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# FURTHER ASYMPTOTIC LAWS OF PLANAR BROWNIAN MOTION[1]

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The asymptotic distributions for large times of a variety of additive functionals of planar Brownian motion $Z$ are derived. Associated with each point in the plane, and with the point infinity, there is a complex Brownian motion governing the asymptotic behavior of windings of $Z$ close to that point. An independent Gaussian field over the plane governs fluctuations in local occupation times of $Z$, while a further independent family of complex Brownian sheets governs finer features of the windings of $Z$. These results unify and extend earlier results of Kallianpur and Robbins, Spitzer, Kasahara and Kotani, Messulam and the authors.

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**0. Introduction.** This paper is a sequel to Pitman and Yor (1986), henceforth referred to as AL*, where results on the asymptotic distributions of winding and crossing numbers were presented as part of a larger framework of asymptotic laws for planar Brownian motion. To follow the present paper in any detail, the reader should have at hand a copy of that earlier work, to which frequent references will be made simply by an asterisk. For example, (1.a)* refers to (1.a) of AL*, Section 1* means Section 1 of AL* and Knight (1971)* refers to the paper by Knight (1971) in the references of AL*. Two corrections to AL* appear at the end of this paper.

We attempted in AL* to unify as well as we could the known results on asymptotic distributions of functionals of planar Brownian motion. Still, the richness of this subject seems unbounded. We now see no end to the possible

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degree of refinement of such asymptotic laws. Our purpose in this article is to present some extensions of results in AL*, linked in various ways to the most basic asymptotic laws for additive functionals considered there. Some of these results were presented without proof in Pitman and Yor (1987). We have chosen to explore the asymptotics of those functionals which seemed to us most natural from either an analytic or geometric point of view, though this by no means exhausts the subject.

A focal point of this paper is the asymptotic behavior as $t \to \infty$ of additive functionals of a complex Brownian motion $Z$ of the form

$$\text{(0.a)} \qquad \text{(i)} \quad \int_0^t f(Z_s)\,ds \quad \text{and} \quad \text{(ii)} \quad \int_0^t f(Z_s)\,dZ_s,$$

for various functions $f$. The two studies are intimately related by Itô's formula, a connection exploited already in similar contexts by Papanicolaou, Stroock and Varadhan (1977)* and Kasahara and Kotani (1979)*.

In Section 1, we consider the asymptotic distribution of the stochastic integral (ii) above in case $f$ is holomorphic in $D_j \setminus \{z_j\}$ for a neighborhood $D_j$ of each point $z_j$, $1 \le j \le n$. The result obtained here, previously announced as Theorem 8.6*, brings out the fundamental role played by the processes $\Phi^j_{\pm}(t)$ of big and small windings about $z_j$, and is an extension of Theorem 6.1* governing the asymptotics of these winding processes.

Section 2 offers some developments of the concept of a *log scaling law*, introduced in Chapter 8* to unify a large body of asymptotic laws. For martingale additive functionals of type (ii) above, subject to a growth condition on $f$ near 0, functionals which obey a log scaling law are characterized, and their limits identified.

Section 3 offers still further refinements for the asymptotics of winding-like functionals. Thus we show that not only does the normalized winding

$$\frac{1}{\log t}\Phi(t) \overset{\text{def}}{=} \frac{1}{\log t}\int_0^t |Z_s|^{-2}(X_s\,dY_s - Y_s\,dX_s)$$

converge in law as $t \to \infty$, but so does

$$\frac{1}{\log t}\int_0^t f(e^{i\Phi_s})\,d\Phi_s$$

for every bounded Borel function $f\colon C \to \mathbb{C}$. (Here, and throughout the paper, we use $C$ for the unit circle and $\mathbb{C}$ for the complex plane.) In particular, the quadruple

$$\frac{1}{\log t}\left(\int_0^t |Z_s|^{-2}X_s\,dY_s;\ \int_0^t |Z_s|^{-2}Y_s\,dX_s;\ \int_0^t |Z_s|^{-2}X_s\,dX_s;\ \int_0^t |Z_s|^{-2}Y_s\,dY_s\right)$$

converges in law as $t \to \infty$ and so does the normalized process of windings in sectors

$$\left(\frac{1}{\log t}\int_0^t 1_{(\arg(Z_s) \in (0,\alpha))}\,d\Phi_s;\ \alpha \in [0, 2\pi]\right).$$

Moreover, as we show in Section 4, the convergence of these integrals of the winding process about one point also holds jointly when one considers the same quantities relative to a finite number of points. Itô's formula then allows us to derive the asymptotic distributions of the normalized Riemann integrals

$$\frac{1}{\log t}\int_0^t \frac{ds}{|Z_s - z_j|^2} f_j(e^{i\Phi_s^j}), \qquad 1 \le j \le n,$$

for bounded Borel functions $f_j$: $C \to \mathbb{C}$ such that

$$\int_0^{2\pi} da\, f_j(e^{ia}) = 0,$$

where $\Phi_s^j$ is the winding number of $Z$ around $z_j$ up to time $s$ for $n$ distinct points $z_1, \dots, z_n$ distinct also from the starting point $z_0$ of the complex Brownian motion $Z$. Section 5 provides a study of a different character for the asymptotics of occupation times of variously positioned discs in the plane. A striking feature here in the limit is the whole Ray–Knight process of Markovian local times of one-dimensional Brownian motion.

Section 6 starts by spelling out the connection between results of Kasahara and Kotani (1979)* for additive functionals of bounded variation and those of Messulam and Yor (1982)* for martingale additive functionals. It is then shown how these "second order" results are linked to the "first order" winding results in particular and log scaling laws in general.

A key to many of our results is a criterion for the asymptotic independence of the Brownian motions associated with two continuous local martingales. This criterion, stated in an Appendix, is a less restrictive version of a criterion developed in Le Gall and Yor (1986)* and AL*. We expect this simple criterion to find applications in other problems involving the asymptotic behavior of additive functionals of diffusions.

**1. Asymptotic residue theorem.** We begin by proving the following theorem, stated as Theorem 8.6*, which is an extension of the asymptotic joint distribution of windings.

THEOREM 1.1. *Let $(Z_t, t \ge 0)$ be a complex Brownian motion started at $z_0$, and suppose that $z_0, z_1, \dots, z_n$ is a finite set of distinct points in $\mathbb{C}$. Suppose that $f$ is a complex valued function such that*:

(i) *$f$ is holomorphic in $D_j \setminus \{z_j\}$ for a neighborhood $D_j$ of each point $z_j$, $j = 1, \dots, n$.*

(ii) *$f$ is bounded and measurable on the complement of the union of these neighborhoods.*

(iii) *$f$ is holomorphic in a neighborhood of $\infty$ and $\lim_{z\to\infty} f(z) = 0$.*

*Then, as $t \to \infty$,*

$$\text{(1.a)} \qquad \frac{2}{\log t}\int_0^t f(Z_s)\, dZ_s \xrightarrow{d} \sum_j \operatorname{Res}(f, z_j)\left(\frac{\Lambda}{2} + iW_-^j\right) + \operatorname{Res}(f, \infty)\left(\frac{\Lambda}{2} - 1 + iW_+\right),$$

*where* $(W_+, W_-^j, \Lambda)$ *is an* $(n+2)$*-tuple of real random variables, such that for each* $j$, $(W_+, W_-^j, \Lambda)$ *is distributed as*

$$\left(\int_0^\sigma 1_{(\beta_s \geq 0)}\, d\theta_s, \int_0^\sigma 1_{(\beta_s < 0)}\, d\theta_s, \lambda_\sigma\right),$$

*where* $\beta$ *and* $\theta$ *are two independent Brownian motions,* $\sigma = \inf\{t\colon \beta_t = 1\}$, $(\lambda_t, t \geq 0)$ *is the local time of* $\beta$ *at* 0 *and the variables* $W_+$ *and* $(W_-^j, 1 \leq j \leq n)$ *are conditionally independent given* $\Lambda$.

Before proving this theorem, we remark that, as a particular case, the asymptotic distribution of the large and small windings around $(z_i; 1 \leq i \leq n)$ is easily recaptured from it. Indeed, let

$$f(z) = \sum_{j=1}^n \left(\lambda_j \frac{1_{(|z-z_j| \leq r_j)}}{(z-z_j)} + \mu_j \frac{1_{(|z-z_j| > r_j)}}{(z-z_j)}\right),$$

where $\lambda_j$, $\mu_j$ are arbitrary complex numbers and $r_j$ are fixed positive reals. Let

$$\Phi_-^j(t) = \operatorname{Im}\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(|Z_s - z_j| \leq r_j)}, \qquad \Phi_+^j(t) = \operatorname{Im}\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(|Z_s - z_j| \geq r_j)},$$

$$\Psi^j(t) = \operatorname{Re}\int_0^t \frac{dZ_s}{(Z_s - z_j)} 1_{(|Z_s - z_j| \leq r_j)}.$$

We call $\Phi_-^j$ the process of small windings of $Z$ around $z_j$ and call $\Phi_+^j$ the process of big windings around $z_j$. We then deduce from (1.a) that

$$\text{(1.b)} \qquad \frac{2}{\log t}\left(\Phi_-^j(t), \Phi_+^j(t); \Psi^j(t); 1 \leq j \leq n\right)$$

converges in distribution to

$$\left(W_-^j, W_+, \frac{\Lambda}{2}; 1 \leq j \leq n\right).$$

By Tanaka's formula,

$$\text{(1.c)} \qquad \Psi^j(t) = -\left(\log|Z_t - z_j| - \log r_j\right)^- + \tfrac{1}{2}L^j(t),$$

where $L^j$ is the local time at level $\log r_j$ of the local martingale $(\log|Z_t - z_j|; t \geq 0)$. As a consequence of (1.c), we may replace $\Psi^j(t)$ by $\frac{1}{2}L^j(t)$ in the expression (1.b). Thus

$$\text{(1.b}'\text{)} \qquad \frac{2}{\log t}\left(\Phi_-^j(t), \Phi_+^j(t); L^j(t); 1 \leq j \leq n\right)$$

converges in distribution, as $t \to \infty$, toward

$$\left(W_-^j, W_+, \Lambda; 1 \leq j \leq n\right).$$

This special case of Theorem 1.1, established already as Theorem 6.1*, will be used in the following proof.

**Proof of Theorem 1.1.** 1. During the proof, we shall use several times the fact that for any Borel function $\psi$: $\mathbb{C} \to \mathbb{C}$ which is locally bounded, the properties

$$(1.d)\qquad \frac{1}{\log t}\sup_{s\le t}\left|\int_0^s \psi(Z_u)\,dZ_u\right| \xrightarrow[t\to\infty]{P} 0 \quad\text{and}\quad \frac{1}{(\log t)^2}\int_0^t du|\psi|^2(Z_u) \xrightarrow[t\to\infty]{P} 0$$

are equivalent. This is a particular case of Lemma A.1*. Hence, we deduce from the Kallianpur–Robbins law (1.a)* that for any locally bounded function $\psi \in L^2(\mathbb{C}, dx\,dy)$, the property (1.d) is satisfied.

2. We first assume that $f$ has compact support. We then deduce from 1 and our hypotheses on $f$ that

$$\frac{1}{\log t}\sup_{s\le t}\left|\int_0^s f(Z_u)\,dZ_u - \sum_j \int_0^s f(Z_u)1_{(Z_u\in D_j)}\,dZ_u\right| \xrightarrow[t\to\infty]{P} 0.$$

3. Moreover, for each $j = 1, 2, \dots, n$, there exists a strictly positive number $\varepsilon_j$ such that $f$ restricted to $D(z_j, \varepsilon_j)\setminus\{z_j\}$, where $D(z_j, \varepsilon_j)$ is the open disc with center $z_j$ and radius $\varepsilon_j$, admits a Laurent expansion

$$f(z) = h_j(z) + g_j\left(\frac{1}{z - z_j}\right),$$

with $h_j$ holomorphic in $D(z_j, \varepsilon_j)$ and $g_j$ an entire function with $g_j(0) = 0$. Therefore,

$$(1.e)\qquad g_j(z) = \operatorname{Res}(f, z_j)z + z^2\tilde{g}_j(z),$$

with $\tilde{g}_j$ another entire function.

