3.7 Exercises

1. For $\mathcal{F} = \{\phi, \Omega\}$, show that $E[X|\mathcal{F}] = E[X]$.

2. Give the proof of Proposition 3.2 when $X$ and $Y$ are jointly continuous.

3. If $E[|X_i|] < \infty$, $i = 1, \ldots, n$, show that

$$E\left[\sum_{i=1}^{n} X_i | \mathcal{F}\right] = \sum_{i=1}^{n} E[X_i | \mathcal{F}]$$

4. Prove that if $f$ is a convex function, then

$$E[f(X)|\mathcal{F}] \geq f(E[X|\mathcal{F}])$$

provided the expectations exist.

5. Let $X_1, X_2, \ldots$, be independent random variables with mean 1. Show that $Z_n = \prod_{i=1}^{n} X_i$, $n \geq 1$, is a martingale.

6. If $E[X_{n+1}|X_1, \ldots, X_n] = a_n X_n + b_n$ for constants $a_n, b_n, n \geq 0$, find constants $A_n, B_n$ so that $Z_n = A_n X_n + B_n, n \geq 0$, is a martingale with respect to the filtration $\sigma(X_0, \ldots, X_n)$.

7. Consider a population of individuals as it evolves over time, and suppose that, independent of what occurred in prior generations, each individual in generation $n$ independently has $j$ offspring with probability $p_j, j \geq 0$. The offspring of individuals of generation $n$ then make up generation $n+1$. Assume that $m = \sum_j j p_j < \infty$. Let $X_n$ denote the number of individuals in generation $n$, and define a martingale related to $X_n, n \geq 0$. The process $X_n, n \geq 0$ is called a branching process.

8. Suppose $X_1, X_2, \ldots$, are independent and identically distributed random variables with mean zero and finite variance $\sigma^2$. If $T$ is a stopping time with finite mean, show that

$$\text{Var}\left(\sum_{i=1}^{T} X_i\right) = \sigma^2 E(T).$$
9. Suppose $X_1, X_2, \ldots$, are independent and identically distributed mean 0 random variables which each take value $+1$ with probability $1/2$ and take value $-1$ with probability $1/2$. Let $S_n = \sum_{i=1}^n X_i$. Which of the following are stopping times? Compute the mean for the ones that are stopping times.
   (a) $T_1 = \min\{i \geq 5 : S_i = S_{i-5} + 5\}$
   (b) $T_2 = T_1 - 5$
   (c) $T_3 = T_2 + 10$.

10. Consider a sequence of independent flips of a coin, and let $p_h$ denote the probability of a head on any toss. Let $A$ be the hypothesis that $p_h = a$ and let $B$ be the hypothesis that $p_h = b$, for given values $0 < a, b < 1$. Let $X_i$ be the outcome of flip $i$, and set
   $$Z_n = \frac{P(X_1, \ldots, X_n | A)}{P(X_1, \ldots, X_n | B)}$$
   If $p_h = b$, show that $Z_n, n \geq 1$, is a martingale having mean 1.

11. Let $Z_n, n \geq 0$ be a martingale with $Z_0 = 0$. Show that
   $$E[Z_n^2] = \sum_{i=1}^n E[(Z_i - Z_{i-1})^2]$$

12. Consider an individual who at each stage, independently of past movements, moves to the right with probability $p$ or to the left with probability $1 - p$. Assuming that $p > 1/2$ find the expected number of stages it takes the person to move $i$ positions to the right from where she started.

13. In Example 3.19 obtain bounds on $p$ when $\theta < 0$.

14. Use Wald’s equation to approximate the expected time it takes a random walk to either become as large as $a$ or as small as $-b$, for positive $a$ and $b$. Give the exact expression if $a$ and $b$ are integers, and at each stage the random walk either moves up 1 with probability $p$ or moves down 1 with probability $1 - p$.

15. Consider a branching process that starts with a single individual. Let $\pi$ denote the probability this process eventually dies out. With $X_n$ denoting the number of individuals in generation $n$, argue that $\pi^{X_n}, n \geq 0$, is a martingale.

3.7 Exercises

16. Given $X_1, X_2, \ldots$, suppose for all $E[X_i X_j] = 0$ if $i \neq j$.

17. Suppose $n$ random variables are equal to 1 connecting all

18. Let $X_1, X_2, \ldots$, be discrete random variables on the tern $0, 0, 0, 0, 0$.

19. Repeat Example 3.3, but in a different setting.

20. Let $Z_n, n \geq 1$, have $P(Z_j \leq v_j, \forall j \geq 0$. Prove the Kolmogorov

21. Consider a gambler who at each stage, independently of past movements, moves to the right with probability $p$ or to the left with probability $1 - p$. Assuming that $p > 1/2$ find the expected number of stages it takes the person to move $i$ positions to the right from where she started.

22. What is the in to the scenario?

23. Three gamblers: In each round, a random selection of individuals is made and when the round ends gambling takes place. Three players play (a) Compute
then the sum \( \sum_{i=1}^{n} c_i^2 \) figuring in Proposition 10.5.1 can be bounded by
\[
\sum_{i=1}^{n} c_i^2 \leq (2\sqrt{2})^2 + 4^2 \sum_{i=1}^{n-1} \frac{1}{n-i} \leq 8 + 16(\ln n + 1).
\]
This in turn translates into the Azuma-Hoeffding bound
\[
\Pr[|D_n - E(D_n)| \geq \lambda] \leq 2e^{-\frac{\lambda^2}{4n^2}}.
\]

10.6 Problems

1. Define the random variables \( Y_n \) inductively taking by \( Y_0 = 1 \) and \( Y_{n+1} \) to be uniformly distributed on the interval \((0, Y_n)\). Show that the sequence \( X_n = 2^n Y_n \) is a martingale.

