Lecture 2: The Kelly criterion for favorable games: stock market investing for individuals

David Aldous

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Most adults drive/own a car
Few adults work in the auto industry.

By analogy
Most (middle class) adults will have savings/investments
Few adults work in the banking/finance industries.

In my list of 100 contexts where we perceive chance

(22) Risk and reward in equity ownership refers to the investor’s viewpoint;

(81) Short-term fluctuations of equity prices, exchange rates etc refers to the finance professional’s viewpoint.
Over the last 30 years there has been a huge increase in the use of sophisticated math models/algorithms in finance, and many “mathematical sciences” majors seek to go into careers in finance. There are many introductory textbooks, such as Capinski - Zastawniak *Mathematics for Finance: An Introduction to Financial Engineering*. and college courses such as IEOR221. But these represent first steps toward a professional career; not really relevant to today’s lecture.

Today’s topic is “the stock market from the investor’s viewpoint”. Best treatment is Malkiel’s classic book *A Random Walk Down Wall Street*. But instead of summarizing that book, I will focus on one aspect, and do a little math.
Almost everything you need to know in one slide . . . . .

- The future behavior of the stock market will be statistically similar to the past behavior is some respects but will be different in other respects – and we can’t tell which.

This is true but not helpful! So go in one of two directions.

1. Devise your investment strategy under the assumption that “the future will be statistically similar to the past”, recognizing this isn’t exactly true.

2. Decide (by yourself or advice from others) to believe that the future will be different in certain specific ways, and base your strategy on that belief.

Stock prices represent a consensus view of “true value”, like the spread in Football betting. Wall St makes money mostly by (2), selling advice or speculating with their own money. But there’s overwhelming empirical evidence (course project: survey the literature) that individual investors do not benefit from paying for this advice, so I will focus on (1). In particular I do not discuss choosing individual stocks.
Here is my “anchor” data.
[show IFA page]
Relating any mathematical theory to actual stock market investing raises many issues.

**Understanding the past.** The first is an issue you might not have expected:

> it is very hard to pin down a credible and useful number for the historical long-term average growth rate of stock market investments.

Over the 60 years 1950-2009 the S&P500 index rose at (geometric) average rate 7.2%. Aside from the (rather minor) point that we are using a particular index to represent the market what could possibly be wrong with using this figure? Well,

- it ignores dividends
- it ignores expenses
- it is sensitive to choice of start and end dates; starting in 1960 would make the figure noticably lower, whereas ending in 1999 would make it noticably higher.
- to interpret the figure we need to compare it to some alternative investment, by convention some “risk-free” investment.
- it ignores inflation
- it ignores taxes.
IFA and similar sites start out by trying to assess the individual’s subjective risk tolerance using a questionnaire. The site then suggests one of a range of 21 portfolios, represented on the slightly curved line in the figure. The horizontal axis shows standard deviation of annual return, (3% to 16%), and the vertical axis shows mean annual return (6% to 13%). Of course this must be historical data, in this case over the last 50 years. Notwithstanding the standard “past performance does not guarantee future results” legal disclaimer, the intended implication is that it is reasonable to expect similar performance in future.

Questions: (a) How does this relate to any theory? (b) Should you believe this predicts the actual future if you invested?

Answers: (a) The Kelly criterion says something like this curve must work. (b) Even if future statistically similar to past, any algorithm will “overfit” and be less accurate at predicting the future than the past.
Expectation and gambling.
Recalling some basic mathematical setup, write \( \mathbb{P}(\cdot) \) for probability and \( \mathbb{E}[\cdot] \) for expectation. Regarding gambling, any bet has (to the gambler) some random profit \( X \) (a loss being a negative profit), and we say that an available bet is (to the gambler) favorable if \( \mathbb{E}[X] > 0 \)
unfavorable if \( \mathbb{E}[X] < 0 \)
and fair if \( \mathbb{E}[X] = 0 \).
Note the word fair here has a specific meaning. In everyday language, the rules of team sports are fair in the sense of being the same for both teams, so the better team is more likely to win. For 1 unit bet on team B, that is a bet where you gain some amount \( b \) units if B wins but lose the 1 unit if B loses,

\[
\mathbb{E}[\text{profit}] = bp - (1 - p); \quad p = \mathbb{P}(B \text{ wins})
\]