Using the equivalence of (1.d) again, we obtain, for each $j \le n$,

$$\frac{1}{\log t}\sup_{s\le t}\left|\int_0^s\left[f(Z_u) - g_j\left(\frac{1}{Z_u - z_j}\right)\right]1_{(Z_u\in D_j)}\,dZ_u\right| \xrightarrow[t\to\infty]{P} 0.$$

4. With the help of Tanaka's formula as in (1.c) and (1.b′) above, to prove the theorem in the case where $f$ has compact support it now remains to show that the function $\tilde{g}_j$ in (1.e) does not contribute to the limit. That is to say,

$$(1.f)\qquad \frac{1}{\log t}\int_0^t dZ_u 1_{(|Z_u - z_j|\le\varepsilon_j)}\frac{1}{(Z_u - z_j)^2}\tilde{g}_j\left(\frac{1}{Z_u - z_j}\right) \xrightarrow[t\to\infty]{P} 0.$$

Let $\tilde{G}_j$ be the primitive of $\tilde{g}_j$ such that $\tilde{G}_j(0) = 0$. Then, from Itô's formula

$$\tilde{G}_j\left(\frac{1}{Z_t - z_j}\right) = \tilde{G}_j\left(\frac{1}{z_0 - z_j}\right) - \int_0^t \frac{dZ_u}{(Z_u - z_j)^2}\tilde{g}_j\left(\frac{1}{Z_u - z_j}\right).$$

Since $1/(Z_t - z_j) \xrightarrow[t\to\infty]{P} 0$, we have

$$\frac{1}{\log t}\int_0^t \frac{dZ_u}{(Z_u - z_j)^2}\tilde{g}_j\left(\frac{1}{Z_u - z_j}\right) \xrightarrow[t\to\infty]{P} 0.$$

Consequently, in order to prove (1.f), we may replace $1_{(|Z_u - z_j| \le \varepsilon_j)}$ by $1_{(|Z_u - z_j| \ge \varepsilon_j)}$ in the left-hand side of (1.f). The proof of (1.f) is now ended by remarking that the function of $z$,

$$\frac{1}{(z - z_j)^2} 1_{(|z - z_j| \ge \varepsilon_j)},$$

is bounded, belongs to $L^2(\mathbb{C}, dx\, dy)$ and so satisfies (1.d).

5. In the case where $f$ is holomorphic in a neighborhood of $\infty$, and $\lim_{z \to \infty} f(z) = 0$, the above proof is easily modified by remarking that $f$ may be written as

$$f(z) = -\mathrm{Res}(f, \infty)\frac{1}{z} + \frac{1}{z^2} g\left(\frac{1}{z}\right)$$

with $g$ holomorphic in an open neighborhood of $\{z\colon |z| \le 1/\eta\}$, for some $\eta > 0$. It then remains to prove, as we have just done, that

$$\frac{1}{\log t} \int_0^t \frac{dZ_s}{Z_s^2} g\left(\frac{1}{Z_s}\right) 1_{(|Z_s| \ge \eta)} \xrightarrow[t \to \infty]{P} 0.$$

This completes the proof of Theorem 1.1. □

REMARK. Note that, because of the equivalence (1.d), the integral from 0 to $t$ in (1.f) cannot be replaced by the supremum over $s$ in $(0, t)$ of the modulus of the integral from 0 to $s$.

**2. Log scaling laws.** In the course of obtaining asymptotic distributions for various functionals of complex Brownian motion, we realized that we were performing the same operations again and again, namely a certain time change followed by Brownian scaling. To avoid repetition, and speed up procedure, we introduced the notion of *log scaling laws* (Chapter 8*). We now recall the basic notation related to this notion.

Brownian motion $Z = (Z_t,\ t \ge 0)$, starting at $z_0$, can be expressed as

$$Z_t = z_0 \exp(\zeta(U_t)),$$

where $\zeta = \beta + i\theta$ is a complex-valued Brownian motion started at 0 and

$$U_t = \int_0^t \frac{ds}{|Z_s|^2}$$

is the *logarithmic clock*. A Brownian functional $G(t) = G(t, Z)$ can always be rewritten as

$$G(t, Z) = \Gamma(U_t, \zeta) \tag{8.r)*}$$

for some process $\Gamma(u) = \Gamma(u, \zeta)$. Now, let $\Gamma^{(h)}$ be obtained from $\Gamma$ by the Brownian scaling operation

$$\Gamma^{(h)}(u, \zeta) = \frac{1}{h}\Gamma(h^2 u, \zeta), \qquad h > 0.$$

In Definition 8.3*, we say that the Brownian functional $G$ is *logarithmically attracted* to the process $\gamma = (\gamma(u, \zeta);\ u \geq 0)$ if

$$\Gamma^{(h)}(u, \zeta) - \gamma(u, \zeta^{(h)}) \xrightarrow[h \to \infty]{P} 0, \tag{8.s)*$$

where the convergence is uniform on compact sets. Equivalently, by Brownian scaling

$$\Gamma^{(h)}(\cdot, \zeta^{(1/h)}) \xrightarrow[h \to \infty]{P} \gamma(\cdot, \zeta) \tag{8.t)*$$

in the same sense. We may also say that $\gamma$ is the *logarithmic attractor* of $G$. As a consequence of this definition, we obtain in particular

$$\frac{2}{\log t} G(t, Z) \xrightarrow[t \to \infty]{d} \gamma(\sigma_1, \zeta), \tag{2.a}$$

where $\sigma_a = \inf\{u\colon \beta(u) = a\}$. See Theorem 8.4* for more consequences.

We turn now to the question of what processes $\gamma$ may arise as logarithmic attractors, and what functionals $G$ are attracted to them. We restrict our attention to continuous processes $G$. Roughly speaking, the attractors $\gamma$ are functions of $\zeta$ which commute with Brownian scaling.

PROPOSITION 2.1. *A continuous process $\gamma$ is the logarithmic attractor of some Brownian functional $G$ with continuous paths if and only if there exists a random variable $\hat{\gamma}$ such that*

$$\gamma(u, \zeta) = \sqrt{u}\, \hat{\gamma}(\zeta^{(\sqrt{u})}) \quad \textit{for all } u \textit{ a.s.} \tag{2.b}$$

PROOF. Suppose $\gamma$ is the logarithmic attractor of $G$. From (8.t)*, for each fixed $u$,

$$\frac{1}{h} \Gamma(uh^2, \zeta^{(1/h)}) \xrightarrow{P} \gamma(u, \zeta). \tag{2.c}$$

Therefore, for every fixed $k > 0$,

$$\frac{1}{hk} \Gamma(uh^2k^2, \zeta^{(1/hk)}) \xrightarrow{P} \gamma(u, \zeta).$$

By Brownian scaling, this implies

$$\frac{1}{hk} \Gamma(uh^2k^2, \zeta^{(1/h)}) - \gamma(u, \zeta^{(k)}) \xrightarrow{P} 0. \tag{2.d}$$

Replacing $u$ by $u/k^2$, (2.c) and (2.d) yield

$$\gamma(u, \zeta) = k\gamma(u/k^2, \zeta^{(k)}) \quad \text{for all } u \text{ a.s.}$$

Finally, (2.b) follows by taking $k = \sqrt{u}$.

Conversely, if a continuous process $\gamma$ satisfies (2.b), then for all $h > 0$,

$$\frac{1}{h} \gamma(uh^2, \zeta^{(1/h)}) = \gamma(u, \zeta), \qquad u \geq 0 \text{ a.s.},$$

indicating that the process $\gamma$ satisfies

$$\gamma^{(h)}(\zeta) = \gamma(\zeta(h)), \quad \text{a.s.}$$

Thus $G(t) = \gamma(U_t, \zeta)$ is logarithmically attracted to $\gamma$. $\square$

To illustrate the above proposition, suppose for example that the process $\gamma$ is of the form

$$\gamma(u, \zeta) = \int_0^u d\beta_v \, \eta(v, \zeta)$$

with $\eta(v, \zeta)$ a continuous process adapted to the filtration of $\zeta$, such that

$$E\left(\int_0^u dv \, \eta(v, \zeta)^2\right) < \infty, \qquad u > 0.$$

Then the identity (2.b) implies that $\gamma$ is the logarithmic attractor of some continuous process $G$ iff for every $v > 0$,

$$\eta(u, \zeta) = \eta\left(u/v, \zeta^{(\sqrt{v})}\right), \qquad du \text{ a.s.},$$

so by continuity of $\eta(\cdot, \zeta)$, for every $v > 0$,

$$\eta(v, \zeta) = \eta\left(1, \zeta^{(\sqrt{v})}\right) \quad \text{a.s.} \tag{2.e}$$

Conversely, if $\eta(1, \zeta) \in L^2(\sigma(\zeta_u, \ u \le 1))$, then by the monotone class theorem there exists a modification of

$$(v, \zeta) \to \eta\left(1, \zeta^{(\sqrt{v})}\right)$$

which is predictable and the process

$$\gamma(u, \zeta) = \int_0^u d\beta_v \, \eta\left(1, \zeta^{(\sqrt{v})}\right)$$

is a logarithmic attractor. In case $\eta(u, \zeta) = f(\beta_u)$, it is necessary for (2.e) that for every $v > 0$

$$f(x) = f(x/\sqrt{v}), \qquad dx \text{ a.e.},$$

which implies that

$$f(x) = f_- 1_{(x \le 0)} + f_+ 1_{(x \ge 0)}, \qquad dx \text{ a.e.} \tag{2.f}$$

for some constants $f_-$ and $f_+$.

The following theorem, which was stated as Theorem 8.5*, provides a further development.

THEOREM 2.2. *Let*

$$G(t) = \int_0^t f(Z_s) \frac{dZ_s}{Z_s} \quad \textit{for a bounded Borel function } f.$$

*Then the following are equivalent:*

(i) $G$ *is logarithmically attracted to some process* $\gamma$.
(ii) $f[\exp(h(x+iy))]$ *converges in* $L^1_{\mathrm{loc}}(dx\,dy)$ *as* $h \to \infty$.
(iii) *There exist constants* $p_+$ *and* $p_-$ *such that as* $R \to \infty$,

$$\frac{1}{\log R}\int_{D(R,\pm)} \frac{dx\,dy}{|z|^2}|f(z) - p_\pm| \to 0,$$

*where*

$$D(R,+) = \{z\colon 1 \le |z| \le R\} \quad \text{and} \quad D(R,-) = \{z\colon R^{-1} \le |z| \le 1\}.$$

*If these conditions are satisfied, then the logarithmic attractor* $\gamma$ *is*

$$\gamma(u) = \int_0^u p(\beta_v)\,d\zeta_v,$$

*where*

$$p(x) = p_+1(x \ge 0) + p_-1(x \le 0)$$

*and there are the alternative formulas*

$$\begin{aligned} p_\pm &= \lim_{R\to\infty}\frac{1}{2\pi\log R}\int_{D(R,\pm)}\frac{dx\,dy\,f(z)}{|z|^2} \\ &= \lim_{R\to\infty}\frac{1}{\log R}\int_{I(R,\pm)}\frac{dr}{r}\left\{\frac{1}{2\pi i}\int_{C_r}\frac{dz}{z}f(z)\right\}, \end{aligned}$$

*where* $I(R,+) = [1,R]$, $I(R,-) = [R^{-1},1]$ *and* $C_r = \{z\colon |z| = r\}$.

REMARK. A discussion of the similarities and differences between Theorems 1.1 and 2.2 is given in AL*, before Theorem 8.6*.

