Rationality of Strategies in Games Theory

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Abstract

This is an extended write-up of analyzing a well-known game interested both psychologists and game theorists called "guess 2/3 of the average"; all the players choose an integer from 1-100 simultaneously; whose choice is the closest to 2/3 of the average of all these numbers will win and get 50 dollars as a prize. This paper also analyses two variations of the original game. One variation changes the way of calculating the winning number, and the other changes the prize amount. The analysis focuses on two perspectives. One is breakdown the game and analyze different types of strategies on the psychological side to illustrate different levels of rationality of these strategies. The other one is trying to come up with an optimal strategy that will win the game. I build a simple model for the optimal strategy using it to simulate data and compare the result with a larger dataset from the previous experiment with New York Times readers.

1. Introduction

The game "Guess 2/3 of the Average" was created by Alain Ledoux in 1981. He used this game as a tie breaker to select the winner among 4000 magazine readers, who reached the same amount of points in the previous puzzle. Then Rosemarie Nagel revealed that this kind of games can be used to disclose participants' "depth of reasoning". Also, due to the analogy of Keynes's comparison of newspaper beauty contests and stock market investments, the guessing game is known as the Keynesian beauty contest. Now, the beauty contest game becomes famous in experimental economics to illustrate the difference of perfect rationality and the common knowledge of rationality.

In this paper, I analyze one of the most widely-spread versions of "Guess 2/3 of the Average", which asks players to choose integers from 1 to 100, and provide the reason why they want to make these choice. Also, two variations of this classic game are included. In the analysis of these three versions of the game, I breakdown and categorize all the players' strategies to illustrate different levels of rationality of these strategies. There is also a model, which is a combination of all these strategies, elucidates the optimal strategy. The final step in this study is comparing the simulated data of the optimal strategy using R with previous data from the New York Times readers.

2. Background and Game Description

38 people ranging from age 18 to age 56 and forming a variety of educational backgrounds were invited to play this game. Besides the original game, players were also asked for two more questions which could be considered as variations of the primary game. As a result, each person played the game three times with slightly different rules each time, also their answers and strategies of making their choices were recorded.

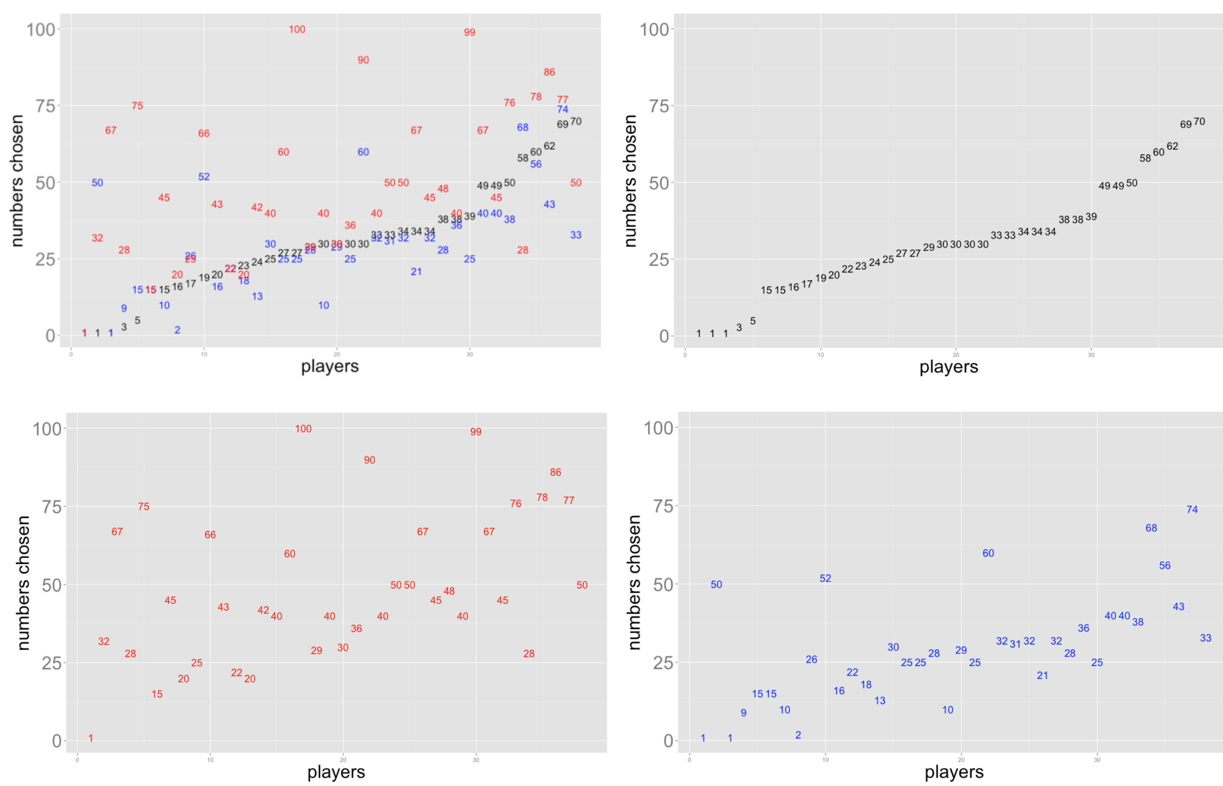
For the original game, all the participants in this game choose an integer from 1-100 simultaneously. Then the average of these numbers will be calculated. Whose choice is the closest to 2/3 of the average will be the winner and get 50 dollars as a prize.

