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Probabilités/*Probability Theory*

Ranked functionals of Brownian excursions

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Abstract. It was shown, in our previous work, that the law of the sequence of normalized ranked lengths of Brownian excursions considered up to a random time T is the same for a large class of random times T.
We present now some results about (unnormalized) ranked heights of Brownian excursions, which although quite different from those obtained for the lengths, have led us to extend the scope of both studies.

Fonctionnelles ordonnées des excursions browniennes

Résumé. *Nous avons montré, dans un travail précédent, que pour de nombreux temps aléatoires T, la loi de la suite des longueurs ordonnées, et normalisées, des excursions browniennes jusqu'en l'instant T, est la même pour une large classe de temps T.*
Nous présentons maintenant certains résultats sur les hauteurs ordonnées des excursions browniennes qui, bien que très différents des résultats obtenus pour les longueurs, nous amènent à étendre le cadre de ces deux études à une classe générale de fonctionnelles des excursions browniennes.

Version française abrégée

0.1. On considère ici un processus de Markov fort $B = (B(t), t \geq 0)$, à valeurs dans $\mathbb{R}$ ou $\mathbb{R}^d$, valant *0* en $t = 0$; on suppose de plus que *0* est régulier pour lui-même (relativement à B). Le processus B admet donc un temps local $(L_t; t \geq 0)$ en *0*, continu en t.

Nous faisons maintenant l'hypothèse supplémentaire : il existe $\beta \in \mathbb{R}$, tel que B soit auto-similaire d'indice β, c'est-à-dire : pour tout $c > 0$, $\{B_{ct}, t \geq 0\} \overset{\text{(loi)}}{=} \{c^\beta B_t, t \geq 0\}$.
L'inverse continu à droite $\{\tau_\ell, \ell \geq 0\}$ du temps local $\{L_t ; t \geq 0\}$ est alors un subordinateur d'indice α, pour un certain $\alpha \in (0,1)$.

Note présentée par Jean-Pierre KAHANE.

0.2. Nous définissons maintenant (les lois de) deux processus associés à B : le pont $(B_t^{(\mathrm{p})}, t \le 1)$ et l'excursion $(B_t^{\mathrm{ex}}, t \le 1)$: soit, pour tout $T \ge 0$, éventuellement aléatoire,

$$G_T = \sup\{s < T : B_s = 0\} \quad \text{et} \quad D_T = \inf\{s \ge T : B_s = 0\}$$

On définit :

$$B^{(\mathrm{p})} = \left\{ \frac{1}{G_1^\beta} B(uG_1) \, ; \, u \le 1 \right\} \quad \text{et} \quad B^{\mathrm{ex}} = \left\{ \frac{1}{(D_1 - G_1)^\beta} B(G_1 + u(D_1 - G_1)) \, ; \, u \le 1 \right\}$$

Il découle de la propriété d'auto-similarité de B que, si l'on remplace dans les définitions précédentes G_1 et D_1 par G_T et D_T respectivement, lorsque $T > 0$ est une v.a. indépendante de B, les lois des processus ainsi obtenus ne dépendent pas de la loi de T.

0.3. Notre résultat principal est le suivant :

THÉORÈME. – *Soit* $F \ge 0$ *une fonctionnelle mesurable des excursions* $e = (e_t, 0 \le t \le V_e)$ *du processus* B *telle que :*

$$F(e_t, 0 \le t \le V_e) = V_e^\gamma F(V_e^{-\beta} e_{uV_e}, u \le 1)$$

pour un certain $\gamma > 0$. *On suppose* $E[(F(B^{\mathrm{ex}}))^{\alpha/\gamma}] < \infty$. *Soit* $F_1^{(\mathrm{p})} > F_2^{(\mathrm{p})} > \cdots > 0$ *la suite décroissante des valeurs strictement positives prises par* $F(e)$, *lorsque* e *décrit l'ensemble des excursions de* $B^{(\mathrm{p})}$.

