In the example in the introduction to this chapter, the null and alternative hypotheses each completely specify the probability distribution of the number of heads, as binomial(10,0.5) or binomial(10,0.7), respectively. These are called **simple hypotheses.** The Neyman-Pearson Lemma shows that basing the test on the likelihood ratio as we did is optimal:

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Suppose that H_0 and H_1 are simple hypotheses and that the test that rejects H_0 whenever the likelihood ratio is less than c and has significance level α. Then *any other test* for which the significance level is less than or equal to α has power less than or equal to that of the likelihood ratio test. ■

The point is that there are many possible tests. Any partition of the set of possible outcomes of the observations into a set that has probability less than or equal to α when the null hypothesis is true and its complement, and that rejects when the observations are in the complement has significance level less than or equal to α by construction. Among all such possible partitions, that based on the likelihood ratio maximizes the power.

Proof

Let $f(x)$ denote the probability density function or frequency function of the observations. A test of $H_0 : f(x) = f_0(x)$ versus $H_1 : f(x) = f_A(x)$ amounts to using a decision function $d(x)$, where $d(x) = 0$ if H_0 is accepted and $d(x) = 1$ if H_0 is rejected. Since $d(X)$ is a Bernoulli random variable, $E(d(X)) = P(d(X) = 1)$. The significance level of the test is thus $\alpha = P_0(d(X) = 1) = E_0(d(X))$, and the power is $P_A(d(X) = 0) = E_A(d(X))$. Here E_0 denotes expectation under the probability law specified by H_0, etc.

Let $d(X)$ correspond to the likelihood ratio test: $d(x) = 1$ if $f_0(x) < cf_A(x)$ and $E_0(d(X)) = \alpha$. Let $d^*(x)$ be the decision function of another test satisfying $E_0(d^*(X)) \leq E_0(d(X)) = \alpha$. We will show that $E_A(d^*(X)) \leq E_A(d(X))$. This will follow from the key inequality

$$d^*(x)[cf_A(x) - f_0(x)] \leq d(x)[cf_A(x) - f_0(x)]$$

which holds since if $d(x) = 1$, $cf_A(x) - f_0(x) > 0$ and if $d(x) = 0$, $cf_A(x) - f_0(x) \leq 0$. Now integrating (or summing) both sides of the inequality above with respect to x gives

$$cE_A(d^*(X)) - E_0(d^*(X)) \leq cE_A(d(X)) - E_0(d(X))$$

and thus

$$E_0(d(X)) - E_0(d^*(X)) \leq c[E_A(d(X)) - E_A(d^*(X))]$$

The conclusion follows since the left-hand side of this inequality is nonnegative by assumption. ■