We have seen two kinds of “purely random” process:

- coin-tossing, dice-throwing, etc – IID sequences.
- Poisson process of random points in $d$ dimensions.

A third kind one might imagine is

- a “purely random” continuous function.

It is not obvious what this means. To start, consider simple symmetric random walk

$$S_n = \sum_{i=1}^{n} \xi_i; \quad \mathbb{P}(\xi_i = \pm 1) = 1/2.$$ 

For large $n$, how can we draw the graph of $(S_0 = 0, S_1, \ldots, S_n)$ on “unit size” paper?

[board]

Know $\mathbb{E}S_n = 0$ and $\text{s.d.}(S_n) = \sqrt{n}$. 
The picture suggests that the “scaling limit” [jargon!] of the random walk is a process \((B(t), 0 \leq t < \infty)\) with the property

1. \(B(t)\) has Normal(0,t) distribution.

It turns out there is a mathematical object called “standard Brownian motion” (BM) with properties (1) and

2. \(B(t) - B(s)\) has Normal(0,t-s) distribution \((s < t)\).

3. For \(s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k\) the increments \(B(t_1) - B(s_1), \ldots, B(t_k) - B(s_k)\) are independent.

4. The sample paths \(t \to B(t)\) are (random) continuous functions of \(t\).

Keep in mind this is a model – looking at some time-varying real-world quantity, it may or may not behave like this BM model.

One example which does fit the model quite well is short-term stock prices [show].

As the stock price graphs suggest, although continuous the sample paths are irregular – not differentiable.
BM is important because

1. one can do many explicit calculations.
2. it is a “building block” for defining other random processes.

This part of the course is more “mathematical” in the sense of emphasizing calculations rather than toy models.

Here is a very simple instance of (1).
Given parameters $-\infty < \mu < \infty$ and $0 < \sigma < \infty$ we can define

$$X(t) = \mu t + \sigma B(t)$$

called Brownian motion with drift rate $\mu$ and variance rate $\sigma^2$. In particular $X(1)$ has Normal($\mu, \sigma^2$) distribution.

The appearance of the Normal distribution is not arbitrary (cf. the Poisson distribution in the Poisson process), as shown by the next theorem.

**Theorem**

*If a process $X(t), 0 \leq t < \infty$ has $X(0) = 0$, continuous sample paths, and (for each $a > 0$) the increments $X(a), X(2a) - X(a), X(3a) - X(2a), \ldots$ are IID, then the process must be BM with some drift and variance rates.*

This is a consequence of the CLT.
Explicit formulas for BM will ultimately be based on the Normal density formula, but first let us see some “structural properties” of BM. Write $\equiv_d$ for “equal in distribution”.

1: Symmetry. $(B(t), 0 \leq t < \infty) \equiv_d (-B(t), 0 \leq t < \infty)$.

2: Markov. Given $t_0$ and the “past” $(B(t), 0 \leq t \leq t_0)$, the “future” process $\tilde{B}(u) = (B(t_0 + u) - B(t_0), 0 \leq u < \infty)$ has the distribution of BM and is independent of the past process.

3: Scaling. For $c > 0$ the “scaled” process $\tilde{B}(u) = (c^{-1/2}B(cu), 0 \leq u < \infty)$ has the distribution of BM. [show Wikipedia demo]

4: Martingale. $(B(t), 0 \leq t < \infty)$ is a martingale.
Recall facts about the Normal density.

\[ \phi(z) = (2\pi)^{-1/2} \exp(-z^2/2), \; -\infty < z < \infty \]

is the density of a "standard Normal(0,1)" RV \( Z \). Write the distribution function as

\[ \mathbb{P}(Z \leq z) = \Phi(z) = \int_{-\infty}^{z} \phi(x)dx. \]

We have

\[ \mathbb{E}Z = 0, \quad \text{var}(Z) = 1. \]

A few lines of calculus show [board]

\[ \mathbb{E}|Z| = \sqrt{2/\pi}. \]

Calculations with \( B(t) \) easily done by scaling: \( B(t) =_d t^{1/2}Z \) and so (for instance) \( \mathbb{E}|B(t)| = \sqrt{2t/\pi} \). The general relation

\[ f_{cY}(y) = c^{-1}f_Y(y/c) \]

gives the density of \( B(t) \)

\[ f_{B(t)}(x) = t^{-1/2}\phi(t^{-1/2}x) = (2\pi t)^{-1/2} \exp(-x^2/(2t)), \; -\infty < x < \infty. \]