PROOF. Time changing $G$ via the logarithmic clock $U$,

$$G(t,Z) = \Gamma(U_t,\zeta),$$

where

$$\Gamma(u,\zeta) = \int_0^u f(\exp\zeta_v)\,d\zeta_v.$$

According to (8.t)*, if $G$ is logarithmically attracted to some $\gamma$ as $h \to \infty$, the process

$$\Gamma^{(h)}(u,\zeta^{(1/h)}) = h\int_0^u f(\exp(h\zeta_v))\,d\zeta_v$$

converges, uniformly on compact sets, in probability, to $\gamma(u)$. By Lemma A.1* such convergence takes place iff

$$\text{(2.g)} \qquad \int_0^s du\left|\phi(h\beta_u;\, e^{ih\theta_u}) - \phi(k\beta_u;\, e^{ik\theta_u})\right|^2 \xrightarrow{P} 0 \quad \text{as } h,k \to \infty,$$

where we have used the notation $\phi(x;\, e^{i\theta}) = f(\exp(x+i\theta))$ and $\zeta_u = \beta_u + i\theta_u$.

The proof is easily completed using the following lemma, which indicates the only possible limits in $L^2([0, s], du)$ for processes $\phi(h\beta_u, e^{ih\theta_u})$. □

LEMMA 2.3. *Let $\phi$: $\mathbb{R} \times C \to \mathbb{R}$ be bounded. The condition* (2.g) *is satisfied if and only if there exist two reals $p_+$ and $p_-$ such that, for $p(x) = p_+1(x > 0) + p_-1(x < 0)$,*

$$(2.\mathrm{g}') \qquad \int_0^s du \left| \phi\left(h\beta_u; e^{ih\theta_u}\right) - p(\beta_u) \right|^2 \xrightarrow{P} 0 \quad \textit{as } h \to \infty.$$

PROOF. Since $\phi$ is bounded, (2.g) is equivalent to

$$E\left( \int_0^s du \left| \phi\left(h\beta_u; e^{ih\theta_u}\right) - \phi\left(k\beta_u, e^{ik\theta_u}\right) \right|^2 \right) \to 0 \quad \text{as } h, k \to \infty.$$

This expectation is identical to

$$\int_{\mathbb{R}^2} dx\, dy\, \Delta\left( \frac{|z|^2}{s} \right) \left| \phi(hx; e^{ihy}) - \phi(kx; e^{iky}) \right|^2,$$

with

$$(2.\mathrm{h}) \qquad \Delta(r) = \frac{1}{2\pi} \int_r^\infty \frac{du}{u} \exp\left( -\frac{u}{2} \right)$$

a strictly positive function in $L^1(dr)$. Therefore, there exists a function $p(x, y)$ defined a.s. $dx\, dy$ such that for all compact subsets $K$ of $\mathbb{R}^2$,

$$\int_K dx\, dy \left| \phi(hx; e^{ihy}) - p(x, y) \right| \to 0 \quad \text{as } h \to \infty.$$

Replacing $h$ by $h/t$ for $t > 0$ and letting $h \to \infty$, we obtain

$$(2.\mathrm{i}) \qquad \text{for all } t > 0, \qquad p(x, y) = p(tx, ty), \qquad dx\, dy \text{ a.s.}$$

Much in the same vein, since $y \to \phi(x; e^{iy})$ has period $2\pi$,

$$(2.\mathrm{j}) \qquad p(x, y) = p(x, y + 2\pi), \qquad dx\, dy \text{ a.s.}$$

It remains to show that for $p$ satisfying (2.i) and (2.j), there exist $p_+$ and $p_-$ such that

$$(2.\mathrm{k}) \qquad p(x, y) = p_+1(x > 0) + p_-1(x < 0), \qquad dx\, dy \text{ a.s.}$$

Clearly it is enough to deal with the existence of $p_+$. From (2.i), we deduce

$$\int_0^\infty dx \int_{-\infty}^\infty dy \int_0^\infty dt \left| p(x, y) - p(tx, ty) \right| = 0.$$

Make the change of variable $u = tx$ and then change the order of integration to obtain

$$\int_0^\infty du \int_0^\infty dx \int_{-\infty}^\infty dy \left| p(x, y) - p(u, u(y/x)) \right| = 0,$$

so that there exists at least $u_+$ such that

$$p(x, y) = p(u_+, u_+(y/x)), \qquad dx\,dy \text{ a.s., } x > 0.$$

Let $p_+(r) = p(u_+, u_+ r)$ for $r \in \mathbb{R}$. Now, using (2.j),

$$p_+\left(\frac{y}{x}\right) = p_+\left(\frac{y + 2\pi}{x}\right), \qquad dx\,dy \text{ a.s., } x > 0,$$

so that

$$\int_0^\infty dx \int_{-\infty}^\infty dy \left| p_+\left(\frac{y}{x}\right) - p_+\left(\frac{y + 2\pi}{x}\right)\right| = 0.$$

Change $x$ into $(1/t)$ to get

$$\int_0^\infty dt \int_{-\infty}^\infty dy |p_+(ty) - p_+(t(y + 2\pi))| = 0,$$

so that

$$\int_0^\infty dt \int_{-\infty}^\infty dy |p_+(y) - p_+(y + t)| = 0.$$

Hence

$$\int_{-\infty}^\infty dy \int_y^\infty d\lambda |p_+(y) - p_+(\lambda)| = 0.$$

Switching the roles of $\lambda$ and $y$, then adding the results, gives

$$\int_{-\infty}^\infty \int_{-\infty}^\infty dy\,d\lambda |p_+(y) - p_+(\lambda)| = 0.$$

Finally, for at least one $\lambda$, $p_+(y) = p_+(\lambda)$, $dy$ a.s. This proves (2.k), and the rest of the proof of the lemma is routine. □

REMARK. The above proof shows that the characterization given in the lemma has little to do with Brownian motion and may simply be understood as a variant of the following fact:

Let $\phi$: $\mathbb{R}_+ \to \mathbb{R}$ or $\mathbb{C}$ be locally integrable. Then $\phi(h\,\cdot)$ converges in $L^1([0,1], dx)$ as $h \to \infty$ iff there exists a constant $\bar{\phi}$ such that

$$\frac{1}{h}\int_0^h dx |\phi(x) - \bar{\phi}| \to 0 \text{ as } h \to \infty.$$

The last condition is reminiscent of the following basic property of an almost periodic function $\phi$:

$$\frac{1}{h}\int_0^h dx\, \phi(x) \text{ converges as } h \to \infty.$$

However, if $\phi$ is almost periodic, so is $\phi(\cdot) - a$, for any constant $a$, and also $|\phi(\cdot) - a|$. But unless $\phi(\cdot) - a$ is identically 0, the limit of $(1/h)\int_0^h dx |\phi(x) - a|$ is strictly positive. See, e.g., Katznelson (1976).

**3. Refinements of the asymptotic laws for windings.** The asymptotic distribution of

$$\frac{1}{\log t}\int_0^t g(Z_s)\,dZ_s$$

as $t \to \infty$, for a complex-valued function $g(z)$, can be radically different from that described in Theorems 1.1 and 2.2. We now suppose that $zg(z)$ is a function of the argument of $z$.

THEOREM 3.1. *Let $f$ be a bounded measurable complex-valued function defined on the unit circle $C$, with $\int_0^{2\pi} f(e^{ia})\,da = 0$. Then*

$$\frac{2}{\log t}\int_0^t \frac{dZ_u}{Z_u} f(e^{i\Phi_u}) \xrightarrow{d} \int_0^\sigma \int_0^{2\pi} d\Gamma_{(s,a)}\, f(e^{ia}),$$

*where $\sigma = \inf\{u\colon \beta(u) = 1\}$ is defined in terms of the real part $\beta$ of $\zeta = \beta + i\theta$, $\Gamma$ is a complex-valued Brownian sheet with intensity $ds\,da/2\pi$ and $\zeta$ and $\Gamma$ are independent.*

REMARKS. (i) This convergence holds jointly with all log scaling laws governed by $\zeta$.

(ii) In case $f$ does not have mean 0, after writing $f = f_C + (f - f_C)$, where

$$f_C = \frac{1}{2\pi i}\int_C \frac{dz}{z} f(z) = \frac{1}{2\pi}\int_0^{2\pi} da\, f(e^{ia}),$$

the constant term gives an extra contribution in the limit of $f_C\zeta_\sigma$, due to the asymptotics of windings. Stated in this manner, Theorem 3.1 now appears as an extension of Spitzer's theorem (1.c)*.

(iii) As in the case of windings, this limit theorem can be split into action at 0 and action at $\infty$, and this is the basis of extending results to several points of origin. (See next section.) More precisely, our method shows that for two bounded Borel functions $f^-$ and $f^+$ on the circle, each with mean 0,

$$\frac{2}{\log t}\left(\int_0^t \frac{dZ_u}{Z_u} f^-(e^{i\Phi_u}) 1_{(|Z_u| \le r)}, \int_0^t \frac{dZ_u}{Z_u} f^+(e^{i\Phi_u}) 1_{(|Z_u| > r)}\right)$$

$$\xrightarrow{d} \left(\int_0^\sigma \int_0^{2\pi} d\Gamma^-_{(s,a)}\, f^-(e^{ia}) 1_{(\beta_s < 0)}, \int_0^\sigma \int_0^{2\pi} d\Gamma^+_{(s,a)}\, f^+(e^{ia}) 1_{(\beta_s > 0)}\right),$$

where $\Gamma^-$ and $\Gamma^+$ are two independent copies of $\Gamma$. This limit could also be written with $d\Gamma_{(s,a)}$ twice instead of $d\Gamma^-_{(s,a)}$ and $d\Gamma^+_{(s,a)}$, but we find the $\pm$ presentation more convenient for the extension to several points.

(iv) An interesting aspect of Theorem 3.1 is that it gives the joint limit in law of a familiy of functionals of the complex Brownian motion. However, if one is interested in the convergence result with respect to just one function $f$, the next corollary may be of some interest, if only for checking constants.

COROLLARY 3.2. *Let $f, g\colon V \to \mathbb{C}$ be two functions which are holomorphic in a neighborhood $V$ of the unit disc and such that $f(0) = g(0) = 0$. Then, using the notation of Theorem 3.1, the triple*

$$\left(\frac{2}{\log t}\int_0^t \frac{dZ_u}{Z_u}, \frac{2}{\log t}\int_0^t d(\log|Z_u|)\, f(e^{i\Phi_u}), \frac{2}{\log t}\int_0^t d\Phi_u\, g(e^{i\Phi_u})\right)$$

*converges in law, as $t \to \infty$, to*

$$\left(\zeta_\sigma, \frac{1}{\sqrt{2}}\|f\|_2\gamma_\sigma, \frac{1}{\sqrt{2}}\|g\|_2\delta_\sigma\right),$$

*where $\zeta$, $\gamma$ and $\delta$ are independent complex Brownian motions and*

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} da |f|^2(e^{ia})\right)^{1/2}.$$

PROOF. From Theorem 3.1, an expression of the limit in law for the triple is

$$\left(\zeta_\sigma, \int_0^\sigma \int_0^{2\pi} dB_{(t,a)}\, f(e^{ia}), \int_0^\sigma \int_0^{2\pi} dD_{(t,a)}\, g(e^{ia})\right),$$

where $\Gamma = B + iD$. Now, the corollary follows from the fact that, say

$$\delta_t(g) = \int_0^{2\pi} d_a D_{(t,a)}\, g(e^{ia})$$

is a Gaussian complex-valued martingale which admits a continuous version. Moreover, if we write $g_1(z) = \operatorname{Re} g(z)$ and $g_2(z) = \operatorname{Im} g(z)$, then for any $i, j \in \{1,2\}$,

$$\langle \delta_\cdot(g_i), \delta_\cdot(g_j)\rangle_t = \frac{t}{2\pi}\int_0^{2\pi} da (g_i g_j)(e^{ia}).$$

However, since $g$ is holomorphic and $g(0) = 0$, we have

$$\begin{aligned} 0 = g^2(0) &= \frac{1}{2\pi}\int_0^{2\pi} da\, g^2(e^{ia}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} da\left[g_1^2(e^{ia}) - g_2^2(e^{i\theta}) + 2i(g_1 g_2)(e^{ia})\right] \end{aligned}$$

so that

$$\int_0^{2\pi} da\, g_1^2(e^{ia}) = \int_0^{2\pi} da\, g_2^2(e^{ia}) \quad\text{and}\quad \int_0^{2\pi} da (g_1 g_2)(e^{ia}) = 0.$$

Finally, $(\sqrt{2}/\|g\|_2)\delta_t(g)$ is a standard complex Brownian motion, from which the statement of the corollary clearly follows. □

REMARK. Assuming $f$ and $g$ satisfy the hypotheses of Corollary 3.2, the processes $(\delta_t(f), t \geq 0)$ and $(\delta_t(g), t \geq 0)$ ae independent if and only if both

$$\int_0^{2\pi} da\, f(a) g(a) = 0 \quad\text{and}\quad \int_0^{2\pi} da\, f(a)\bar{g}(a) = 0,$$

where $\bar{g}(a)$ is the complex conjugate of $g(a)$. For example, the processes $(\sqrt{2}\int_0^{2\pi} d_a D_{(t,a)} e^{ina},\ t \geq 0)$ for $n = 1, 2, \dots$ are independent complex Brownian motions, from which the entire sheet $D$ can be recovered by a Fourier series.