2. An urn contains \( b \) black balls and \( w \) white balls. Each time we randomly withdraw a ball, we replace it by \( c + 1 \) balls of the same color. Let \( X_n \) be the fraction of white balls after \( n \) draws. Demonstrate that \( X_n \) is a martingale.

3. Let \( Y_1, Y_2, \ldots \) be a sequence of independent random variables with zero means and common variance \( \sigma^2 \). Let \( X_n = Y_1 + \cdots + Y_n \), then show that \( X_n^2 - n \sigma^2 \) is a martingale.

4. Let \( Y_1, Y_2, \ldots \) be a sequence of i.i.d. random variables with common moment generating function \( M(t) = E(e^{tY_1}) \). Prove that
\[
X_n = M(t)^{-n} e^{t(Y_1 + \cdots + Y_n)}
\]
is a martingale whenever \( M(t) < \infty \).

5. Let \( Y_n \) be a finite-state, discrete-time Markov chain with transition matrix \( P = (p_{ij}) \). If \( v \) is a column eigenvector for \( P \) with nonzero eigenvalue \( \lambda \), then verify that \( X_n = \lambda^{-n} v Y_n \) is a martingale, where \( v Y_n \) is coordinate \( Y_n \) of \( v \).

6. Suppose \( Y_n \) is the number of particles at the \( n \)th generation of a branching process. If \( s_{\infty} \) is the extinction probability, prove that \( X_n = s_{\infty}^n \) is a martingale. (Hint: If \( Q(s) \) is the progeny generating function, then \( Q(s_{\infty}) = s_{\infty} \).

7. In Example 10.3.2, show that \( \text{Var}(X_{\infty}) = \frac{s^2}{\mu(\alpha-1)} \) by differentiating equation (10.12) twice. This result is consistent with the mean square convergence displayed in equation (10.9).
8. In Example 10.3.2, show that the fractional linear transformation

$$L_{\infty}(t) = \frac{pt - p + q}{qt - p + q}$$

solves equation (10.12) when $Q(s) = \frac{p}{1-qs}$ and $\mu = \frac{q}{p}$. Also verify equation (10.13).

9. In Example 10.2.2, suppose that each $Y_i$ is equally likely to assume the values $\frac{1}{2}$ and $\frac{3}{2}$. Show that $\prod_{i=1}^{\infty} Y_i \equiv 0$, but $\prod_{i=1}^{\infty} E(Y_i) = 1$ [19]. (Hint: Apply the strong law of large numbers to the sequence in $Y_n$.)

10. Given $X_0 = \mu \in (0,1)$, define $X_n$ inductively by

$$X_{n+1} = \begin{cases} \alpha + \beta X_n, & \text{with probability } X_n \\ \beta X_n, & \text{with probability } 1 - X_n. \end{cases}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Prove that $X_n$ is a martingale with (a) $X_n \in (0,1)$, (b) $E(X_n) = \mu$, and (c) $\text{Var}(X_n) = [1-(1-\alpha^2)^n]\mu(1-\mu)$. Also prove that Proposition 10.3.2 implies that $\lim_{n \to \infty} X_n = X_{\infty}$ exists with $E(X_{\infty}) = \mu$ and $\text{Var}(X_{\infty}) = \mu(1-\mu)$. (Hint: Derive a recurrence relation for $\text{Var}(X_{n+1})$ by conditioning on $X_n$.)

11. Let $S_n = X_1 + \cdots + X_n$ be a symmetric random walk on the integers $\{-a, \ldots, b\}$ starting at $S_0 = 0$. For the stopping time

$$T = \min\{n : S_n = -a \text{ or } S_n = b\},$$

prove that $\text{Pr}(S_T = b) = a/(a + b)$ by considering the martingale $S_n$ and that $E(T) = ab$ by considering the martingale $S_n^2 - n$. (Hints: Apply Proposition 10.4.1 and Problem 3.)

12. In the Wright-Fisher model of Example 10.2.6, show that

$$Z_n = \frac{X_n(1 - X_n)}{\left(1 - \frac{1}{2m}\right)^n}$$

is a martingale with values on $[0,1]$. In view of Proposition 10.3.2, $\lim_{n \to \infty} Z_n = Z_{\infty}$ exists. Thus, $X_n(1 - X_n) \approx \left(1 - \frac{1}{2m}\right)^n Z_{\infty}$ for $n$ large. In other words, $X_n$ approaches either 0 or 1 at rate $1 - \frac{1}{2m}$.

13. In Proposition 10.4.1 suppose we can write either

$$X_n = B_n + I_n \quad \text{or} \quad X_n = B_n - I_n$$

for $n \leq T$, where $|B_n| \leq c$ and $0 \leq I_{n-1} \leq I_n$ when $n \leq T$. In other words for all times up to $T$, the $B$ process is bounded and the $I$ process is increasing. Show that $E(X_T) = \mu$ holds without making assumptions (b) and (c) of the proposition. (Hints: Show that $E(X_{T\wedge n}) = \mu$ for $T \wedge n = \min\{T, n\}$. Apply the bounded convergence theorem to $B_{T\wedge n}$ and the monotone convergence theorem to $I_{T\wedge n}$.)

14. Let $Y_1, \ldots, Y_n$ be $\epsilon$-cess probability $\mu$ (10.22) to Chebyshev's

when $\mu = 1/2$. W 3 favor Chebyshev's

15. Suppose that $v_i$, $Y_1, \ldots, Y_n$ be indi the two-point set that