Soit, de plus, Γ_α *une variable, indépendante de* B, *de densité :*

$$P(\Gamma_\alpha \in dt) = \frac{t^{\alpha-1} e^{-t} \, dt}{\Gamma(\alpha)} \qquad (t \ge 0)$$

Alors, la loi de la suite $\{F_j^{(\mathrm{p})} ; j = 1, 2, \ldots\}$ *est caractérisée par l'identité :*

$$\{\mu(\Gamma_\alpha^\gamma F_j^{(\mathrm{p})}) ; j = 1, 2, \ldots\} \overset{(loi)}{=} \{T_j^* ; j = 1, 2, \ldots\}$$

où $T_j^* = \left(\sum_{i=1}^{j} \varepsilon_i\right)/\varepsilon_0$, *la suite* $(\varepsilon_n ; n = 0, 1, 2, \ldots)$ *étant constituée de variables exponentielles, indépendantes, de paramètre 1*, et $\mu(x) \overset{def}{=} \int_0^\infty \frac{\alpha \, dt}{\Gamma(1-\alpha) \, t^{\alpha+1}} e^{-t} P(F(B^{\mathrm{ex}}) > x/t^\gamma)$.

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1. On ranked lengths of Brownian excursions

In this paragraph, we review some of our results about the sequences:

$$V_1(t) > V_2(t) > \ldots > V_n(t) > \ldots$$

of ranked lengths of excursions of Brownian motion up to time t, among which, if $g_t < t$, the length $(t - g_t)$ of the final meander $\{B_{g_t+u}, u \le t - g_t\}$ is hidden.

Of course, there is no such "residual" term in case t is a zero of the trajectory $\{B_u(\omega), u \ge 0\}$, which is the case, in particular, if $t = \tau_s \equiv \inf\{u : \ell_u > s\}$ for (ℓ_u) the local time at *0* of (B_u).

THEOREM 1 ([5], [6]). – *The law of the sequence* $\left\{\frac{V_n(T)}{T}\,;\, n = 1, 2, \cdots\right\}$ *is the same for each of the following random times :*

i) $T = s$ *for fixed* $s > 0$*;* ii) $T = \tau_u$ *for fixed* $u > 0$*;* iii) $T = H_m(a) \stackrel{def}{=} \inf\{t : V_m(t) > a\}$ *for fixed* $a > 0$ $(m \in \mathbb{N})$.

This identity in law when T is a fixed time, and an inverse local time provides a nice, a posteriori, explanation of Lévy's remarkable result (*see* [2]):

$$\frac{1}{s} A_s^+ \stackrel{(\text{law})}{=} \frac{1}{\tau_u} A_{\tau_u}^+, \quad \text{for given } s \text{ and } u \tag{1}$$

where $A_t^+ = \int_0^t du\, 1_{(B_u > 0)}$, both sides of (1) being arc sine distributed.

The above theorem motivated our definition of an admissible random time, as a time such that the law of $\left\{\frac{V_n(T)}{T}\,;\, n = 1, 2, \ldots\right\}$ is the common law encountered in the theorem. See [3] for a more complete discussion of admissible and non-admissible random times, and [4] for extensions to linear diffusions.

As a companion to Theorem 1, there is the following description of the law of the rank N_s of the meander length $(s - g_s)$ among the sequence $\{V_k(s)\}_{k \geq 1}$, i.e. : $s - g_s = V_{N_s}(s)$:

$$P(N_s = n | V_k(s)\,;\, k = 1, 2, \ldots) = \frac{V_n(s)}{s} \tag{2}$$

a formula from which the unconditioned law of N_s is easily deduced; *see*, e.g., formula (118) in [6].

2. On ranked heights of Brownian excursions

2.1. In this paragraph, we consider for fixed, or suitable random t, the sequence:

$$M_1(t) > M_2(t) > \cdots > M_n(t) > \cdots$$

of ranked heights of excursions of the absolute value of Brownian motion up to time t, in which the meander height: $\sup_{g_t \leq u \leq t} |B_u|$ is hidden.

2.2. We start with the following consequence of the Poisson properties of excursions considered up to τ_1, together with Williams' path decomposition at the maximum.