and so to make the bet is fair we must have \( b = (1 - p)/p \). (Confusingly, mathematicians sometimes say “fair game” to mean each player has chance 1/2 to win, but this is sloppy language).
Several issues hidden beneath this terminology should be noted. Outside of games we usually don’t know probabilities, so we may not know whether a bet is favorable, aside from the common sense Sky Masterson principle that most bets offered to us will be unfavorable to us. The terminology comes from the law of large numbers fact that if one could repeat the same bet with the same stake independently, then in the long run one would make money on a favorable bet but lose money on an unfavorable bet. Such “long run” arguments ignore the issues of (rational or irrational) risk aversion and utility theory, which will be discussed later. In essence, we are imagining settings where your possible gains or losses are small, in your own perception.
Unfavorable bets.
Roughly speaking, there are two contexts in which we often encounter unfavorable bets. One concerns most activities we call *gambling*, e.g. at a casino, and the other concerns insurance. Regarding the former, mathematicians often say ridiculous things such as

*Gambling against the house at a casino is foolish, because the odds are against you and in the long run you will lose money.*

What’s wrong is the *because*. Saying

*Spending a day at Disneyland is foolish, because you will leave with less money than you started with*

is ridiculous, because people go to Disneyland for entertainment, and know they have to pay for entertainment. And the first quote is equally ridiculous. Casino gamblers may have irrational ideas about chance and luck, but in the U.S. they typically regard it as entertainment with a chance of winning, not as a plan to make money. So it’s worth being more careful and saying
Gambling against the house at a casino and expecting to make money is foolish, because the odds are against you and in the long run you will lose money.

The second context is that buying insurance is mathematically similar to placing an unfavorable bet – your expected gain is negative, because the insurance company wants to cover its costs and make a profit. But the whole point of buying insurance is risk aversion, so this needs to be treated in the setting of utility theory and psychology of probability (a later Lecture).
So where can I find a favorable bet?
The wiseacre answer “start your own casino or insurance company” is not so practical, but a variant of the latter is. For those who can, following the advice

*increase your insurance deductibles to the maximum you can comfortably afford to lose*

is a favorable bet, likely to save you money over a lifetime. In this lecture we consider investing in the stock market as mathematically similar to making a sequence of favorable bets (and letting your winnings ride). Exactly *why* one could consider this a favorable bet could be debated endlessly – standard economic theory asserts that investors need to be rewarded for taking risk rather than using alternative risk-free investments, while empiricists observe that, in countries without anti-capitalist revolutions, the historical performance of stock markets actually has been better than those alternatives.
**long term** versus **short term**.
In everyday language, a job which will only last six months is a short term job; someone who has worked for a company for fifteen years is a long term employee. Joining a softball team for a summer is a short term commitment; raising children is a long term commitment. We judge these matters relative to human lifetime; *long term* means some noticeable fraction of a lifetime.

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One of several possible notions of *long term* in financial matters is “the time span over which compounding has a noticeable effect”. Rather arbitrarily interpreting “noticeable effect” as “10% more” and taking the 7% interest rate, this suggests taking 8 years as the cut-off for *long term*. Being about 10% of a human lifetime, this fortuitously matches reasonably well the “noticeable fraction of a lifetime” criterion above. And indeed in matters pertaining to individuals, financial or otherwise, most writers use a cut-off between 5 and 10 years for “long term”.
The theme of this lecture is the nature of compounding when gains and losses are unpredictable.

The relevant arithmetic is **multiplication** not addition: a 20% gain followed by a 20% loss combine to a 4% loss, because
\[ 1.2 \times 0.8 = 0.96. \]

Let’s move on to some mathematics . . . . .
First let us make explicit the type of model used implicitly above. A “return” \( x = 0.2 \) or \( x = -0.2 \) in a year means a 20\% gain or a 20\% loss.

**The IID model.**

Write \( X_i \) for the return in year \( i \). *Suppose* the \( (X_i) \) are IID random variables. Then the value \( Y_n \) of your investment at the end of year \( n \) is

\[
Y_n = Y_0 \prod_{i=1}^{n} (1 + X_i)
\]

where \( Y_0 \) is your initial investment.

