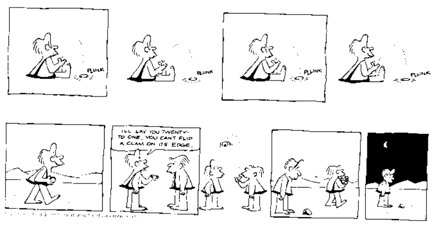
Probability

A Mathematical Model of Randomness



Where Are We Going?

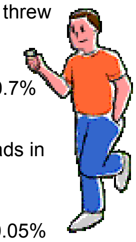
Where are we going?

- Meaning of "probability"
  - long run frequency
  - personal belief
- Games of chance
- Drawing tickets from a box
  - with replacement
  - without replacement
- Multiplying probabilities
- Conditional Probability
- Independence
- Adding probabilities

Probability as Long Run Frequency

In the eighteenth century, Compte De Buffon threw 2048 heads in 4040 coin tosses.

$$\text{Frequency} = \frac{2048}{4040} = .507 = 50.7\%$$



Around 1900, Karl Pearson threw 12,012 heads in 24,000 tosses

$$\text{Frequency} = \frac{12{,}102}{24{,}000} = .5005 = 50.05\%$$

We can think of probability as the long run average of the number of times an event occurs in a sequence of independent trials.

Each individual outcome is unpredictable, but the long run average is.

“Probability” and “chance” will be used interchangeably and may be expressed as decimals (e.g. .51) or percents (51%).

Field Goal Records of Philadelphia 76ers for 48 home games in the 1980-81 Season

Richardson	50%	(248)
Erving	52%	(884)
Hollins	46%	(419)
Cheeks	56%	(339)
C. Jones	47%	(272)
Toney	46%	(451)
B. Jones	54%	(433)
Mix	52%	(351)
Dawkins	62%	(403)

Probability as a measure of belief

Probability measures what odds you would be willing to accept on a bet.

With this interpretation, your probability may be different than mine.

This is a “subjective” notion, as compared to the “objective” notion of probability as long run relative frequency.

The subjective interpretation is known as “Bayesian” and the objective is called “frequentist.”

The Reverend Thomas Bayes

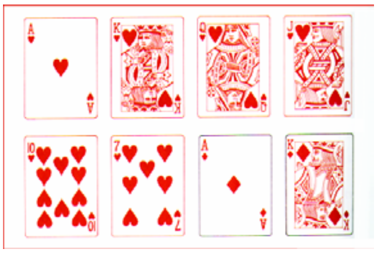
There have been bitter disputes between the “frequentists” and the “Bayesians.”

Some Properties

- The chance an event happens is between 0 and 100%.
- The chance an event does not happen is 100% minus the chance that it does happen.

Probability and Games of Chance

Shuffle these cards and choose one at random:



What is the probability that the card is:

- King of diamonds?
- Not King of diamonds?
- A heart?
- An ace?
- A face card?

If all outcomes are equally likely, the probability an event occurs is the number of ways it can occur divided by the total number of possible outcomes.

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A roulette wheel has 38 compartments:

0 and 00 are green

1--36: half red and half black

What is the probability of:

The number 7?

A black number?

Any number from 1--12?

A green number?

A red number, if all 26 preceding numbers were black?*

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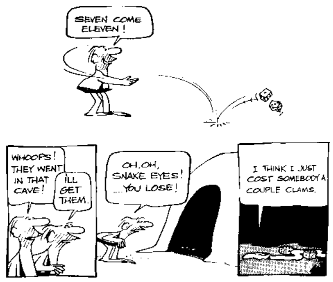
*This actually happened on August 18, 1913 at the Casino in Monte Carlo

Dice



Craps

Rules of Craps

- Throw two dice
- Lose on first round if you throw a 2, 3, or 12
- Win on first round if you throw a 7 or 11
- If you don’t win or lose on the first round the game continues in a way we won’t go into.