PROOF OF THEOREM 3.1. This is a straightforward consequence of the following theorem, which is a slight modification of Theorem 3.5 in Yor (1983). See also Borodin (1986) and Csáki, Földes and Kasahara (1987). □

Let $\mathbb{P}_{2\pi}$ be the set of bounded Borel functions $f\colon \mathbb{R} \to \mathbb{R}$, which are periodic, with period $2\pi$. We use the notations $\|f\|_2$ and $f_C$ for functions $f \in \mathbb{P}_{2\pi}$ as if they were functions defined on the unit circle $C$, as considered above. To illustrate, for $f \in \mathbb{P}_{2\pi}$,

$$f_C = \frac{1}{2\pi}\int_0^{2\pi} f(a)\, da \quad \text{and} \quad \|f - f_C\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} da\big(f(a) - f_C\big)^2\right)^{1/2}.$$

THEOREM 3.3. *Let $\beta$ and $\theta$ be two independent real-valued Brownian motions, each starting from 0. Let $f, g \in \mathbb{P}_{2\pi}$. Then:*

(i) *As $c \to \infty$, the continuous processes in $t \in \mathbb{R}_+$,*

$$\left(\beta_t, \theta_t, \int_0^t d\beta_s\, f(c\theta_s), \int_0^t d\theta_s\, g(c\theta_s)\right),$$

*converge in law to*

$$\left(\beta_t, \theta_t, f_C\beta_t + \int_0^t\int_0^{2\pi} dB_{(s,a)}\big[f(a) - f_C\big], g_C\theta_t + \int_0^t\int_0^{2\pi} dD_{(s,a)}\big[g(a) - g_C\big]\right),$$

*where $\beta$, $\theta$, $B$, and $D$ are independent and $B$ and $D$ are Brownian sheets indexed by $\mathbb{R}_+ \times [0, 2\pi]$ whose associated Gaussian measures have intensity $ds\, da/2\pi$.*

(ii) *In particular, as $c \to \infty$, the quadruple*

$$\left(\beta_t, \theta_t, \int_0^t d\beta_s\, f(c\theta_s), \int_0^t d\theta_s\, g(c\theta_s)\right)$$

*converges in law toward*

$$(\beta_t, \theta_t, f_C\beta_t + \|f - f_C\|_2\delta_t, g_C\theta_t + \|g - g_C\|_2\varepsilon_t),$$

*where $(\beta, \theta, \delta, \varepsilon)$ is a four-dimensional Brownian motion, starting from 0.*

(iii) *For $a \in [0, 2\pi]$ let $S_a$ be the sector $0 \leq \arg(z) \leq a$. As $c \to \infty$, the quadruple of continuous processes in $(t, a)$*

$$\left(\beta_t, \theta_t, \int_0^t d\beta_s 1\big(e^{ic\theta_s} \in S_a\big), \int_0^t d\theta_s 1\big(e^{ic\theta_s} \in S_a\big)\right)$$

*converges in law toward*

$$\left(\beta_t, \theta_t, \frac{a}{2\pi}\beta_t + \tilde{B}_{(t,a)}, \frac{a}{2\pi}\theta_t + \tilde{D}_{(t,a)}\right), \tag{3.a}$$

*where*

$$\tilde{B}_{(t,a)} = B_{(t,a)} - \frac{a}{2\pi} B_{(t,2\pi)}, \qquad \tilde{D}_{(t,a)} = D_{(t,a)} - \frac{a}{2\pi} D_{(t,2\pi)}.$$

REMARKS. (i) The processes $\tilde{B}_{(t,a)}$ and $\tilde{D}_{(t,a)}$ are Brownian motions in $t$ and Brownian bridges in $a$. The third and fourth components of the limit in (3.a) are independent Brownian sheets in $(t, a)$, both with intensity $dt\,da/2\pi$. For future reference, we introduce the notation

$$E_{(t,a)} = \frac{a}{2\pi}\theta_t + \tilde{D}_{(t,a)} \tag{3.b}$$

for the fourth component in (3.a).

(ii) The convergences in law refer to the weak convergence of the associated distributions on $C(S, \mathbb{R}^d)$, equipped with the topology of uniform convergence on compact subsets of $S$, where $S = \mathbb{R}_+$ or $\mathbb{R}_+ \times [0, 2\pi]$.

(iii) The proof of Theorem 3.3 [or that of Theorem 3.5 in Yor (1983)] hinges on Knight's theorem (1971)* and the basic fact that

$$\text{for } f \in \mathbb{P}_{2\pi}, \qquad \int_0^t ds\, f(c\theta_s) \xrightarrow{\text{a.s.}} t f_C \quad \text{as} \quad c \to \infty. \tag{3.c}$$

*Applications of Theorem* 3.1. Recall that the differential of the winding number $\Phi(t)$ derived from $Z_t = X_t + iY_t$ is

$$d\Phi_t = \frac{X_t\,dY_t - Y_t\,dX_t}{|Z_t|^2}.$$

Spitzer's theorem (1.c)* asserts the convergence in distribution of $2\Phi_t/\log t$, as $t \to \infty$, to a standard Cauchy variable. We show, in the next theorem, that the two-dimensional variables

$$\frac{1}{\log t}\left(\int_0^t \frac{X_s\,dY_s}{|Z_s|^2}, \int_0^t \frac{Y_s\,dX_s}{|Z_s|^2}\right)$$

converge in law, and we identify the limit, thereby reinforcing Spitzer's result (1.c)*. In fact, let

$$a_t = \int_0^t X_s\,dX_s/|Z_s|^2, \qquad b_t = \int_0^t Y_s\,dY_s/|Z_s|^2,$$

$$c_t = \int_0^t X_s\,dY_s/|Z_s|^2, \qquad d_t = \int_0^t Y_s\,dX_s/|Z_s|^2.$$

THEOREM 3.4. *There exists a four-dimensional Brownian motion* $(\beta_t, \theta_t, \delta_t, \varepsilon_t;$ $t \geq 0)$ *such that, as* $t \to \infty$, *the* 4-*tuple*

$$\frac{2}{\log t}(a_t, b_t, c_t, d_t)$$

*converges in law to*

$$\tfrac{1}{2}(1+\delta_\sigma; 1-\delta_\sigma; \theta_\sigma+\varepsilon_\sigma; -\theta_\sigma+\varepsilon_\sigma),$$

*where* $\sigma = \inf\{t\colon \beta_t = 1\}$.

REMARKS. (i) $\beta + i\theta$ is the usual $\zeta$ for log scaling laws and the convergence holds jointly with such laws.

(ii) From the theorem, we recover, in particular, the log scaling laws

$$\frac{2}{\log t}\log|Z_t| = \frac{2}{\log t}(a_t + b_t) \xrightarrow{d} 1 \quad \text{as } t \to \infty,$$

$$\frac{2}{\log t}\Phi_t = \frac{2}{\log t}(c_t - d_t) \xrightarrow{d} \theta_\sigma \quad \text{as } t \to \infty.$$

PROOF OF THEOREM 3.4. Linear operations on the identities

$$\frac{X_u\,dX_u + Y_u\,dY_u}{|Z_u|^2} + i\frac{X_u\,dY_u - Y_u\,dX_u}{|Z_u|^2} = \frac{dZ_u}{Z_u},$$

$$\frac{X_u\,dX_u - Y_u\,dY_u}{|Z_u|^2} + i\frac{X_u\,dY_u + Y_u\,dX_u}{|Z_u|^2} = \frac{dZ_u}{Z_u}\left(\frac{Z_u}{|Z_u|}\right)^2,$$

give formulas for $da_u$, etc., in terms of the right-hand differentials above. Hence, if we use the notation in Theorem 3.1 and introduce the standard complex Brownian motion

$$\delta_t + i\varepsilon_t = \int_0^{2\pi} d_a \Gamma_{(t,a)} e^{2ia},$$

we find as a consequence of Theorem 3.1 that

$$\frac{2}{\log t}(a_t, b_t, c_t, d_t)$$

converges in law toward

$$\tfrac{1}{2}(1+\delta_\sigma, 1-\delta_\sigma, \theta_\sigma+\varepsilon_\sigma, -\theta_\sigma+\varepsilon_\sigma). \qquad \square$$

Here is a second application of Theorem 3.1:

THEOREM 3.5. *Let* $r > 0$. *The pair of continuous processes in* $a \in [0, 2\pi]$,

$$\frac{2}{\log t}\left(\int_0^t d\Phi_s 1(|Z_s| < r, Z_s \in S_a); \int_0^t d\Phi_s 1(|Z_s| > r, Z_s \in S_a)\right),$$

*converges in distribution as $t \to \infty$ to*

$$\left(\int_0^\sigma \int_0^a dE_{(s,u)} 1(\beta_s < 0);\ \int_0^\sigma \int_0^a dE_{(s,u)} 1(\beta_s > 0)\right),$$

*where we use the same notation as in* (3.b) *above and where*

$$\sigma = \inf\{a: \beta_a = 1\}.$$

*In particular, the finite-dimensional distributions of the continuous process*

$$\frac{2}{\log t}\int_0^t d\Phi_s 1(Z_s \in S_a), \qquad a \in [0, 2\pi],$$

*converge as $t \to \infty$ to those of*

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} B_a, \qquad a \in [0, 2\pi],$$

*where $(B_a,\ a \in [0, 2\pi]$ is a Brownian motion independent of $(\beta_t,\ t > 0)$, hence also independent of $\sigma$.*

REMARK. Let $X_a = (\sigma/2\pi)^{1/2} B_a$, $a \in [0, 2\pi]$. Then for $f \in L^2([0, 2\pi], da)$,

$$E\left(\exp i\int_0^{2\pi} f(a)\, dX_a\right) = \exp - \| f \|_2.$$

In particular, for $0 < u < v < 2\pi$, $X_v - X_u$ has a Cauchy distribution with parameter $((v - u)/2\pi)^{1/2}$.