PROPOSITION 1. – *Let* $X_1 < X_2 < \cdots < X_n < \cdots$ *denote the sequence of successive jump times of a standard Poisson process, i.e.* $X_n = e_1 + \cdots + e_n$, $n \geq 1$, *where the* (e_i) *are independent exponential variables with parameter 1. Then* a) $\{(M_n(\tau_1))^{-1};\, n \geq 1\} \stackrel{(law)}{=} \{X_n\,;\, n \geq 1\}$. *Consequently, if* $\rho_n \stackrel{def}{=} \frac{M_{n+1}(\tau_1)}{M_n(\tau_1)}$, $n \geq 1$, *then the variables* $\{\rho_n^n;\, n \geq 1\}$ *are independent random variables uniformly distributed on (0,1).*

b) *The law of* τ_1*, conditionally on* $\{M_n(\tau_1)\,;\, n \geq 1\}$ *is characterized by the formula: for* $\lambda \in \mathbb{R}$, $\lambda \neq 0$,

$$E\left[\exp\left(-\frac{\lambda^2}{2}\tau_1\right) \Big| M_n(\tau_1), n \geq 1\right] = \prod_{n=1}^{\infty} \left(\frac{\lambda M_n(\tau_1)}{\operatorname{sh} \lambda M_n(\tau_1)}\right)^2$$

Note. – Above, and in the sequel, we use the abbreviations: sh, ch, th for the hyperbolic functions.

As a consequence of the Proposition, we obtain the following extension of an identity due to Knight [1] in the case $n = 1$.

COROLLARY. – *For every integer* $n \geq 1$*, and every* $\lambda > 0$,

$$E\left[\exp\left(-\frac{\lambda^2}{2}\frac{\tau_1}{(M_n(\tau_1))^2}\right)\right] = \left(\frac{2\lambda}{e^{2\lambda} - 1}\right)^{n-1} \left(\frac{\lambda}{\operatorname{sh}\lambda}\right)^2 \left(\frac{\operatorname{th}\lambda}{\lambda}\right)^n$$

2.3. Our next aim is to describe the law of the ranked heights $\{M_n^{\mathrm{br}}\,;\,n \geq 1\}$ of excursions of the absolute value of the standard Brownian bridge. Such a description will follow from:

THEOREM 2. – *Let S be an exponential time, with parameter 1/2, independent of $(B_u,\, u \geq 0)$. Denote $k(x) = \coth(x)$.*
Then, the sequence:

$$\{\ell_S,\, \ell_S\,(k\,(M_1\,(g_S)) - 1), \ldots, \ell_S\,(k\,(M_{n+1}\,(g_S)) - k\,(M_n\,(g_S))), \ldots\}$$

consists of independent exponential variables, with parameter 1.
As a consequence, the law of the sequence $\{M_n\,(g_S)\,;\,n \geq 1\}$ may be described as:

$$\{M_n\,(g_S);\, n \geq 1\} \overset{(law)}{=} \left\{\frac{1}{2}\log\left(1 + \frac{2}{T_n^*}\right);\, n \geq 1\right\}$$

where $T_n^* = \left(\sum_{i=1}^{n} \varepsilon_i\right)/\varepsilon_0$, the ε_i's, $i = 0, 1, 2, \ldots$, being independent exponentials, with parameter 1.

We now use the fact that $g_S \overset{(\mathrm{law})}{=} |N|$, where N is a centered, standard Gaussian variable, and $\left\{\frac{1}{\sqrt{g_S}}\,B_{u g_S},\, u \leq 1\right\}$ is a standard Brownian bridge, independent of g_S. Thus, we obtain the following:

COROLLARY. – *The sequence $\{|N|M_n^{\mathrm{br}}\,;\,n \geq 1\}$ is Markovian, with one-dimensional distributions given by:*

$$P\,(|N|M_n^{\mathrm{br}} \geq x) = (1 - \operatorname{th} x)^n, \quad n = 1, 2, 3, \ldots$$

and inhomogeneous transition probabilities:

$$P\,(|N|M_n^{\mathrm{br}} \leq x\,|\,|N|M_{n-1}^{\mathrm{br}} = y) = \left(\frac{\operatorname{th} x}{\operatorname{th} y}\right)^n, \quad \text{for} \quad x \leq y \quad \text{and} \quad n = 2, 3, \ldots$$

2.4. Our final aim is to describe the law of the ranked heights $\{M_n\,(t)\,;\,n = 1, 2\ldots\}$ of the standard reflecting Brownian motion, as well as that of J_t, the rank of the meander height: $\sup_{g_t \leq u \leq t} |B_u|$ among the sequence $\{M_n\,(t)\,;\,n = 1, 2\ldots\}$.