Probability of Winning?

Win if throw a 7 or 11

White die \ Red die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Probability of Losing?

Lose if throw 2, 3, or 12

White die \ Red die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Galileo (1562-1642)

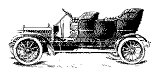
He was asked by gamblers about probabilities in a game in which there were 3 dice.

To figure out the answers, he listed all possibilities.

$6 \times 6 \times 6 = 6^3 = 216$ possibilities

If all possibilities are equally likely, you can find the chance of something happening by counting the number of ways it can happen and dividing by the total number of possible outcomes.

The Monty Hall Game

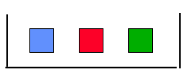


The rules:

- You choose a box
- I open an empty box (one of the other two)
- You can either stay with your original choice or switch
- You get contents of the box you have chosen

What strategy should you use? What is your chance of winning?

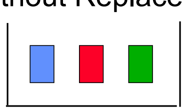
Drawing Tickets from a Box



With replacement: Put the ticket back in before you draw the next one.

Without replacement: Don't put it back in

Without Replacement

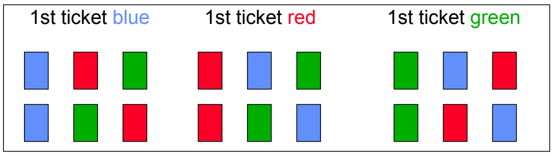


Probability that the first ticket is blue?

Probability that the first ticket is red?

Probability that the second is red?

Without replacement: probability that second is red?



What's the chance the first sock is red **and** the second sock is red?

Draw twice without replacement.

What would happen if we were to do this 300 times?

How many times would we expect to draw a red on the first draw? 200 $\frac{200}{300}$

Of these 200 how many times would we expect to draw a red on the second draw? 100 $\times \frac{100}{200}$

We thus expect to succeed 100 times out of 300, so the chance is 33.3%

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

The Multiplication Rule

The chance two things will happen equals the chance of the first times the chance that the second will happen given that the first has happened.

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Conditional Probability

Probability that second sock is red **given** that the first is red? This is called a **conditional probability.** We saw that it equals 50%.

A conditional probability is denoted by P(B|A): the probability that event B occurs given that event A has occurred.

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The Multiplication Rule

The chance two things will happen equals the chance of the first times the chance that the second will happen given that the first has happened.

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

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Two cards drawn without replacement from a 52 card deck

What's the chance that the first is red and the second is red?

Chance first is red =

Chance the second is red given that the first is red =

Chance they are both red =

Two cards drawn without replacement from a 52 card deck. What is the chance they are both aces?

Chance the first is an ace:

Chance that second is an ace given that the first is:

Three cards drawn without replacement from a 52 card deck. What is the chance they are all aces?

Chance the first is an ace:

Chance that second is an ace given that the first is:

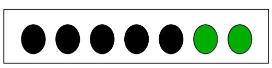
Chance that third is an ace, given that first two are:

The Olive Game

From *Never Give a Sucker an Even Bet,* by John Fisher (1976)

Get a bottle and put in two green olives and five black olives. Ask the sucker to shake them and bet him that he will not be able to get out three olives without having a green one among them.

Calculations:



Chance the first olive is black = _ Would you bet on this happening?

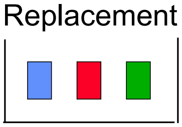
Chance the second olive is black given that the first one is black = __would you bet on this happening?

Chance that the first two olives are black = _____? Would you bet on this?

Chance that the third olive is black given that the first two are = ______? Would you bet on it?

Chance that the first two are black and the third is black = ____ ? Would you bet on it?

Two Tickets Drawn *with* Replacement



What is the chance that the first is blue?

What is the chance second is red?

What is the chance that the second is red given that the first is blue?

What is the chance that the first is blue and the second is red?