Finally, we give an application of Theorem 3.1 to the asymptotic distribution of functionals of the type

$$\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s})$$

for certain bounded Borel functions $f: C \to \mathbb{C}$. Recall the notation

$$f_C = \frac{1}{2\pi i}\int_C f(z)\frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ia})\, da.$$

THEOREM 3.6. (i) *Let $f: C \to \mathbb{C}$ be a bounded Borel function such that $f_C = 0$. Then*

$$\frac{1}{\log t}\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s}) - \frac{2}{\log t}\int_0^t (F_C - F)(\Phi_s)\, d\Phi_s \xrightarrow{P} 0, \tag{3.d}$$

*where $F(a) = \int_0^a f(e^{ib})\, db$, $a \in \mathbb{R}$, is a $2\pi$-periodic function and*

$$F_C = \frac{1}{2\pi}\int_0^{2\pi} F(a)\, da = -\frac{1}{2\pi}\int_0^{2\pi} a f(e^{ia})\, da.$$

(ii) *Moreover, one may incorporate in both the Riemann and the stochastic integrals in* (3.d) *either the indicator $1_{(|Z_s| \le r)}$ or the indicator $1_{(|Z_s| \ge r)}$.*

(iii) *Consequently, if* $f, g\colon C \to \mathbb{C}$ *satisfy the hypothesis stated for* $f$ *alone in* (i) *and if* $0 < r < r' < \infty$, *then the* $\mathbb{C}^2$-*valued random vector*

$$\frac{1}{\log t}\left(\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s}) 1_{(|Z_s| \le r)}, \int_0^t \frac{ds}{|Z_s|^2} g(e^{i\Phi_s}) 1_{(|Z_s| \ge r')}\right) \tag{3.e}$$

*converges in law toward*

$$\left(\int_0^\sigma \int_0^{2\pi} dD_{(s,a)} 1_{(\beta_s \le 0)}(F_C - F)(a), \int_0^\sigma \int_0^{2\pi} dD_{(s,a)} 1_{(\beta_s \ge 0)}(G_C - G)(a)\right), \tag{3.f}$$

*where we use the notation in Theorem* 3.1, *with* $D = \operatorname{Im} \Gamma$.

PROOF. (i) The fact that $F$ is $2\pi$-periodic is an immediate consequence of the hypothesis $f_C = 0$. Now, let $\tilde{F}(a) = \int_0^a dx\, F(x)$, $a \in \mathbb{R}$. We have, from Itô's formula,

$$\tilde{F}(\Phi_t) = \tilde{F}(\Phi_0) + \int_0^t F(\Phi_s)\, d\Phi_s + \frac{1}{2}\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s}). \tag{3.g}$$

Due to the periodicity of $F$, $\tilde{F}(x) - xF_C$ is $2\pi$-periodic and continuous, hence a bounded function, so (3.g) immediately implies (3.d).

(ii) In view of (3.d) we need only consider $\int_0^t (ds/|Z_s|^2) f(e^{i\Phi_s}) 1_{(|Z_s| \le r)}$. Since $f_C = 0$, the Kallianpur–Robbins law (1.a)* implies that as far as the limit in law of

$$\frac{1}{\log t}\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s})$$

is concerned, the indicator $1_{(|Z_s| \le r)}$ can be replaced by $\chi(\log|Z_s|)$, where $\chi\colon \mathbb{R} \to \mathbb{R}_+$ is a $C^2$ function such that $\chi(x) = 1$ for $x \le \log r$ and $\chi(x) = 0$ for $x \ge \log r + \varepsilon$, for some $\varepsilon > 0$. Keeping the notation from the beginning of the proof, apply Itô's formula to the product $\chi(\log|Z_t|)F^\sharp(\Phi_t)$, where $F^\sharp(x) = \tilde{F}(x) - xF_C$, to obtain

$$\begin{aligned} &\chi(\log|Z_t|)F^\sharp(\Phi_t) - \chi(\log|z_0|)F^\sharp(\Phi_0) \\ &\quad = \int_0^t \chi(\log|Z_s|)\left[(F - F_C)(\Phi_s)\, d\Phi_s + \frac{ds}{2|Z_s|^2} f(e^{i\Phi_s})\right] \\ &\qquad + \int_0^t F^\sharp(\Phi_s)\, d[\chi(\log|Z_s|)]. \end{aligned} \tag{3.h}$$

Now divide both sides of this identity by $(\log t)$. Clearly, the left-hand side does not contribute to the limit. Next,

$$\frac{1}{\log t}\int_0^t F^\sharp(\Phi_s)\, d[\chi(\log|Z_s|)] \xrightarrow[t\to\infty]{P} 0. \tag{3.i}$$

Indeed, we have

$$d[\chi(\log|Z_s|)] = \chi'(\log|Z_s|)\, d(\log|Z_s|) + \frac{1}{2}\chi''(\log|Z_s|)\frac{ds}{|Z_s|^2}.$$

Two applications of the Kallianpur–Robbins law now show that (a) the stochastic integral [with respect to $d(\log|Z_s|)$] is of order $\sqrt{\log t}$ in law, while (b) the Riemann integral is $o(\log t)$ in law. Indeed, the periodicity of $F^\#$ implies

$$\frac{1}{\log t}\int_0^t F^\#(\Phi_s)\chi''(\log|Z_s|)\frac{ds}{|Z_s|^2}$$

converges in probability to 0 as $t \to \infty$, because the function of $Z_s$ in the integrand has an integral over the whole plane equal to

$$F_C^\# \int_0^\infty \frac{d\rho}{\rho}\chi''(\log \rho) = F_C^\#\{\chi'(+\infty) - \chi'(-\infty)\} = 0$$

since $\chi'$ has compact support. Hence, (3.i) is proved. Going back to (3.h), we now find

$$\frac{1}{\log t}\int_0^t \chi(\log|Z_s|)\left(\frac{ds}{2|Z_s|^2}f(e^{i\Phi_s}) + (F - F_C)(\Phi_s)\,d\Phi_s\right) \xrightarrow[t\to\infty]{P} 0$$

and, much as above, we may replace $\chi(\log|Z_s|)$ by $1_{(|Z_s|\le r)}$.

(iii) The last assertion of the theorem is an immediate consequence of Theorem 3.1. □

The particular case when the functions $f$ and $g$ featured in the statement of Theorem 3.6 are traces on $C$ of functions $f, g\colon V \to \mathbb{C}$ which are holomorphic in a neighborhood $V$ of the unit disc and such that $f(0) = g(0) = 0$ is most interesting. Indeed, for such a function $f$, say $f(z) = \sum_{n\ge1} f_n z^n$, there is the expression

$$F(a) = h_f(e^{ia}), \quad \text{where} \quad h_f(z) = -i\sum_{n\ge1}\frac{f_n}{n}z^n.$$

We find that $f_C = F_C = 0$ and from the discussion in the proof of Corollary 3.2, the limit variables (3.f) may now be represented as

$$\text{(3.j)} \qquad \left(\frac{1}{\sqrt{2}}\|f\|_*\int_0^\sigma d\gamma_s^-\,1_{(\beta_s\le 0)};\ \frac{1}{\sqrt{2}}\|g\|_*\int_0^\sigma d\gamma_s^+\,1_{(\beta_s\ge 0)}\right),$$

where $\gamma^-$ and $\gamma^+$ are two independent complex Brownian motions which are also independent of $\beta$, $\sigma = \inf\{u\colon \beta_u = 1\}$ and $\|f\|_* = (\sum_{n\ge1}(|f_n|^2/n^2))^{1/2}$.

Two simple interesting examples are $f(z) = z$ and $f(z) = z^2$, in which cases we deduce

$$\text{(3.k)} \qquad \frac{2}{\log t}\int_0^t \frac{ds\,Z_s}{|Z_s|^3} \xrightarrow{d} \sqrt{2}\,\gamma'_\sigma$$

and

$$\text{(3.l)} \qquad \frac{2}{\log t}\int_0^t \frac{ds}{Z_s^2} \xrightarrow{d} \frac{1}{\sqrt{2}}\gamma''_\sigma$$

where, according to the remark after Corollary 3.2, $\gamma'$ and $\gamma''$ are two complex Brownian motions independent of each other and of $\sigma$.

*Local times on rays.* We now present a complement to Theorem 3.6, which gives a more geometric interpretation of the asymptotic Brownian sheet $D$. We begin with the fact that if $Z$ starts at $z_0 \neq 0$, there is a jointly continuous process

$$(L_t^a;\ t \geq 0,\ a \in [0,2\pi]),$$

such that for every bounded Borel function $f\colon C \to \mathbb{C}$,

$$\int_0^t \frac{ds}{|Z_s|^2} f(e^{i\Phi_s}) = \int_0^{2\pi} da\, f(e^{ia}) L_t^a. \tag{3.m}$$

Such a process $(L_t^a)$ is defined by

$$L_t^a = \sum_{n\in\mathbb{Z}} l_t^{a+2n\pi}, \tag{3.n}$$

where $(l_t^b;\ t \geq 0,\ b \in \mathbb{R})$ is the jointly continuous version of the local times of the local martingale $(\Phi_t,\ t \geq 0)$. Since $l$ is continuous and has compact support in $b$, it is immediate that the formula (3.n) defines a jointly continuous process.

THEOREM 3.7. *With the notation introduced in Theorem* 3.1, *define*

$$\delta_a = \operatorname{Im}\Gamma_{(\sigma,a)} = D_{(\sigma,a)}, \qquad a \in [0,2\pi].$$

(i) *The finite-dimensional distributions of*

$$\left(\frac{1}{\log t}(L_t^a - L_t^0);\ a \in [0,2\pi]\right)$$

*converge weakly toward those of*

$$\left(\delta_a - \frac{a}{2\pi}\delta_{2\pi};\ a \in [0,2\pi]\right).$$

(ii) *For every* $a \in [0,2\pi]$,

$$\frac{4}{(\log t)^2} L_t^a \xrightarrow{d} \frac{\sigma}{2\pi}. \tag{3.o}$$

REMARK. In view of the approximation of Brownian local times by downcrossing numbers, $aN_t^a \to \frac{1}{2}L_t^0$ as $a \to 0$, so (3.o) is the limiting case as $a \to 0$ of the following result of Burdzy, Pitman and Yor (1988): For every $a \in (0,2\pi]$,

$$\frac{8}{(\log t)^2}(aN_t^a) \xrightarrow[t\to\infty]{d} \frac{\sigma}{2\pi}.$$

Before the proof, we give a slightly heuristic explanation of the first statement in Theorem 3.7, with the help of Theorem 3.6.

Let $f\colon C \to \mathbb{C}$ be a bounded Borel function such that $f_C = 0$. Then, from (3.d), we obtain

$$\frac{1}{\log t}\int_0^{2\pi} da\, f(a)(L_t^a - L_t^0) \xrightarrow[t\to\infty]{d} \int_0^{2\pi} d\delta_a (F_C - F)(a). \tag{3.p}$$

Expressing $F$ in terms of $f$ and using integration by parts we may write (3.p) as

$$\text{(3.p}'\text{)} \qquad \frac{1}{\log t}\int_0^{2\pi} da\, f(a)\left(L_t^a - L_t^0\right) \xrightarrow[t\to\infty]{d} \int_0^{2\pi} da\, f(a)\left(\delta_a - \frac{a}{2\pi}\delta_{2\pi}\right),$$

which renders the first statement of Theorem 3.7 very plausible. A proof of this statement could presumably be obtained following these lines. But we shall give an alternative approach.

PROOF OF THEOREM 3.7. (i) We imitate the proof of the first statement of Theorem 3.6, the role of Itô's formula (3.g) now being played by Tanaka's formula. More precisely, for a given $a \in [0, 2\pi)$, let

$$F_a(x) = \sum_{n\in\mathbb{Z}} 1_{(2n\pi \le x < 2n\pi + a)} \quad \text{and} \quad \tilde{F}_a(x) = \int_0^x dy\, F_a(y).$$

The second derivative of $\tilde{F}_a$, in the sense of Schwartz's distributions, is the measure

$$F_a''(dx) = \sum_{n\in\mathbb{Z}} \left\{-\varepsilon_{2n\pi+a}(dx) + \varepsilon_{2n\pi}(dx)\right\},$$

where $\varepsilon_\xi(dx)$ is the Dirac measure at $\xi \in \mathbb{R}$.