Both descriptions involve the law of $\Sigma \overset{\mathrm{def}}{=} \sup_{g_S \leq u \leq S} |B_u|$.

PROPOSITION 2. – *The joint law of $(|B_S|,\, \Sigma)$ is given by:*

$$P\,(|B_S| \in \mathrm{d}x,\, \Sigma \in \mathrm{d}y) = \frac{\operatorname{sh} x}{(\operatorname{sh} y)^2}\,\mathrm{d}x\,\mathrm{d}y\,1_{(x \leq y)}$$

and the one-dimensional marginals are:

$$P\,(|B_S| \in \mathrm{d}x) = \exp(-x)\,\mathrm{d}x\,;\; P\,(\Sigma \in \mathrm{d}y) = \frac{\mathrm{d}y}{2\operatorname{ch}^2(y/2)}$$

From this proposition, together with our knowledge of the law of $\{M_n\,(g_S)\,;\,n = 1, 2, \ldots\}$ and the scaling property of Brownian motion, we deduce the following:

THEOREM 3. – 1. *The distribution of the ranked sequence $\{M_n\,(S)\,;\,n = 1, 2, \ldots\}$ is determined by the following formula: for every non-negative Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$E\left[\exp\left(-\sum_{n=1}^{\infty} \varphi\,(M_n\,(S))\right)\right] = \frac{N_\varphi}{D_\varphi}$$

where $N_\varphi = \int_0^\infty \frac{\mathrm{d}x}{2\operatorname{ch}^2(x/2)} \exp(-\varphi\,(x))$ and $D_\varphi = \int_0^\infty \mathrm{d}x \left(\frac{1}{x^2} - \frac{\exp(-\varphi\,(x))}{\operatorname{sh}^2(x)}\right)$.

2. *For any integer* $n \geq 1$, *one has:*

$$P(M_n(S) \geq x) = E\left[\exp\left(-\frac{x^2}{2M_n^2(1)}\right)\right] = \frac{\exp(-(n-1)x)}{(\operatorname{ch} x)^n}$$

We then use the previous Proposition 2 to obtain the law of $J_t \left(\overset{(\text{law})}{=} J_1\right)$.

THEOREM 4. – *Let* $J_S \left(\overset{(law)}{=} J_1\right)$ *denote the rank of the meander height* Σ, *among the sequence* $\{M_n(S)\}$. *Then we have* $P(J_S > n|\Sigma) = (1 - \operatorname{th}(\Sigma))^n$. *Consequently:*

$$P(J_1 > n) = \int_0^1 \mathrm{d}t \, \frac{(1-t)^{2n}}{(1+t^2)^n} = \int_0^1 \mathrm{d}x \, \frac{x^n}{\sqrt{2x - x^2}\,(1 + \sqrt{2x - x^2})}$$

References

[1] F.B. Knight. Inverse local times, positive sojourns, and maxima for Brownian motion. Astérisque 157–158 (1988), 233–247.

[2] P. Lévy. Sur certains processus stochastiques homogènes. Compositio Math. 7 (1939), 283–339.

[3] J. Pitman, M. Yor. On the relative lengths of excursions derived from a stable subordinator. Sém. Prob. XXXI, Lect. Notes in Math. 1655, 287–305, Springer, 1997.

[4] J. Pitman, M. Yor. On the lengths of excursions of some Markov processes. Sém. Prob. XXXI, Lect. Notes in Math. 1655, 272–286, Springer, 1997.

[5] J. Pitman, M. Yor. Arc sine laws and interval partitions derived from a stable subordinator. Proc. London Math. Soc. (3), 65 (1992), 326–356.

[6] J. Pitman, M. Yor. The two-parameter Poisson–Dirichlet distribution derived from stable subordinators. Ann. Prob., 25(2) (1997), 455–500.