Conceptually, we are assuming the future is statistically the same as the past, and assuming independence over different time periods.
To analyze this model we take logs and divide by $n$:

$$n^{-1} \log Y_n = n^{-1} \log Y_0 + n^{-1} \sum_{i=1}^{n} \log(1 + X_i)$$

and the law of large numbers says that as $n \to \infty$ the right side converges to $\mathbb{E}[\log(1 + X)]$. We want to compare this to an investment with a non-random return of $r$. For such an investment (interest rate $r$, compounded annually) we would have $Y_n = Y_0 (1 + r)^n$ and therefore $n^{-1} \log Y_n \to \log(1 + r)$. Matching the two cases gives the conclusion

**In the IID model, the long term growth rate is**

$$\exp(\mathbb{E}[\log(1 + X)]) - 1.$$ 

The formula looks strange, because to compare with the IID annual model we are working with the equivalent “compounded annually” interest rate. It is mathematically nicer to use instead the “compounded instantaneously” interest rate, which becomes just $\mathbb{E}[\log(1 + X)].$
The main impact of this result is that what matters “in the long term” about the random return $X$ is not precisely the mean $\mathbb{E}[X]$, but rather its “multiplicative” analog $\mathbb{E}[\log(1 + X)]$. Let us note, but set aside for a while, the points

- Is the model realistic for stock market investing?
- The phrase long term here refers to the applicability of the law of averages as an approximation to finite time behavior – this is a third meaning of the phrase, logically quite distinct from the two previous meanings.

Instead we focus on the conceptual point is that there are many investment possibilities, that is ways to allocate money to different risky or safe assets. Write $\alpha$ for a portfolio, that is a way of investing given proportions of your fortune in different assets.

We can now jump to our first key mathematical point.
The Kelly Criterion. Suppose you have a range of possible investment portfolios $\alpha$, which will produce return $X_\alpha$, with known distribution. Then (assuming the IID model) the way $\alpha$ that maximizes the long term growth rate is the way that maximizes $E[\log(1 + X_\alpha)]$. So choose that way.

The math here is just STAT134, but the implications are rather subtle, as we will see.

Mathematics of the Kelly criterion: one risky and one safe asset. Suppose there is both a “risky” (random return) asset (a “stock”, more realistically a S&P500 index fund) and a risk-free alternative investment (a “bond”) that pays a fixed interest rate $r$. 
Suppose we choose some number 0 ≤ p ≤ 1 and at the start of each year we invest a proportion p of our total “investment portfolio” in the stock market, and the remaining proportion 1 − p in the bond. In this case our return in a year is

\[ X^* = pX + (1 - p)r \]

where X is the return on the stock. The long term growth rate is now a function of p: in the continuous setting

\[ \text{growth}(p) = \mathbb{E}[\log(1 + pX + (1 - p)r)]. \quad (2) \]

The Kelly criterion says: choose p to maximize growth(p). Let’s see two examples. In the first X is large, and we end up with p small; in the second X will be small, and we end up with large p. In these two examples we take the time unit to be 1 day instead of 1 year (which doesn’t affect math formulas).
Example: pure gambling.

Imagine a hypothetical bet which is slightly favorable. Suppose each day we can place a bet of any size $s$; we will either gain $s$ (with probability $0.5 + \delta$) or lose $s$ (with probability $0.5 - \delta$), independently for different days (here $\delta$ is assumed small). Take $r = 0$ for the moment. What proportion $p$ of our portfolio do we want to bet each day?
Here, for small $\delta$, 

$$
\mathbb{E}[\log(1 + pX)] = (\frac{1}{2} + \delta) \log(1 + p) + (\frac{1}{2} - \delta) \log(1 - p)
$$

$$
\approx (\frac{1}{2} + \delta)(p - p^2/2) + (\frac{1}{2} - \delta)(-p - p^2/2)
$$

$$
= 2\delta p - p^2/2.
$$

Thus the asymptotic growth rate is approximately the quadratic function of $p$

$$
G(p) = 2\delta p - p^2/2 \tag{3}
$$

shown in the Figure. The Kelly criterion says to choose $p \approx 2\delta$ and then your long term growth rate will be $\approx 2\delta^2$. 
Now recall that we simplified by taking $r = 0$; when $r > 0$, the fact that a proportion $1 - p \approx 1$ of the portfolio not put at risk each day can earn interest, brings up the optimal growth rate to $r + 2\delta^2$; the quantity $2\delta^2$ represents the extra growth one can get by exploiting the favorable gambling opportunity.