The analogue of Itô's formula (3.g) is now

$$\text{(3.q)} \qquad \tilde{F}_a(\Phi_t) = \tilde{F}_a(\Phi_0) + \int_0^t F_a(\Phi_s)\, d\Phi_s + \tfrac{1}{2}\left(L_t^0 - L_t^a\right).$$

We then deduce, much as in the proof of Theorem 3.6, that

$$\frac{1}{\log t}\left(L_t^a - L_t^0\right) \sim \frac{2}{\log t}\int_0^t \left(F_a - (F_a)_C\right)(\Phi_s)\, d\Phi_s,$$

meaning that the difference between the two sides converges in probability to 0. We immediately deduce, with the help of Theorem 3.1, that

$$\frac{1}{\log t}\left(L_t^a - L_t^0\right) \xrightarrow[t\to\infty]{d} \delta_a - \frac{a}{2\pi}\delta_{2\pi}.$$

Consideration of linear combinations gives convergence of finite-dimensional distributions as $a$ varies.

(ii) For any bounded Borel $f\colon C \to \mathbb{C}$, we have, with the notation of (2.h)* and (3.h)* and $h = \frac{1}{2}\log t$,

$$\frac{4}{(\log t)^2}\int_0^t \frac{ds}{|Z_s|^2} f\left(e^{i\Phi_s}\right) = \frac{1}{h^2}\int_0^{U_t} ds\, f\left(e^{i\theta_s}\right) = \int_0^{(1/h^2)U_t} dv\, f\left(e^{ih\theta_v^{(h)}}\right),$$

which, from (3.c), converges in law toward $f_C\sigma$. On the other hand,

$$\frac{4}{(\log t)^2}\int_0^t \frac{ds}{|Z_s|^2} f\left(e^{i\Phi_s}\right) = \int_0^{2\pi} da\, f\left(e^{ia}\right)\frac{1}{h^2}\left(L_t^a - L_t^0\right) + \frac{1}{h^2}\left(\int_0^{2\pi} da\, f\left(e^{ia}\right)\right)L_t^0.$$

Since the first integral converges to 0 in probability, we obtain (3.o). □

**4. Extensions to several origins.** Our first aim in this section is to obtain an extension of Theorem 3.1 to stochastic integrals whose integrands have singularities at $n$ distinct points $z_1, z_2, \dots, z_n$, assumed also distinct from the starting point $z_0$ of the complex Brownian motion $Z$. Let $\Phi_u^j$ be the winding number of $Z$ around $z_j$ up to time $u$ and let $f_j\colon C \to \mathbb{C}$, $1 \le j \le n$, be a sequence of bounded Borel functions. We want to show, under some suitable assumptions on the $f_j$'s, that the random vector

$$\left(\frac{2}{\log t}\int_0^t \frac{dZ_u}{Z_u - z_j} f_j(e^{i\Phi_u^j}),\ 1 \le j \le n\right) \tag{4.a}$$

converges in law as $t \to \infty$, and we want to describe the limit law.

The case when the $f_j$'s are constant was the focal point of our study in AL*. The result then may be summarized as follows. Introduce $2n$ strictly positive real numbers $r_j$, $r_j'$, $1 \le j \le n$, and let

$$D_j^- = \{z\colon |z - z_j| \le r_j\}, \qquad D_j^+ = \{z\colon |z - z_j| > r_j'\}.$$

Then there exists a continuous $\mathbb{C}^n$-valued process $\zeta$ consisting of $n$ complex Brownian motions $\zeta_j = \beta_j + i\theta_j$, $1 \le j \le n$, whose joint law is described in Theorem (6.2)* (where we used a superscript $\infty$ notation, which we now drop), such that

$$\left(\frac{2}{\log t}\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(Z_s \in D_j^-)},\ \frac{2}{\log t}\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(Z_s \in D_j^+)};\ 1 \le j \le n\right) \tag{4.b}$$

converges in law toward

$$\left(\int_0^{\sigma_j} d\zeta_j(s)\, 1_{(\beta_j(s) \le 0)},\ \int_0^{\sigma_j} d\zeta_j(s)\, 1_{(\beta_j(s) \ge 0)};\ 1 \le j \le n\right), \tag{4.c}$$

where $\sigma_j = \inf\{t\colon \beta_j(t) = 1\}$. [Note that we have already presented the convergence in law of the imaginary parts of (4.b) in (1.b′) above.] The study of the limit law of (4.a) thus reduces to the case where $f_i$ has mean 0 for each $j$. In this case we have

THEOREM 4.1. *Let $f_j, g_j\colon C \to \mathbb{C}(1 \le j \le n)$ be $2n$ Borel bounded functions such that*

$$\int_0^{2\pi} d\theta\, f_j(e^{i\theta}) = \int_0^{2\pi} d\theta\, g_j(e^{i\theta}) = 0 \quad \textit{for every } j. \tag{4.d}$$

*Then, the $\mathbb{C}^{2n}$-valued random vector*

$$\frac{2}{\log t}\left(\int_0^t \frac{dZ_u}{Z_u - z_j} f_j(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^-)},\ \int_0^t \frac{dZ_u}{Z_u - z_j} g_j(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^+)},\ 1 \le j \le n\right)$$

*converges in law toward*

$$\left(\int_0^{\sigma_j}\int_0^{2\pi} d\Gamma_{s,\theta}^{j,-}\, 1_{(\beta_j(s) \le 0)} f_j(e^{i\theta}),\ \int_0^{\sigma_1}\int_0^{2\pi} d\Gamma_{s,\theta}^{+} 1_{(\beta_1(s) \ge 0)} g_j(e^{i\theta}),\ 1 \le j \le n\right), \tag{4.e}$$

*where $\beta_j$ and $\sigma_j$ are as in* (4.c), *and* $\Gamma^{j,-}$, $1 \le j \le n$, *and* $\Gamma^+$ *are* $(n+1)$ *independent complex-valued Brownian sheets, with intensity* $ds\, d\theta/2\pi$, *independent of the* $\zeta$ *process.*

Before proving Theorem 4.1, we describe in more detail in a particular case the law of the random vector in (4.e). Assume now that the $n$ functions $g_j$ are identical to a single function $g$, that $f_j$ for $1 \le j \le n$ and $g$ are the traces on $C$ of functions which are holomorphic on a neighbourhood of the unit disc and that (4.d) holds. We also assume, without loss of generality, that $\|f_j\|_2 = \|g\|_2 = 1$. Let $\Delta_-^j$ denote the left-hand component in (4.e), $\Delta_+$ the common right-hand component in (4.e) with $g$ instead of $g_j$ and $\Lambda$ the value (which does not depend on $j$) of the local time at 0 of $\beta_j$ at time $\sigma_j$. Then the $n+2$ complex-valued random variables $(\Delta_-^j,\ 1 \le j \le n,\ \Delta_+,\ \Lambda)$ are such that for each $j$ the triple $(\Delta_-^j, \Delta_+, \Lambda)$ is distributed as

$$\left(\int_0^\sigma 1_{(\beta_s \le 0)}\, d\delta_s,\ \int_0^\sigma 1_{(\beta_s > 0)}\, d\delta_s,\ \lambda_\sigma\right), \tag{4.f}$$

where $\beta$ and $\delta$ are independent real- and complex-valued Brownian motions respectively, both starting at 0, $\sigma = \inf\{t: \beta_t = 1\}$, $(\lambda_t,\ t \ge 0)$ is the local time of $\beta$ at 0 and the $n+1$ variables $\Delta_-^j$, $1 \le j \le n$, and $\Delta_+$ are mutually conditionally independent given $\Lambda$. This dependence structure, which is very similar to that described in Theorem 6.1*, comes from the fact that the Brownian motions $\beta_j$ have independent negative excursions but identical positive excursions, as described in Theorem 6.2*.

First step in the proof of Theorem 4.1. Let

$$W_+^k(g,t) = \int_0^t (dZ_u/(Z_u - z_k))\, g(e^{i\Phi_u^k}) 1_{(Z_u \in D_k^+)}.$$

As a first step in the proof, we shall show that, for all $j, k \le n$, and all bounded Borel functions $g$,

$$\frac{1}{\log t} \sup_{s \le t} \left| W_+^k(g,s) - W_+^j(g,s) \right| \xrightarrow[t \to \infty]{P} 0. \tag{4.g}$$

(Note that it is not necessary to suppose $g$ has mean 0 for this step. The mean 0 assumption is made in the theorem just to focus attention to the contribution of the Brownian sheets.)
From (1.d), (4.g) is equivalent to

$$\frac{1}{(\log t)^2} \int_0^t ds\, f_{k,j}(Z_s) \xrightarrow[t \to \infty]{P} 0; \tag{4.h}$$

where

$$f_{k,j}(z) = \left| \frac{1}{(z - z_k)} g((z - z_k)') 1_{(z \in D_k^+)} - \frac{1}{(z - z_j)} g((z - z_j)') 1_{(z \in D_j^+)} \right|^2$$

and we write simply $\xi'$ for $\xi/|\xi|$. We may as well replace $f_{k,j}$ by

$$\tilde{f}_{k,j}(z) = \left| \frac{1}{(z - z_k)} g((z - z_k)') - \frac{1}{(z - z_j)} g((z - z_j)') \right|^2 1_{(|z| \geq R)}$$

for $R$ large enough, since the difference $f_{k,j} - \tilde{f}_{k,j}$ is a bounded, integrable function. But the function

$$z \to \left( \frac{1}{z - z_k} - \frac{1}{z - z_j} \right) 1_{|z| \geq R}$$

is bounded and belongs to $L^2(\mathbb{C})$. Therefore, it suffices to show (4.h) with $f_{k,j}$ replaced by

$$f^{\#}_{k,j}(z) = \frac{1}{|z - z_j|^2} \left| g((z - z_j)') - g((z - z_k)') \right|^2 1_{(|z| \geq R)}.$$

In case *$g$ is the trace on $C$ of a continuously differentiable function $\tilde{g}$ on a neighborhood $V$ of the unit disc*, we may write

$$\begin{aligned}
&\left| g((z - z_j)') - g((z - z_k)') \right| \\
&\quad \leq \gamma \left| (z - z_j)' - (z - z_k)' \right| \quad \left[ \text{where } \gamma = \sup_{|\xi| \leq 1} |\nabla \tilde{g}(\xi)| \right] \\
&\quad \leq \frac{\gamma \left| (z - z_j)|z - z_k| - (z - z_k)|z - z_j| \right|}{|z - z_j|\, |z - z_k|} \\
&\quad = O\left( \frac{1}{|z|} \right) \quad \text{as } |z| \to \infty.
\end{aligned}$$

Thus $f^{\#}_{k,j}(z) = O(1/|z|^4)$ as $|z| \to \infty$ and $f^{\#}_{k,j}$ is therefore integrable. The case where $g$: $C \to \mathbb{C}$ is only assumed to be Borel bounded is more delicate to handle, for the following reason: Under the hypotheses we have made up until now, much more than (4.h) is true. In fact,

$$\frac{1}{\log t} \int_0^t ds\, f_{k,j}(Z_s) \text{ converges in law.}$$

In the general case, which we now turn to, we shall only be able to prove that

$$\text{(4.i)} \qquad \frac{1}{(\log t)^2} \int_0^t ds\, f^{\#}_{k,j}(Z_s) \xrightarrow[t \to \infty]{L^1} 0.$$

Our main tool to prove this result will be

PROPOSITION 4.2. *Let $(Z_t)$ be Brownian motion in $\mathbb{C}$ starting from $z_0$, with $|z_0| < R$. Then, for every Borel function $u$: $\mathbb{C} \to \mathbb{R}_+$, which is locally integrable in $\{z: |z| \geq R\}$, the following inequality holds*:

$$\limsup_{t \to \infty} \frac{1}{(\log t)^2} E\left[ \int_0^t ds\, 1_{(|Z_s| \geq R)}\, u(Z_s) \right] \leq \frac{1}{4\pi} \limsup_{r \to \infty} \left\{ r^2 \int_0^{2\pi} d\theta\, u(re^{i\theta}) \right\}.$$

*The reverse inequality holds with* lim sup *replaced by* lim inf.