For variation 1, the way of calculating the average is different. After the data has been collected, the highest 5% choices of numbers will be considered outliers. In other words, the biggest 5% choices will be removed first, and only the left 95% will be used to do the subsequent calculation, so the mean will be a trimmed mean. The rest of the game remains the same.

For variation 2, almost all the rules are the same with the original game (not variation 1). The only difference is that the prize amount. The winner will get \$x rather than \$50, where x is the number that the winner chooses.

3. Data

All the players choose 3 integers from 1-100. In the first plot (top left), every 3 numbers on one vertical line are the numbers chosen by the same person, which shows how people change their answers in different scenarios. Black numbers are answers of the original game, and blue numbers for variation 1, and red numbers for variation 2. The data is sorted increasingly based on answers of the original game. The other three plots are the separate data plots for each game, which indicate the range and distribution of each game more clearly.



4. Analysis of the Original Game

4.1 Categorizing

All the players were asked for the reason why they chose that number, and players with similar strategies were put together as a group. Overall, there are 5 different groups based on the players’ strategies.

- Naive players
- Common-rational players I
- Common-rational players II
- Hyper-rational players
- The best players

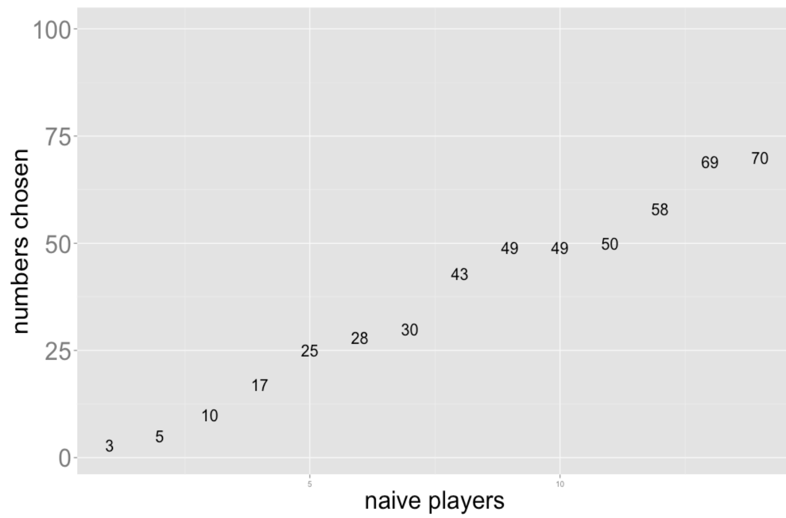
In the following analysis for each group, I will show the recorded information including common answers for the players in the same group, their common responses, and how many players are assigned to this group. Also I will analyze their strategies and state a method to simulate similar answers using all these information.

