In the last class, we discussed the problem of nonparametrically estimating a spectral density. The natural estimator is:

\[ I(\lambda) := \sum_{h:|h|<n} \hat{\gamma}(h)e^{-2\pi i \lambda h} \quad \text{for } -1/2 \leq \lambda \leq 1/2 \]

which for \( \lambda = j/n \in (0,1/2] \) coincides with the periodogram:

\[ I(j/n) = \frac{|b_j|^2}{n} \quad \text{where } b_j = \sum_t x_t \exp \left(-\frac{2\pi ijt}{n}\right) \]

The key result about the periodogram is that under some regularity conditions which hold for all ARMA processes under the gaussian noise, it can be shown that when \( n \) is large, the random variables:

\[ \frac{2I(j/n)}{f(j/n)} \quad \text{for } 0 < j < n/2 \]

are approximately independently distributed according to the \( \chi^2_2 \) distribution. As a result, \( I(\lambda) \) is not a good estimator of \( f(\lambda) \).

We studied two modifications of the periodogram:

1. **Moving average smoothing**: Choose an integer \( m \geq 1 \) and estimate \( f(j/n) \) by

\[ \hat{f}(j/n) := \frac{1}{2m+1} \sum_{k=-m}^{m} I \left( \frac{j+k}{n} \right) \]

or more generally

\[ \hat{f}(j/n) := \sum_{k=-m}^{m} W(k)I \left( \frac{j+k}{n} \right) \]

where \( W(k) \) are nonnegative weights summing to one. This estimator is based on the approximate representation \( I(j/n) \approx f(j/n) + U_j f(j/n) \) for \( 0 < j < n/2 \) where \( \{U_j\} \) is white noise.

2. **Lag Window Spectral Density Estimator**: Choose an integer \( r \geq 1 \) and estimate \( f(j/n) \) by

\[ \hat{f}(j/n) := \sum_{h:|h|\leq r} \hat{\gamma}(h) \exp(-2\pi i \lambda h) \]

or more generally

\[ \hat{f}(j/n) := \sum_{h:|h|\leq r} w \left( \frac{h}{r} \right) \hat{\gamma}(h) \exp(-2\pi i \lambda h) \]
where \( w(x) \) is a symmetric i.e., \( w(x) = w(-x) \) function satisfying \( w(0) = 1, |w(x)| \leq 1 \) and \( w(x) = 0 \) for \( |x| > 1 \). This estimator is based on the formula:

\[
f(\lambda) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \lambda h}
\]

and the idea that \( \gamma(h) \) for large \( h \) become close to zero (because \( \sum_{h} |\gamma(h)| < \infty \)) and they are also difficult to estimate from the data.

1 Equivalence of these Two Estimators

We shall now show that these two ways of improving the periodogram: by smoothing it and the lag window spectral density estimator are essentially the same. To see this, we first need an inverse relationship between \( I(\lambda) \) and \( \hat{\gamma}(h) \). We have defined \( I(\lambda) := \sum_{h:|h|<n} \hat{\gamma}(h) e^{-2\pi i \lambda h} \) for \(-1/2 \leq \lambda \leq 1/2\).

It is possible to invert this formula to write \( \hat{\gamma}(k) \) in terms of \( I(\lambda) \). Fix an integer \( k \) with \( |k| < n \) and multiply both sides of the above formula by \( e^{2\pi i \lambda k} \). Integrating the resulting expression with respect to \( \lambda \) from \(-1/2\) to \(1/2\), we get

\[
\int_{-1/2}^{1/2} e^{2\pi i \lambda k} I(\lambda) d\lambda = \sum_{h:|h|<n} \hat{\gamma}(h) \int_{-1/2}^{1/2} e^{2\pi i \lambda (k-h)} d\lambda = \hat{\gamma}(k).
\]

This therefore implies

\[
\hat{\gamma}(k) = \int_{-1/2}^{1/2} e^{2\pi i \lambda k} I(\lambda) d\lambda. \quad (1)
\]

In other words, the function \( I(\lambda) \) is the spectral density corresponding to the sample autocorrelation function. Using the formula (1), we can write the lag window spectral density estimator as

\[
\tilde{f}(\lambda) = \sum_{h:|h| \leq r} w\left(\frac{h}{r}\right) \hat{\gamma}(h) e^{-2\pi i \lambda h}
\]

\[
= \sum_{h:|h| \leq r} w\left(\frac{h}{r}\right) \int_{-1/2}^{1/2} e^{2\pi i \rho h} I(\rho) d\rho e^{-2\pi i \lambda h}
\]

\[
= \int_{-1/2}^{1/2} I(\rho) \sum_{h:|h| \leq r} w\left(\frac{h}{r}\right) e^{2\pi i (\rho - \lambda) h} d\rho.
\]

By the change of variable \( \rho = \lambda + u \), we get

\[
\tilde{f}(\lambda) = \int I(\lambda + u) \sum_{h:|h| \leq r} w\left(\frac{h}{r}\right) e^{2\pi i u h} du.
\]

Letting

\[
W(u) = \sum_{h:|h| \leq r} w\left(\frac{h}{r}\right) e^{2\pi i u h},
\]

we get that

\[
\tilde{f}(\lambda) = \int I(\lambda + u) W(u) du.
\]

Thus the lag window spectral density estimator \( \tilde{f} \) can also be thought of as obtained by smoothing the periodogram.
2 Approximate Confidence Intervals for $f(j/n)$

Recall that the random variables

$$\frac{2I(j/n)}{f(j/n)} \quad \text{for } 0 < j < n/2$$

are approximately independently distributed according to the $\chi^2$ distribution.

Therefore, approximately

$$\hat{f}(j/n) = \frac{1}{2m+1} \sum_{k=-m}^{m} I \left( \frac{j+k}{n} \right) \approx \frac{f(j/n)}{2(2m+1)} \sum_{k=-m}^{m} \frac{2I((j+k)/n)}{f((j+k)/n)}.$$ 

This would allow us to approximate the distribution of $\hat{f}(j/n)$ in the following way:

$$2(2m+1)\frac{\hat{f}(j/n)}{f(j/n)} \sim \chi^2_{2(2m+1)}.$$ 

If $\chi^2_{2(2m+1)}(\alpha/2)$ and $\chi^2_{2(2m+1)}(1 - \alpha/2)$ satisfy

$$\mathbb{P} \left\{ \chi^2_{2(2m+1)}(\alpha/2) \leq \chi^2_{2(2m+1)}(1 - \alpha/2) \right\} = 1 - \alpha,$$

then we conclude that approximately

$$\mathbb{P} \left\{ \chi^2_{2(2m+1)}(\alpha/2) \leq 2(2m+1)\frac{\hat{f}(j/n)}{f(j/n)} \leq \chi^2_{2(2m+1)}(1 - \alpha/2) \right\} \approx 1 - \alpha.$$ 

This would lead to the following confidence interval for $f(j/n)$ of level approximately $1 - \alpha$:

$$2(2m+1)\frac{\hat{f}(j/n)}{\chi^2_{2(2m+1)}(1 - \alpha/2)} \leq f(j/n) \leq 2(2m+1)\frac{\hat{f}(j/n)}{\chi^2_{2(2m+1)}(\alpha/2)}.$$