In fact, we shall prove (4.i) by using the two following straightforward consequences of Proposition 4.2.

COROLLARY 4.3. (i) *If* $u\colon \mathbb{C} \to \mathbb{R}_+$ *is a locally bounded Borel function such that*

$$\lim_{r\to\infty} r^2 \int_0^{2\pi} d\theta\, u(re^{i\theta}) = 0,$$

*then*

$$\lim_{t\to\infty} \frac{1}{(\log t)^2} E\left[\int_0^t ds\, u(Z_s)\right] = 0.$$

(ii) *Let* $(Z_t)$ *be complex Brownian motion starting from* $z_0$ *with* $|z_0| < R$ *and let* $Z_t' = Z_t/|Z_t|$. *Then, for every positive Borel function* $u\colon C \to \mathbb{R}_+$,

$$\limsup_{t\to\infty} \frac{1}{(\log t)^2} E\left[\int_0^t \frac{ds}{|Z_s|^2} 1_{(|Z_s| \geq R)} u(Z_s')\right] \leq \frac{1}{4\pi} \int_0^{2\pi} d\theta\, u(e^{i\theta}).$$

*The reverse inequality holds with* lim sup *replaced by* lim inf.

We now prove (4.i). In the case when $g\colon C \to \mathbb{C}$ is continuous, the function $u = f_{k,j}^{\#}$ clearly satisfies the hypothesis of part (i) of Corollary 4.3, and this gives (4.i) in this case.

Consider now the case when $g$ is only assumed to be bounded Borel. Plainly, it is sufficient to show

$$\text{(4.j)} \qquad \frac{1}{(\log t)^2} \int_0^t ds\, h(Z_s) \xrightarrow[t\to\infty]{L^1} 0,$$

where $(Z_t)$ is complex Brownian motion starting at 0 and

$$h(z) = \frac{1}{|z|^2} |g((z+a)') - g(z')|^2 1_{(|z| \geq R)} \quad \text{for some } a \neq 0.$$

Now approach $g$ in $L^2(C, d\theta)$ by a sequence $(g_p)$ of continuous functions. Let $h_p$ be the function $h$ with $g$ replaced by $g_p$ and let

$$l_{p,R}(z) = \frac{1}{|z|^2} |g_p(z') - g(z')|^2 1_{(|z| \geq R)}.$$

Then, for some universal constant $c$,

$$\text{(4.k)} \qquad h(z) \leq c\{l_{p,R-a}(z+a) + h_p(z) + l_{p,R}(z)\}.$$

Finally, let

$$I(h) = \limsup_{t\to\infty} \frac{1}{(\log t)^2} E\left[\int_0^t ds\, h(Z_s)\right].$$

Then, from (4.k) and part (ii) of Corollary 4.3, we obtain

$$I(h) \le c\left(\frac{1}{2\pi}\int_0^{2\pi} d\theta |g - g_p|^2(e^{i\theta}) + I(h_p)\right).$$

Since $g_p$ is continuous, we already know that $I(h_p) = 0$; moreover, as

$$\int_0^{2\pi} d\theta |g - g_p|^2(e^{i\theta})$$

can be made arbitrarily small, we have $I(h) = 0$, which proves (4.j), hence (4.i) in full generality. □

PROOF OF PROPOSITION 4.2. Define

$$I_t(u) = E\left[\int_0^t ds\, 1_{(|Z_s| \ge R)} u(Z_s)\right].$$

Then,

$$I_t(u) = \int_R^\infty r\,dr \int_0^{2\pi} d\theta\, u(re^{i\theta}) \Delta\left(\frac{|re^{i\theta} - z_0|^2}{t}\right),$$

where

$$\Delta(x) = \frac{1}{2\pi}\int_x^\infty \frac{dr}{r} e^{-r/2}$$

is the same function as in (2.h). Since $\Delta$ is a decreasing function,

$$I_t(u) \le \int_R^\infty r\,dr \int_0^{2\pi} d\theta\, u(re^{i\theta}) \Delta\left(\frac{(r - |z_0|)^2}{t}\right).$$

Now, let $R' > R$. Then,

$$I_t(u) \le \int_R^{R'} r\,dr \int_0^{2\pi} d\theta\, u(re^{i\theta}) \Delta\left(\frac{(R - |z_0|)^2}{t}\right)$$
$$+ \sup_{r \ge R'}\left(r^2 \int_0^{2\pi} d\theta\, u(re^{i\theta})\right) \int_{R'}^\infty \frac{dr}{r} \Delta\left(\frac{(r - |z_0|)^2}{2}\right).$$

Now, it is easily seen that

$$\frac{1}{\log t}\Delta\left(\frac{(R - |z_0|)^2}{t}\right) \xrightarrow[t \to \infty]{} \frac{1}{2\pi}$$

and

$$\frac{1}{(\log t)^2}\int_{R'}^\infty \frac{dr}{r} \Delta\left(\frac{(r - |z_0|)^2}{t}\right) \xrightarrow[t \to \infty]{} \frac{1}{4\pi}$$

so that, making use of the local integrability of $u$, we obtain

$$\limsup_{t\to\infty}\frac{1}{(\log t)^2}I_t(u)\le\frac{1}{4\pi}\sup_{r\ge R'}\left(r^2\int_0^{2\pi}d\theta\,u(re^{i\theta})\right).$$

The proof of the first inequality is completed by letting $R'$ tend to $\infty$.

On the other hand, we have

$$I_t(u)\ge\int_R^\infty r\,dr\int_0^{2\pi}d\theta\,u(re^{i\theta})\Delta\left(\frac{(r+|z_0|)^2}{t}\right)$$

and, for $R'>R$,

$$I_t(u)\ge\int_R^{R'}r\,dr\int_0^{2\pi}d\theta\,u(re^{i\theta})\Delta\left(\frac{(R'+|z_0|)^2}{t}\right)$$
$$+\inf_{r\ge R'}\left(r^2\int_0^{2\pi}d\theta\,u(re^{i\theta})\right)\int_{R'}^\infty\frac{dr}{r}\Delta\left(\frac{(r+|z_0|)^2}{t}\right),$$

which, much as before, implies

$$\liminf_{t\to\infty}\frac{1}{(\log t)^2}I_t(u)\ge\frac{1}{4\pi}\liminf_{r\to\infty}\left(r^2\int_0^{2\pi}d\theta\,u(re^{i\theta})\right).\qquad\square$$

SECOND STEP IN THE PROOF OF THEOREM 4.1. This second half of the proof is very similar to the proof of Theorem 6.1*, so we go at a quick pace. Thanks to the equivalence (1.d) and the Kallianpur–Robbins law, we may assume $r_j'$ to be so small and $r_1'$ to be so large that the $(n+1)$ sets $D_j^-$, $j=1,2,\dots,n$, and $D_1^+$ are disjoint. Also, from the first half of the proof, we only have to consider the $\mathbb{C}^{n+1}$-valued random vector

$$\left(\frac{2}{\log t}\int_0^t\frac{dZ_u}{Z_u-z_j}f_j(e^{i\Phi_u^j})1_{(Z_u\in D_j^-)},\ 1\le j\le n;\ \frac{2}{\log t}\int_0^t\frac{dZ_u}{Z_u-z_1}g(e^{i\Phi_u^1})1_{(Z_u\in D_1^+)}\right)$$

for $n+1$ real-valued functions $f_j$, $g$ which satisfy (4.d). Let

$$M_-^j(t)=\int_0^t\frac{dZ_u}{(Z_u-z_j)}f_j(e^{i\Phi_u^j})1_{(Z_u\in D_j^-)},$$

$$M_+(t)=\int_0^t\frac{dZ_u}{(Z_u-z_1)}g(e^{i\Phi_u^1})1_{(Z_u\in D_1^+)}.$$

The processes $M_-^j$, $1\le j\le n$, and $M_+$ are conformal martingales [see Getoor and Sharpe (1972)*], hence time changes of complex Brownian motions which we denote by $m_-^j$ and $m_+$. Because the sets $D_-^j$ and $D_+$ are disjoint, $m_-^j$, $1\le j\le n$, and $m_+$ are $n+1$ independent complex Brownian motions, from Knight's theorem (1971)*. Moreover, if we denote by $\xi^j$ the complex Brownian motion which is the time change of $\int_0^t(dZ_s/(Z_s-z_j))$, then the random vectors

$$\boldsymbol{\xi}=(\xi^1,\dots,\xi^n)\quad\text{and}\quad\mathbf{m}==(m_-^1,\dots,m_-^n,m_+)$$

have the asymptotic property

$$(\xi^h, \mathbf{m}^h) \xrightarrow[h\to\infty]{d} (\zeta, \mathbf{m}^\infty),$$

where the superscript $h$ indicates rescaling space by $h$ and time by $h^2$, as in Theorem 6.2*, $\zeta$ and $\mathbf{m}^\infty$ are independent, $\mathbf{m}^\infty$ is a Brownian motion in $\mathbb{C}^{n+1}$ and the distribution of $\zeta$ is described in Theorem 6.2*.

In order to prove this result, it suffices—following the proof of Theorem 6.1*—to replace the vector $\xi$ by $(\xi_-^1, \dots, \xi_-^n, \xi_+)$ which is the $\mathbb{C}^{n+1}$-valued Brownian motion obtained by time-changing the conformal martingales

$$\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(Z_s \in D_-^j)}, \qquad \int_0^t \frac{dZ_s}{(Z_s - z_1)} 1_{(Z_s \in D_+^1)}, \qquad 1 \le j \le n,$$

with their respective increasing processes, and to show $((\xi_-^{1,h}, \dots, \xi_-^{n,h}, \xi_+^h); \mathbf{m}^h)$ converges in law, as $h \to \infty$, to a $\mathbb{C}^{2(n+1)}$-valued Brownian motion. In fact, thanks to the orthogonality properties of the various martingales involved, and with the help of our appendix, this all boils down to problems involving only one singularity which have already been dealt with in Section 3.

Next, the normalized vector of clocks,

$$\frac{4}{(\log t)^2}\left(\int_0^t \frac{du}{|Z_u - z_j|^2}|f_j|^2(e^{i\Phi_u^j})1_{(Z_u \in D_j^-)}, 1 \le j \le n;\right.$$
$$\left.\int_0^t \frac{du}{|Z_u - z_1|^2}|g|^2(e^{i\Phi_u^1})1_{(Z_u \in D_1^-)}\right)$$

converges in law toward

$$\left(\|f_j\|_2^2 \int_0^{\sigma_j} ds\, 1_{(\beta_j(s) \le 0)}, 1 \le j \le n; \|g\|_2^2 \int_0^{\sigma_1} ds\, 1_{(\beta_1(s) \ge 0)}\right).$$

Putting all these results together, the limit in law of

$$\frac{2}{\log t}\left(M_-^j(t), 1 \le j \le n; M_+(t)\right)$$

may be expressed as

$$\left(\|f_j\|_2 \int_0^{\sigma_j} d\delta_j^-(s) 1_{(\beta_j(s) \le 0)}, 1 \le j \le n; \|g\|_2 \int_0^{\sigma_1} d\delta^+(s) 1_{(\beta_1(s) \ge 0)}\right),$$

where $\delta_j^-, \delta^+$ $(1 \le j \le n)$ are $(n+1)$ independent complex Brownian motions, independent of the $\zeta$-process. Finally, a linearity argument enables us to present this in terms of a Brownian sheet, as in the statement of Theorem 4.1. □

*Asymptotic distributions for some Riemann integrals.* To make our story shorter, we shall only consider the extension of Theorem 3.6 to functions with several singularities in the case when the functions are holomorphic. The relationship of the next Theorem 4.4 to Theorem 4.1 is the same as that of Theorem 3.6 to Theorem 3.1: In both cases, what is achieved is the reduction of the asymptotic study of a Riemann integral to that of a stochastic integral.