4.2 Native players

Generally, native players can choose any number as their answers, because they are simply guessing. Their common responses can be “I am just guessing”, or “I feel like this number is right”. Out of 38, 14 people said they guessed the answer.

Naive players in fact have no strategy, and they choose the number randomly. These players usually do not read the game description carefully. Sometimes, although they read, they do not quite understand what the problem is asking. Most of the time, naïve players just pick numbers they like or they feel is right, so I try to use uniform [1,100] to simulate these kind of answers.

Then further analysis of native players shows that the result should not be this simple. First, if a player spends a few minutes to think about the question, it is easy to tell that the highest winning number can only be 67, which happens when everyone chooses 100. Therefore, I was expecting that no one would chose numbers greater than 67 before I saw the data, but the highest answer was 70 as the following plot shows. It means that naïve players did not realize this situation.



However, besides the highest number, another occurrence is obvious in this plot. Among all the choices of 14 naive players, the lowest answer can be as small as 3, but no one chose any number bigger than 70 even the range is 1 to 100. This shows that even naïve players said they were purely guessing, uniform [1,100] may not be a good approximation. Hence, I asked these players if they had thought about $100 \times 2/3$ is 67. Some of them simply said no, but some said that they felt like they shouldn't choose a number too big, because they saw a 2/3 and also the word average in the description. Although they could not clearly explain why these words should influence their answers, these players did not purely guess. At least they got some information from the description. Therefore, uniformly on [1, 67] should work better than uniform on [1,100] for those players who were influenced by the word "average" and "2/3". I modified my simulation to be uniformly distributed on [1,67] for these players, and uniformly on [1,100] for others.

4.3 Common-rational players I

One of the most common strategies leads players to choose numbers like 33 or 34. When talking about their reasons for making these choices, they always say something like, "from 1 to 100, the average should be around 50, and 2/3 of it is 33". Out of all 38 players, 12 players used this strategy.

These players use a simple strategy. Their strategy is based on an assumption that other answers are uniform and the average is around 50, so they will win if they choose 2/3 of 50. Since some players who chose answers like 30 also gave similar reasons, these answers are not exactly uniform.

I apply Normal (33.5, 1) to simulate answers of using this strategy.

One problem about this strategy is that the assumption, which is others choose numbers uniformly on [1,100], only happens when all other players have no specific strategies. Unfortunately, this assumption is not true. Even the naïve players in this experiment are not all choosing numbers uniformly. For these common-rational players, if their opponents are random number generators, to be more rigorous a large number of random number generators, their strategy will be a good one. In practice, players who use this strategy, oversimplify the game.

4.4 Common-rational players II

Another common strategy is used by players who choose 22 or 23 as their answers. The typical reason is like, "numbers bigger than 67 are unlikely to win, so the problem is actually asking for choosing numbers on [1, 67] rather than [1, 100]. The average on [1, 67] is 34, and 2/3 of the average is 22". This sounds reasonable, and 5 players used this strategy.

These players realize that 2/3 of 100 cannot be bigger than 67. It is certainly right, because even if all the players in this game choose 100, the final answer can only be as big as 67. Normally, a player should not expect all the other people to choose 100, unless this player can control others' mind. Then the answer can be any number the player wants it to be. The player should not worry about choosing numbers. Actually, there is one circumstance for the winning number to be greater than 67. It happens when all the answers are greater than 67, and the smallest one among them will win. In this case, because only the smallest number can win, all the players want to keep reducing their answers to be the smallest one, and they will reach numbers under 67 eventually. The conclusion is that answers greater than 67 are not plausible to win. In game theory, we say that these numbers are dominated.

In this case, rational choices should be in the region [1, 67]. Assume players choose uniformly on [1, 67]. Then the average should be around 34, and 2/3 of the average should be around 22.5. Similar to the last one, Normal (22.5, 1) is used to simulate these answers.

Players who use this strategy are making the same mistake as pervious common-rational players. After they have done their first step analysis, they assume that all the other answers are uniformly distributed on [1, 67]. Because there are many native players who do not realize the domination, this assumption certainly cannot be guaranteed. Besides native players, players use this strategy also underestimate some hyper-rational players who think through more steps.