To give a more concrete mental picture, suppose $\delta = 1\%$. The model matches either of the two following hypothetical scenarios.

(a) To attract customers, a casino offers (once a day) an opportunity to make a roulette-type bet with a 51% chance of winning.
(b) You have done a statistical analysis of day-to-day correlations in some corner of the stock market and have convinced yourself that a certain strategy (buying a portfolio at the start of a day, and selling it at the end) replicates the kind of favorable bet in (a).
In either scenario, the quantity \(2 \delta^2 = 2/10,000\) is the “extra” long term growth rate available by taking advantage of the risky opportunity. Note this growth rate is much smaller than 2% “expected gain” on one bet. On the other hand we are working “per day”, and in the stock market case there are about 250 days in a year, so the growth rate becomes about 5% per year; recalling this is “5% above the risk-free interest rate”, it seems a rewarding outcome. But if \(\delta\) were instead 0.5% then the extra growth rate becomes \(1\frac{1}{4}\%\), and (taking into account transaction costs and our work) the strategy hardly seems worth the effort.

Implicit in the Figure (back 2 pages) is a fact that at first strikes everyone as counter-intuitive. The curve goes negative when \(p\) increases above approximately \(4\delta\). So even though it is a favorable game, if you are too greedy then you will lose in the long run!

This setting was artificially simple; here is a first step towards a more realistic setting.
Example: what proportion of your portfolio to put into the “stock”?  
As before, suppose there is both a “risky” (random return) asset (a “stock”, more realistically a S&P500 index fund) and a risk-free alternative investment (a “bond”) that pays a fixed interest rate $r$. And suppose we choose some number $0 \leq p \leq 1$ and at the start of each year we invest a proportion $p$ of our total “investment portfolio” in the stock market, and the remaining proportion $1 - p$ in the bond. We know that the long term growth rate is a function of $p$:

$$growth(p) = \mathbb{E}[\log(1 + pX + (1 - p)r)].$$

There is a nicer algebraic way of dealing with the interest rate $r$. Set

$$X = r + (1 + r)X^*$$

and interpret $X^* = (X - r)/(1 + r)$ as “return relative to interest rate”. Then a couple of lines of algebra let us rewrite the formula as

$$growth(p) = (1 + r)\exp(\mathbb{E}[\log(1 + pX^*)]) - 1 \quad (4)$$

and the optimization problem now doesn’t involve any $r$. 
If we imagine the stock market on a daily time-scale and suppose changes $X^*$ are small, with mean $\mu$ and variance $\sigma^2$, then we can use the series approximation

$$\log(1 + pX^*) \approx pX^* - \frac{1}{2}(pX^*)^2$$

to calculate

$$\mathbb{E}[\log(1 + pX^*)] \approx p\mu - \frac{1}{2}p^2(\mu^2 + \sigma^2) \approx p\mu - \frac{1}{2}p^2\sigma^2$$

(the latter because $\mu$ and $\sigma^2$ are in practice of the same order, so $\mu^2$ is of smaller order than $\sigma^2$). So the Kelly criterion says: choose $p$ to maximize $p\mu - \frac{1}{2}p^2\sigma^2$, that is choose

$$p = \mu/\sigma^2.$$  \hspace{1cm} (5)

This is another remarkable formula, and let us discuss some of its mathematical implications.
1. The formula is (as it should be) *time-scale free*. That is, writing \( \mu_{\text{day}}, \mu_{\text{year}}, \sigma_{\text{day}}^2, \sigma_{\text{year}}^2 \) for the means and variances over a day and a \( N \)-day year, then (because compounding has negligible effect over a year) \( \mu_{\text{year}} \approx N\mu_{\text{day}} \) and \( \sigma_{\text{year}}^2 \approx N\sigma_{\text{day}}^2 \), so we get the same value for \( \mu/\sigma^2 \) whether we work in days or years.