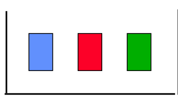
Independence

The outcomes in drawing with replacement are independent. The chances of the second do not depend on how the first turned out.

Chance they both happen = chance that first happens times chance that the second happens.

You can multiply probabilities of independent events to find probability they both happen.

$$P(A \text{ and } B) = P(A) \times P(B)$$



Draw with replacement

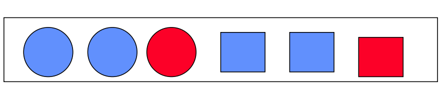
What's the chance none of three draws are green?

Independent or Dependent?

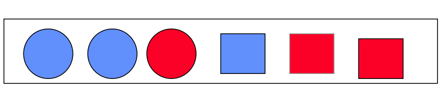
Suppose you know one outcome? Does this knowledge affect the chance of the other?

Example: Are shape and color independent?

Box 1



Box 2



Do basketball players shoot in streaks?

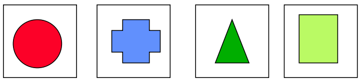
	Chance of a hit	Chance of a hit after a hit
Richardson	50%	49%
Erving	52%	53%
Hollins	46%	46%
Cheeks	56%	55%
C. Jones	47%	45%
Toney	46%	43%
B. Jones	54%	53%
Mix	52%	51%
Dawkins	62%	57%

What do random numbers look like: a string of 99 random 0's and 1's

1	0	1	0	0	0	1	1	1	1	0
0	1	1	0	1	1	0	1	0	0	0
1	0	1	1	1	1	0	0	0	1	0
1	1	0	1	0	0	1	1	0	1	1
0	0	1	1	0	1	0	0	0	0	0
0	1	0	0	1	1	0	0	0	1	0
0	1	1	1	0	0	0	0	0	1	1
1	0	1	0	0	1	1	0	0	1	0
1	1	0	0	0	0	1	0	1	0	0

What's the Chance?

ESP Experiment



Card is drawn at random. Chance of guessing the symbol by luck? 1/4

Deck is shuffled between draws. Chance of guessing three in a row?

Test is given to 500 people. If one of them guesses three in a row correctly, is that evidence that he has ESP?

Chance one person is wrong =

Chance that all 500 are wrong

The chance of at least one person getting them all right is

The Addition Rule: What's the chance one event happens **or** another event happens?

"Either one or both"
"At least one"

Win if throw a 7 or 11

White die \ Red die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
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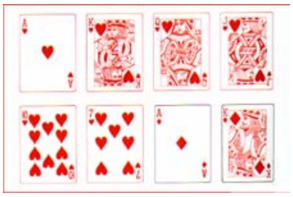
Chance of 7 =

Chance of 11 =

Chance of 11 or 7 =



What is the probability that the card is:

- A heart?
- An ace?
- A heart or an ace?

The Addition Rule

Two events are **mutually exclusive** if when one occurs the other cannot. That was the case in the first example (craps) but not in the second (cards).

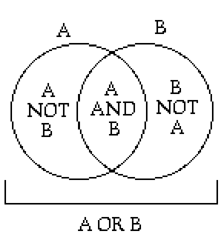
The chance that at least one (either one or both) of the two things will happen is the sum of the chances, **if** they are mutually exclusive.

P(A or B) = P(A) + P(B)

Otherwise

P(A or B) = P(A) + P(B) – P(A and B)

P(A or B) = P(A) + P(B) – P(A and B)



Example: The chance of failing an exam is 5%. The chance of getting an A is 15%. Is the chance of getting an A or failing = 20%?

The chance of passing is 95%. Is the chance of passing or getting an A = 110%?

The chance of getting a B is 35%. What is the chance of getting an A or a B?

Los Angeles Times

(August 24, 1987)

Several studies of sexual partners of people infected with AIDS show that a single act of vaginal intercourse has a surprisingly low risk of infecting the uninfected partner -- perhaps 1 in 100 or 1 in 1000. For an average, consider the risk to be 1 in 500. If there are 100 acts of intercourse with an infected partner, the odds of infection rise to 1 in 5.