4.5 Hyper-rational players

In many different versions of these guess average games, there are always players choose the smallest choice as their answer. Here it is 1. Their reasons are usually like this, "first, all the numbers bigger than 2/3 of 100 are dominated, so we only need to look at the numbers on [1, 66]. Then, do the same analysis for these numbers. It leads to numbers on [1, 44]. Keep doing this again and again. The answer will end up with 1". Comparing to all the previous answers, their strategy sounds pretty reasonable, and 3 out of 38 players gave this kind of reason and the answer 1.

Hyper-rational players can also be called perfect-rational players, but it does not mean their answers are perfect. It means that these players do perfect-rational analysis only tenable in theory. These players are smart, and also some of them have learnt game theory before. They solve this problem by first assuming that all the players are hyper-rational, which is exactly the assumption people usually make when doing a game theory problem. After that, hyper-rational players do much further analysis than those 2 different kinds of common-rational players mentioned before. In the situation where all the players are hyper-rational, they would all realize that the numbers greater than 67 are dominated. This means now they are playing a new game which is choosing a number from 1 to 67. Because hyper-rational players know that all the players should get this idea, they can go one more step further. That is all the numbers greater than 2/3 of 67 are also dominated. Then the game becomes choosing a number from 1 to 45. All the players do the similarly analysis again and again. Each time the upper limit of the valid interval goes down by 1/3 of it, until they get the hyper-rational answer, 1. In game theory, we call this answer as a Nash Equilibrium. For these players, their answers have no potential to deviate, so no complicated simulation is needed. Simply the number 1 is used to simulate the answers for hyper-rational players.

However, hyper-rational players forget that this is not an ideal situation that everyone in this game is hyper-rational. Not all the players can think through these many steps. The naive players and common-rational players choose numbers much bigger than one. Their choices push the average up, so the final answer is not plausible to be 1.

I asked the hyper-rational players if they had ever thought about that even though their logical inference was totally right in theory, choosing 1 was still unlikely to win. In fact, they did know that, but they told me 1 is the "correct" answer for this problem. Whether they will win is not important. What important is their inference is perfect. I then realized that they chose 1, just because they wanted to prove that they understand the game, and they are rational.

Hyper-rational players give good analysis and answers. However, they are still not the best players, and also not the winners. The best players should not only be rational themselves, but also understand that not all the people are rational like them. Hence, the best players should choose a larger number rather than 1.

4.6 The best players

The best players usually say "I cannot know the correct answer of this problem, before I know who the other players are. You cannot use the same strategy when you are playing this game with scientists or with children". In this experiment, people who provided this kind of reasons finally gave answers between 15 and 25. There were 4 out of 38 people used this strategy.

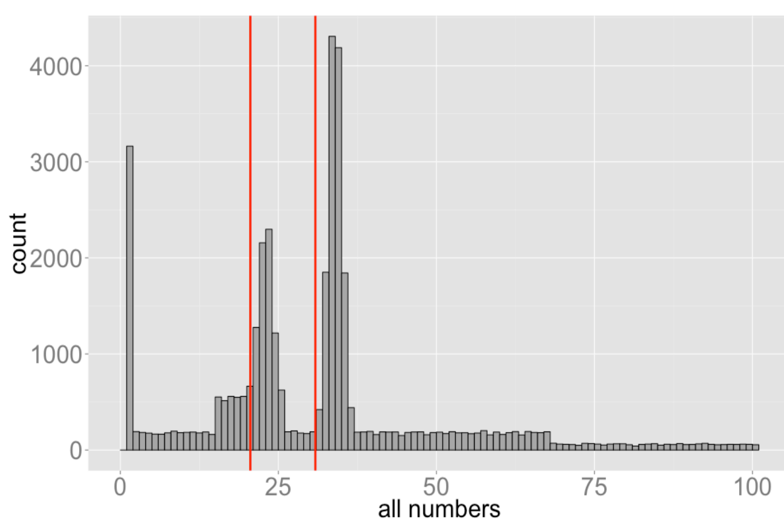
The best players use a mixed strategy to optimize their answer. They are rational themselves, but different from the hyper-rational players. The best players know the theoretical answer is 1, but they also take account of what levels of rationality other players are. Some players may be naive, and some players maybe hyper-rational, and also some players maybe thinking about using a mixed strategy just like them. They know that the answer should be different for different composition of the population. The best players give the answer based on their estimated rationality of the population. However, this analysis cannot be done quantitatively without the data. Simply following what the data shows for the best players' answers, Uniform [15, 25] is used to simulate their answer. A more rigorous way of finding an optimal answer like this will be shown in next section.