2. Even though we introduced the setup by stating that \( 0 \leq p \leq 1 \), the model and its analysis make sense outside that range. Economic theory and experience both say that the case \( \mu < 0 \) doesn’t happen (investors are risk averse and so would buy no stock; this would cause the current price of stock to drop), but if it did then the formula \( p = \mu/\sigma^2 < 0 \) says that not only should be invest 100% of our wealth in the bond, but also we should “sell short” (i.e. borrow) stock and invest the proceeds in the bond.

3. More interesting is the case \( p > 1 \). Typical values given for the S&P500 index (as noted earlier, stating meaningful historical values is much harder than one might think) are (interest-rate-adjusted) \( \mu = 5.6\% \) and \( \sigma = 20\% \), in which case the Kelly criterion says to invest a proportion \( p = 140\% \) of your wealth in the stock market, i.e. to borrow money (at fixed interest rate \( r \)) and invest your own and the borrowed money in the stock market.
What about the not-so-long term?
We started with the multiplicative model, which assumes that returns in different time periods are IID. This is not too realistic, but the general idea behind the Kelly criterion works without any such assumption, as we now explain.

Going back to basics, the idea

\[ \text{to invest successfully in the stock market, you need to know whether the market is going to go up or go down} \]

is just wrong. Theory says you just need to know the probability distribution of a future return. So suppose (a very big SUPPOSE, in practice!) at the beginning of each year you could correctly assess the probability distribution of the stock return over the coming year, then you can use the Kelly criterion (2) to make your asset allocation. The fact that the distribution, and hence your asset allocation, would be different in different years doesn’t make any difference – this strategy is still optimal for long-term growth.
The second key insight from mathematics

The numbers for growth rates that come out of the formula of course depend on the distributions of each next year’s returns, but there’s one aspect which is “universal”. In any situation where there are sensible risky investments, following the Kelly strategy means that you accept a short-term risk which is always of the same format:

- 40% chance that at some time your wealth will drop to only 40% of what you started with.

The magical feature of this formula is that the percents always match: so there is a 10% chance that at some time your wealth will drop to only 10% of what you started with.

The math here is from the theory of diffusions or stochastic calculus; a little too hard to explain here.
x\% chance that at some time your wealth will drop to only x\% of what you started with.

For an individual investor, it is perfectly OK to be uncomfortable with this level of medium-term risk and to be less aggressive (in investment jargon) by using a partial Kelly strategy, that is using some smaller value of

\[ p = \text{proportion of your assets invested in stocks} \]

than given by the Kelly criterion. Theory predicts you will thereby get slower long-term growth but with less short-term volatility.

A memorable quote

*The Kelly strategy marks the boundary between aggressive and insane investing.*
How one might expect this theory (based on assuming known true probability distributions for the future, and on seeking to optimize long-term growth rate) to relate to the actual stock market is not obvious, but one can certainly look at what the actual percentages have been.

The next figure shows the historical distribution (based on hypothetical purchase of S&P500 index on first day of each year 1950-2009 and on subsequent monthly closing data) of the minimum future value of a 100 investment.
This is obviously very different from the “Kelly” prediction of a flat histogram over $[0, 100]$. This data is not adjusted for inflation or for comparison with a risk-free investment, and such adjustments (course project?) would make the histogram flatter, but still not close to the Kelly prediction. We mentioned before that over the historical long term it has been more profitable to borrow to invest more than 100% of your assets in the market. Both observations reflect the fact that the stock market fluctuates less than would the fortune of a Kelly-optimizing speculator.
Wrap up

The math I’ve shown is still a long way away from explaining the IFA graphic. They have many different “asset classes”, each represented by an index fund [show IFA]. We want to construct a portfolio by weighting over each asset class (note we still avoid individual stocks). Any possible portfolio has (historical data) some average and some SD of annual return. We just choose a range of portfolios that maximize average for given SD (this exploits past pattern of correlations as well as averages and SDs).

The standard textbook math is Modern portfolio theory which is essentially the short-term Normal approximation for price fluctuations. The reason I emphasize Kelly instead is that (in principle) you could model “Black Swans” (rare severe shocks) or at least use actual historic annual return distributions rather than assuming Normal.
The two books most related to our “Kelly criterion” are
William Poundstone *Fortune’s Formula*: a history of precisely this topic

A memorable quote from Brown: Kelly enables you *to get rich exponentially slowly*.