\((B(t), 0 \leq t < \infty)\) is standard Brownian motion (BM),

Consider \(M(t) = \max_{0 \leq s \leq t} B(s)\). The joint density of \((M(t), B(t))\) is

\[
f_{M(t),B(t)}(a, b) = \frac{2(2a - b)}{\sqrt{2\pi} t^{3/2}} \exp\left(-\frac{(2a - b)^2}{2t}\right); \quad a \geq 0, \quad a \geq b.
\]

This formula is complicated, but there are two interesting consequences.

**Proposition**

\[
\mathbb{P}(M_1 > a | B(1) = 0) = \exp(-2a^2), \quad a > 0.
\]
\[
\mathbb{P}(B(1) \leq -b | M(1) = 0) = \exp\left(-\frac{b^2}{2}\right), \quad b > 0.
\]
Other calculations we can do involve

\[ L = \sup\{ t \leq 1 : B(t) = 0 \}; \quad R = \inf\{ t \geq 1 : B(t) = 0 \}. \]

(Jargon: the **excursion** containing time 1 happens over the interval \([L, R]\).) The results are

\[ f_R(t) = \frac{1}{\pi(t - 1)^{1/2}t}, \quad 1 < t < \infty \]

\[ \mathbb{P}(L \leq s) = 2\pi^{-1} \arcsin s^{1/2}, \quad 0 < s < 1. \]  \( (1) \)

\[ f_L(s) = \frac{1}{\pi s^{1/2}(1 - s)^{1/2}}, \quad 0 < s < 1 \quad (\text{arcsine distribution}). \]

On the board I will show

\[ \mathbb{P}(L \leq s) = \int_{-\infty}^{\infty} g_s(x)\phi_s(x) \, dx \]

\[ g_s(x) = \mathbb{P}(T_{|x|} > 1 - s) = 1 - 2\Phi(|x|/\sqrt{1-s}), \quad \phi_s(\cdot) \text{ is density of } B(s). \]

Then (1) is a hard calculus exercise!