4.7 Optimal Strategy and Result Comparison

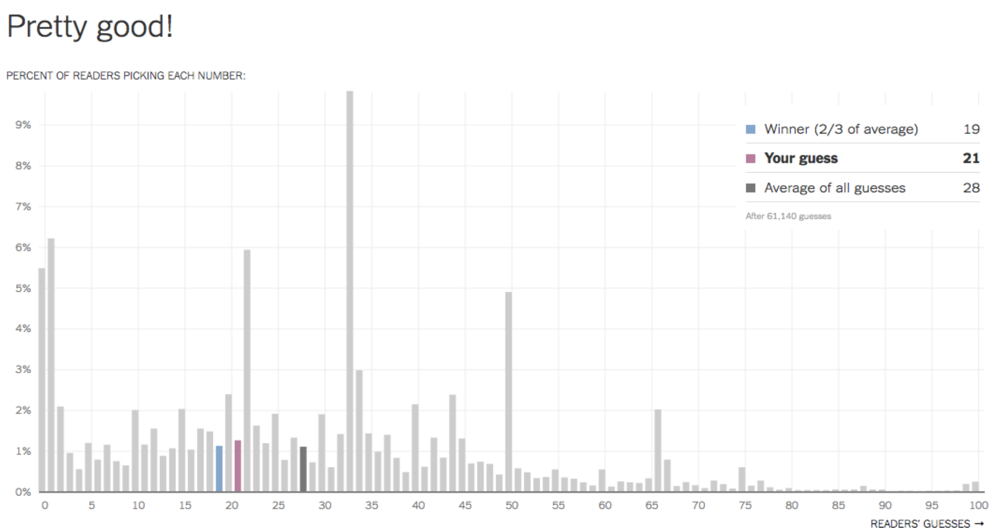
The following form is a summary of previously mentioned simulation. All these data and simulation are used to build an optimal strategy.

Players Type	Number chosen	Percentage
Naive Players	Uniform [1,100] Uniform [1,67]	6/38 8/38
Common-Rational Players I	Normal (33.5,1)	12/38
Common-Rational Players II	Normal (22.5,1)	5/38
Hyper-Rational Players	1	3/38
The Best Players	Uniform [15,25]	4/38

The following plot visualize the simulation. It involves 1000 groups (38000 numbers in total) simulated data. Their average is 30.85 (red line on the right in the plot), and 2/3 of the average is 20.56 (red line on the left in the plot), so the answer according to my optimal strategy is 21. Compare it to the real data collected before, the real average is 30.53, and 2/3 of the average is 20.35, so the winning number should be 20. Even though this is the optimal strategy, the result still misses the right answer. Actually the simulated answer is pretty close to the winning number, which indicates the optimal strategy is not bad. However, even the optimal strategy cannot guarantee a success. This game does need a bit of luck.



The following picture is comparing the simulated result using optimal strategy to a previous experiment. This is an experiment with 61,139 New York Times readers, so the data is much larger and more representative. Notice that their final answer is 19, which is not that close to the simulated answer 21. One reason is that 0 is included as a valid choice in this game, so it is reasonable that the final answer is a little smaller. The other reason can be that my data involves too many native players. When doing the simulation, my model generates too many numbers on [80, 95], which the New York Time readers rarely chose.

