the hitting probabilities are

\[
h_{in} = \begin{cases} 
\left( \frac{q}{p} \right)^{i-1}, & p \neq q \\
\frac{i}{n}, & p = q
\end{cases}
\]

and the mean hitting times are

\[
t_i = \begin{cases} 
\frac{n}{p-q} \left( \frac{q}{p} \right)^{i-1} - \frac{i}{p-q}, & p \neq q \\
i(n-i), & p = q.
\end{cases}
\]

(Hint: First argue that

\[
t_i = 1 + \sum_{k=1}^{m} p_{ik} t_k
\]

in the notation of Section 7.5.)

19. An acceptance function \( a : (0, \infty) \to [0, 1] \) satisfies the functional identity \( a(x) = xa(1/x) \). Prove that the detailed balance condition

\[
\pi_i q_{ij} a_{ij} = \pi_j q_{ji} a_{ji}
\]

holds if the acceptance probability \( a_{ij} \) is defined by

\[
a_{ij} = a \left( \frac{\pi_j q_{ji}}{\pi_i q_{ij}} \right)
\]

in terms of an acceptance function \( a(x) \). Check that the Barker function \( a(x) = x/(1 + x) \) qualifies as an acceptance function and that any acceptance function is dominated by the Metropolis acceptance function in the sense that \( a(x) \leq \min\{x, 1\} \) for all \( x \).