Statistically, 500 acts of intercourse with one infected partner or 100 acts with 5 partners lead to a 100% probability of being infected (statistically, not necessarily in reality).

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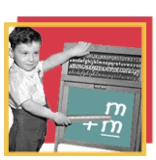
The lesson to be learned:



You cannot add probabilities *unless* the events are mutually exclusive: when one happens the other cannot.

Don't thoughtlessly add probabilities!

Can we calculate the chance of infection correctly?

Assume: chance of infection is the same in every act (1/500) and is *independent* in all 500 acts.

Chance of infection = 1 – chance of no infection

Review: Addition and Multiplication

To find the chance of two things *both happening*, you can multiply their chances if they are independent. If they are not independent, you can multiply the probability of the first one happening times the probability of the second one happening given that the first one has.

To find the chance of *at least one of two things happening*, you can add their chances **if** they are mutually exclusive.

Some Problems

1. Two draws are made *with replacement* from a box containing 10 tickets, numbered 1-10. What is the chance that ticket numbered 9 is drawn both times?

2. Two draws are made *with replacement* from a box containing 10 tickets, numbered 1-10. What is the chance the same ticket is drawn both times?

3. A box contains 5 tickets, one of which is blue and the others green. Two tickets are drawn with replacement. What is the chance that the second ticket drawn is blue?

3 (cont). A box contains 5 tickets, one of which is blue and the others green. Two tickets are drawn *without* replacement. What is the chance that the second ticket drawn is blue?

4. Five cards are dealt from a deck. What is the probability that the fifth card is the ace of spades?

5. What is the chance that the seventeenth card dealt is red?

6. What is the chance that the seventeenth card is red and the twentieth card is black?

7. A box contains 6 red tickets and 4 blue tickets. Three draws are made without replacement. What is the chance that:

(a) All are red

(b) None are red

(c) Not all are red

(d) At least one is red

9. What is the chance that a poker hand contains at least one ace?

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10. A card is drawn from a shuffled deck. What is the chance that it is red or a face card?

P(red)

P(face card)=

P(red or face card)=

Mutually exclusive?

P(red or face card) =

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Where Have We Been?

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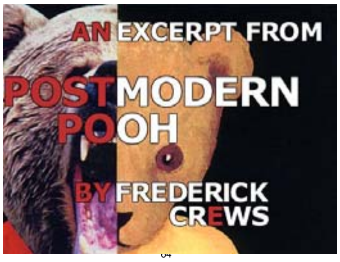
Where Have We Been?

- Meaning of "probability"
  - long run frequency
  - personal belief
- Games of chance.
  - Useful for illustrating ideas of probability. Can often calculate probabilities by counting. If all basic outcomes are equally likely, the chance something occurs equals the number of ways it can occur divided by the total number of outcomes.

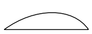
- Drawing tickets from a box
  - A useful model for randomness
  - with replacement
  - without replacement
- Conditional Probability. The chance that something occurs given that something else has occurred.
- Multiplying probabilities
  - Independence. If events are independent, you can multiply their probabilities to find the chance they *both* happen.

- Adding probabilities
  - If events are mutually exclusive, you can add their probabilities to find the chance that *at least one* of them happens.

How to write an impressive postmodern essay



Review Problem: Shimbui



Used in in Taoist temples in Taiwan for praying for your desires

Drop two on the ground.

If both flat sides are up, god is laughing at you.

If land one flat side up and one rounded side up three times in a row, the god may grant your wish.

What is the chance that first one lands flat side up?

What is the chance that both land flat side up?

What is the chance that first one lands rounded side up?

What is the chance that both land rounded side up?

What is the chance that two are different?

What is the chance that two are different three times in a row?