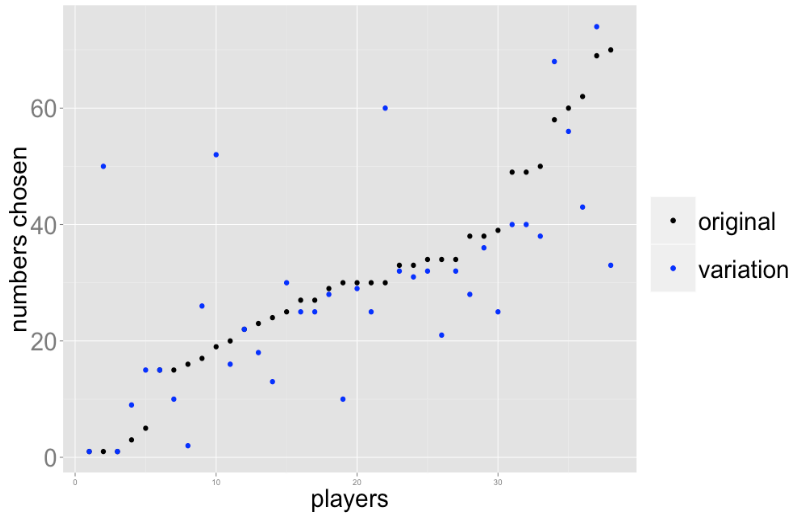


5. Analysis of Variation 1

5.1 Comparing Data to the Original Game

For the first variation, remove the highest 5% of the numbers makes two difference. One is that it removes some big numbers, which are chosen by some naive players who do not have any idea about what is happening. Also, because being the highest 5% cannot win the game, people certainly do not what their answers to be high and get removed. This condition makes players to choose a smaller number. Both these two conditions lead to a lower average, also a lower final answer.

The following is the plotted data comparing the data of the original game and the data of the variation. Most people chose a smaller number than their previous answer, except a few people who guessed.



5.2 Strategy Analysis

Keep groups the same as the original game. The typical answers of players in each group are different for the new version.

- Native players

  Usually they are going to guess a number for this variation, just like what they do for the primary game. I expect these players to choose some number smaller, but the data does not turn out to be like this. Their answers are random, and the simulation remains the same.

- Common-rational players I & II

  What these players will do is that slightly reduce the number they pick for the original game. Their answers of the original game are multiplied by 95% to simulate their new answers. Here the correct way to do it should be calculating the quantile and find the trimmed mean. However, in practice a player cannot do this calculation without knowing the real data or using simulation, so the alternative method is to multiply the numbers of the original game by 95%, which was mentioned by most of the players in their answers.

- Hyper-rational players

  In this game rational players want to choose a smaller number, but hyper-rational players do not change their answers, because 1 cannot be smaller.

- The best players

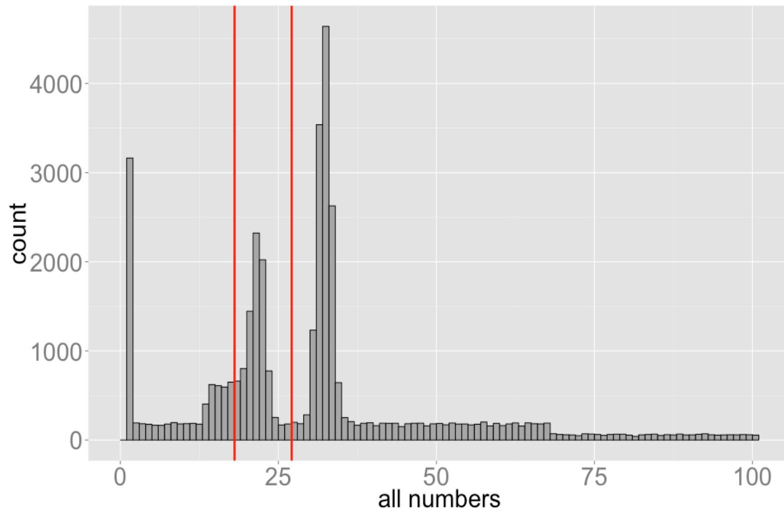
  According to the data, the best players reduced their answers to some new numbers around 90% of their answers of the original game. This shows that they think more carefully than the common-rational players. They realize that not only the mean is trimmed, but also players tend to choose smaller numbers. Their answers of the original game are multiplied by 90% to simulate their new answers

5.3 Optimal Strategy and Result Comparison

The following form and plot is the simulation describes how people choose their answers for this variation. This result is modified from the original simulation based on players' new strategies of this new game.

Players Type	Number chosen	Percentage
Naive Players	Uniform [1,100] Uniform [1,67]	6/38 8/38
Common-Rational Players I	Normal (31.825,1)	12/38
Common-Rational Players II	Normal (21.375,1)	5/38
Hyper-Rational Players	1	3/38
The Best Players	Uniform [13.5,22.5]	4/38

For this variation, the average is 27.09 (red line on the right in the plot), and 2/3 of it is 18.06 (red line on the left in the plot), so the winning number should be 18. Compare it to the real data collected before. The real average is 26.92, and 2/3 of it is 17.94, so winning number should be 18. This time the answer by using the optimal strategy and the real winning number are identical.



6. Analysis of Variation 2

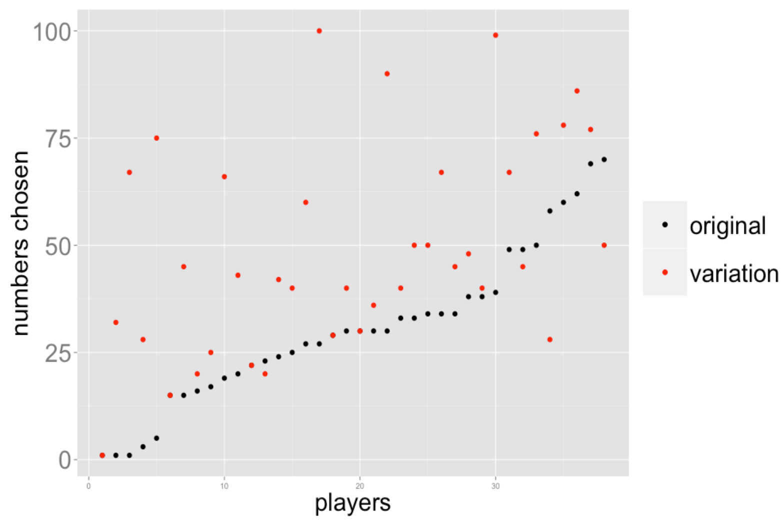
6.1 Comparing Data to the Original Game

For the second variation, players want to choose a bigger number in order to make more money. However, there is one obstruction. If a choice is too big, it cannot win the game. For instance, numbers bigger than 67 can hardly win as said before, so players should choose numbers up to 67 even they want to increase their choices to make more money.

The game now becomes choosing a number from 1 to 67. Does this sound familiar? This is exactly how the hyper-rational players' strategy starts in the original game. Theoretically, players can do all the analysis without the condition that the prize is related to the number they choose. This is because that they have to win first, then start thinking about how much money they can make given they have already won. Especially for hyper-rational players, this problem and the original problem are the same. That added condition is a disturbing term.

However, just as how we overturned hyper-rational players' answers in the original game. Also in this question, not all the players can realize that the condition is useless in theory. In this experiment, many of them followed the new condition seriously to choose a bigger number in order to win more money. Roughly speaking, how much they raised up their answers depends on how greedy they were.

The following is the plotted data. Most people chose numbers much larger than their previous answers, some players even choose numbers as big as 100. This is more prominent for naive players who choose non-rational numbers in the first and the second round.



6.2 Strategy Analysis

- Native players

  Some of them still chose random numbers, but also someone increased their answers a lot. For those who increased their answer, there were also two different types. One is increasing their answers above 50. In general, the reason is that the prize in the original game is 50, and they want to make more money in this variation, so they have to choose a number at least 50. Another one is increasing their answers to extreme numbers like 100. These players just naively want to win 100. Therefore, the simulation is separated to three different uniform distributions as showing in the form in next section.

- Common-rational players I & II

  Usually common-rational players increase their answers by an estimated average increment. They want to win the game, but also they know that an overlarge number cannot win. In this experiment, their choices are approximately 1.5 times of their answers of the original game. The simulation is then adjusted by multiplying parameters with 1.5 .

- Hyper-rational players

  Hyper-rational players should not change their answers, because 1 is still the Nash Equilibrium of this variation. However, this did not actually happen. There were 3 hyper-rational players in the original game, and only 1 of them insisted in his answer 1. The other 2 gave up choosing 1. In this case, they were no longer hyper-rational players in this variation. Their reason was also distinct. They said that they would win at most $1 for being hyper-rational, so insisting on 1 made them look silly. According to their final decisions, I put one of them in the common-rational group, and one of them in the native group, since he was nearly guessing and impatient to tell me his reason.

- The best players

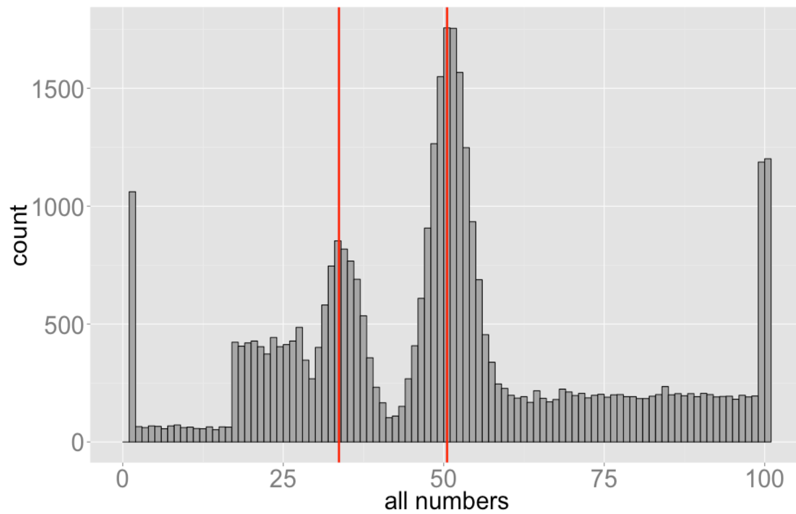
  The best players also increased their answers by an estimated average increment. Generally, comparing to the common-rational players they are more careful. They have a deeper insight that a big number will lead to a failure. According to the data, their choices are approximately 1.2 times of their answers of the original game.

6.3 Optimal Strategy and Result Comparison

The following form and plot is the simulation describes how people choose their answers when they are doing this variation. It is modified from the original simulation based on players' new strategies of this new game. This time not only the numbers they chosen are different, but also the percentage is different.

Players Type	Number chosen	Percentage
Naive Players	Uniform [1,100] Uniform [50,100] Uniform [99,100]	6/38 7/38 2/38
Common-Rational Players I	Normal (50.25,1)	12/38
Common-Rational Players II	Normal (33.75,1)	6/38
Hyper-Rational Players	1	1/38
The Best Players	Uniform [16.5, 27.9]	4/38

For this variation, the average is 50.52 (red line on the right in the plot), and 2/3 of it is 33.68 (red line on the left in the plot), so the winning number should be 34. Compare it to the real data collected before. The real average is 49.26, and 2/3 of it is 32.84, so the winning number should be 33. Again, the simulated answer using my optimal strategy is quite close to the winning number, but it misses the winning number as the original game, which indicates that no matter what the strategy is, it cannot guarantee a win for this kind of game.



7. Deficiency

In this study, there are several methodological deficiencies. First, the sample size is fairly small for only involving 38 players. Also, participants were not randomly selected. In fact, participants are mainly college students, and many of them are Berkeley students. Therefore, the result is skewed toward the younger generation. All the simulation distributions are roughly estimated from the collected data. It is not very accurate. This could be the reason why the optimal answer miss the real winning number.

8. Conclusion

This game is more than just picking numbers. It illustrates the common knowledge of rationality. In other words, what is one's expectation of other players' rational levels. It also highlights the trouble in both natively thinking and thinking too many steps ahead.

A famous real life example of this kind of game is the "Keynesian beauty contest". It is a contest to choose the most beautiful face. In this contest, picking the face that others think is the most beautiful is more likely to win than picking a face that the participant thinks is the most beautiful. In fact, a player can also pick the face others think that others think that others think - and so on - is the most beautiful, just like the hyper-rational players in my experiment. However, comparing to doing the endless others-think inference, start from the composition of the population to construct a mixed strategy is still optimal.

Reference

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