THEOREM 4.4. *Let $f_j$, $1 \le j \le n$, and $g$ be $n+1$ functions from $V$ to $\mathbb{C}$, which are holomorphic in a neighbourhood $V$ of the unit disc, and such that $f_j(0) = g(0) = 0$. Then, the $\mathbb{C}^{2n}$-valued random vector*

$$\frac{2}{\log t}\left(\int_0^t \frac{du}{|Z_u - z_j|^2} f_j(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^-)}, \int_0^t \frac{du}{|Z_u - z_j|^2} g(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^+)}; 1 \le j \le n\right)$$

*converges in law toward*

$$\left(\sqrt{2}\,\|f_j\|_* \int_0^{\sigma_j} d\gamma_s^{j,-} 1_{(\beta_j(s) \le 0)}, \sqrt{2}\,\|g\|_* \int_0^{\sigma_1} d\gamma_s^{+} 1_{(\beta_1(s) \ge 0)}; 1 \le j \le n\right), \tag{4.1}$$

*where $(\gamma^{j,-}, \gamma^+; 1 \le j \le n)$ are $n+1$ independent complex Brownian motions, which are independent of the $\zeta$-process, in terms of which the real Brownian motions $\beta_j$ and the hitting times $\sigma_j$ are defined, as in* (4.c).

PROOF. We have shown, in the proof of Theorem 3.6, that

$$\frac{1}{\log t}\int_0^t \frac{du}{|Z_u - z_j|^2} f_j(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^-)} \underset{t\to\infty}{\sim} \frac{2}{\log t}\int_0^t d\Phi_u^j\, h_j(e^{i\Phi_u^j}) 1_{(Z_u \in D_j^-)},$$

where $h_j$ is the $h$-function associated with $f_j$ as in the discussion following the proof of Theorem 3.6.

An analogous result holds for the integral depending on $g$. The final result now follows from Theorem 4.1, provided we represent the Brownian sheet integrals as we did in Corollary 3.2 and formula (3.j). □

To illustrate Theorem 4.4, we look at the $n$-point extension of the examples (3.k) and (3.l). Then, the $\mathbb{C}^{4n}$-valued random variables

$$\left\{\frac{2}{\log t}\left(\int_0^t \frac{du}{|Z_u - z_j|^3}(Z_u - z_j) 1_{Z_u \in D_j^-)}, \int_0^t \frac{du}{|Z_u - z_j|^3}(Z_u - z_j) 1_{(Z_u \in D_j^+)}\right);\right.$$
$$\left.\frac{2}{\log t}\left(\int_0^t \frac{du}{(Z_u - z_j)^2} 1_{(Z_u \in D_j^-)}, \int_0^t \frac{du}{(Z_u - z_j)^2} 1_{(Z_u \in D_j^+)}\right); 1 \le j \le n\right\}$$

converge in law, as $t \to \infty$, toward

$$\begin{aligned}&\left\{\sqrt{2}\left(\int_0^{\sigma_j} d\gamma_s^{j,-} 1_{(\beta_j(s) \le 0)}, \int_0^{\sigma_1} d\gamma_s^{+} 1_{(\beta_1(s) \ge 0)}\right);\right.\\ &\left.\frac{1}{\sqrt{2}}\left(\int_0^{\sigma_j} d\delta_s^{j,-} 1_{(\beta_j(s) \le 0)}, \int_0^{\sigma_1} d\delta_s^{+} 1_{(\beta_1(s) \ge 0)}\right); 1 \le j \le n\right\},\end{aligned} \tag{4.m}$$

where $(\gamma^{j,-}, \gamma^+, \delta^{j,-}, \delta^+; 1 \le j \le n)$ are $2n+2$ independent complex Brownian motions which are independent of the $\zeta$-process. Moreover, the distribution of the $\mathbb{C}^{n+1}$-valued variable features in each line of (4.m) is that of a constant ($\sqrt{2}$ or $1/\sqrt{2}$) times $(\Delta_-^j, \Delta_+, 1 \le j \le n)$, where we use the notation introduced after Theorem 4.1. These calculations lead to the next theorem, which concerns the asymptotic distribution of

$$\int_0^t ds\, f(Z_s)$$

when $f$ belongs to a class of meromorphic functions. This theorem should be compared with Theorem 1.1, which dictates the asymptotic distribution of

$$\int_0^t dZ_s\, f(Z_s)$$

for another class of meromorphic functions.

In order to fully justify our choice for the class of functions considered in Theorem 4.6, we present the following elementary statement, the proof of which is left to the reader.

LEMMA 4.5. *Let $\mathbb{C}_\infty$ denote the Riemann sphere and let $f$: $\mathbb{C}_\infty \to \mathbb{C}_\infty$ be a meromorphic function such that*

(i) $\lim_{z\to\infty} zf(z) = 0$ *and*
(ii) *the poles of $f$ are of at most second order.*

*Then $f$ has at most a finite number of distinct poles, call them $z_1, \dots, z_n$, and there exist $2n$ complex numbers $r_1, \dots, r_n$, $\rho_1, \dots, \rho_n$, such that*

$$\sum_j r_j = 0 \quad \text{and} \quad f(z) = \sum_j r_j \frac{1}{z - z_j} + \sum_j \rho_j \frac{1}{(z - z_j)^2}.$$

*Moreover*:

(a) $\rho(f, \infty) \overset{\text{def}}{=} \lim_{z\to\infty} z^2 f(z)$ *exists and* $\rho(f, \infty) = \sum_j \rho_j + \sum_j r_j z_j$.
(b) $\{f\} \overset{\text{def}}{=} \lim_{\varepsilon\to 0,\, R\to\infty} \int_{\Sigma_{\varepsilon, R}} \underline{dz}\, f(z)$ *exists*
*where $\Sigma_{\varepsilon, R}$ is the complement of $\bigcup_{j=1}^n \{z\colon |z - z_j| \le \varepsilon\} \cup \{z\colon |z| \ge R\}$ and*

$$\{f\} = 2\pi \sum_j r_j \bar{z}_j.$$

Here the notation $\underline{dz}$ signifies a Lebesgue integral. In the sequel, we shall refer to this class of functions as $M_2$.

THEOREM 4.6. *Let $(Z_t,\ t \ge 0)$ be a complex Brownian motion started at $z_0$ and suppose that $z_0, z_1, \dots, z_n$ is a finite set of distinct points in $\mathbb{C}$. Suppose that $f$ is a complex-valued function such that*:

(i) *In a neighborhood $D_j$ of each point $z_1, \dots, z_n$ and for $z \in D_j \setminus \{z_j\}$,*

$$f(z) = h_j(z) + \rho_j \frac{1}{(z - z_j)^2},$$

*where $h_j$ is integrable in $D_j$.*

(ii) *$f$ is bounded and measurable on the complement of the union of these neighborhoods.*

(iii) *In a neighborhood* $D_\infty$ *of* $\infty$,

$$f(z) = h_\infty(z) + g(z),$$

*where* $h_\infty$ *is integrable in* $D_\infty$, $g$ *is holomorphic in* $D_\infty \cup \{\infty\}$ *and* $\lim_{z \to \infty} zg(z) = 0$. *We denote* $\rho_\infty = \lim_{z\to\infty} z^2 g(z)$. *Then*:

(a) $\{f\} \overset{\text{def}}{=} \lim_{\varepsilon \to 0, R \to \infty} \int_{\Sigma_{\varepsilon, R}} \underline{dz}\, f(z)$ *exists, where* $\Sigma_{\varepsilon, R}$ *is as in Lemma* 4.5.

(b) *As* $t \to \infty$, $(2/\log t)\int_0^t ds\, f(Z_s)$ *converges in law toward*

$$\frac{1}{2\pi}\{f\}\Lambda + \frac{1}{\sqrt{2}}\sum_j \rho_j \Delta_-^j + \frac{1}{\sqrt{2}}\rho_\infty \Delta_+,$$

*where the* $(n+2)$*-tuple* $(\Lambda, \Delta_-^j, \Delta_+; 1 \le j \le n)$ *is distributed as indicated after Theorem* 4.1.

REMARKS. (i) The case where $f$ is bounded and integrable on the entire plane is a particular case of Theorem 4.6. Then $\{f\} = \int_{\mathbb{C}} \underline{dz}\, f(z)$, $\rho_j = \rho_\infty = 0$ and we recover the Kallianpur–Robbins law (1.a)*.

(ii) In the case when $f \in M_2$,

$$\rho_j = \lim_{z \to z_j} (z - z_j)^2 f(z), \qquad \rho_\infty = \lim_{z \to \infty} z^2 f(z).$$

Then, from Lemma 4.5, the limit variable in Theorem 4.6 may be written as

$$\sum_j r_j\left(-\frac{1}{2}\bar{z}_j\Lambda + \frac{1}{\sqrt{2}}z_j\Delta_+\right) + \sum_j \rho_j\left(\frac{1}{\sqrt{2}}\Delta_-^j + \frac{1}{\sqrt{2}}\Delta_+\right). \tag{4.n}$$

PROOF OF THEOREM 4.6. We may choose $\varepsilon$ so small and $R$ so large that

$$h_{\varepsilon, R}(z) = f(z) - \sum_j \rho_j \frac{1}{(z - z_j)^2} 1_{(|z - z_j| \le \varepsilon)} - \rho_\infty \frac{1}{z^2} 1_{|z| \ge R}$$

is an integrable function. Therefore, the Kallianpur–Robbins law combined with the illustration of Theorem 4.4 given above yields part (b) of the theorem with $(1/2\pi)\{f\}$ replaced by $(1/2\pi)\int_{\mathbb{C}} \underline{dz}\, h_{\varepsilon, R}(z)$. Part (a) of the theorem and the equality

$$\{f\} = \int_{\mathbb{C}} \underline{dz}\, h_{\varepsilon, R}(z)$$

are proved by remarking that for $\varepsilon' < \varepsilon$ and $R' > R$,

$$\int_{\Sigma_{\varepsilon', R'}} f(z)\, \underline{dz} = \int_{\Sigma_{\varepsilon', R'}} h_{\varepsilon, R}(z)\, \underline{dz} \underset{\substack{\varepsilon' \to 0 \\ R' \to \infty}}{\longrightarrow} \int_{\mathbb{C}} h_{\varepsilon, R}(z)\, \underline{dz}. \qquad \square$$

The following corollary of Theorem 4.6 plays a key role in the study of the speed of convergence of renormalized local times of intersection of complex Brownian motion toward Varadhan's renormalization, which is undertaken in Yor (1987). This corollary exhibits a family of functionals of complex Brownian motion whose limits in law are the random components in the linear combination (4.n).

COROLLARY 4.7. *The* $\mathbb{C}^{2n}$*-valued variable*

$$\left(\frac{2}{\log t}\int_0^t ds\,\frac{z_j}{(Z_s - z_j)Z_s};\ \frac{2}{\log t}\int_0^t \frac{ds}{(Z_s - z_j)^2};\ 1 \le j \le n\right)$$

*converges in law toward*

$$\left(-\bar{z}_j\frac{\Lambda}{2} + z_j\frac{\Delta_+}{\sqrt{2}};\ \frac{1}{\sqrt{2}}(\Delta_-^j + \Delta_+);\ 1 \le j \le n\right)$$

*with the same notation as in Theorem* 4.6.

PROOF. Let $\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n$ be $2n$ complex numbers. Then

$$f(z) = \sum_{j=1}^{n}\left(\mu_j\frac{z_j}{(z - z_j)z} + \nu_j\frac{1}{(z - z_j)^2}\right)$$

belongs to $M_2$ and $r_j = \mu_j$, $\rho_j = \nu_j$. Now, the result is a consequence of Remark (ii) following Theorem 4.6. □

*Additive functionals derived from singular integrals.* We now apply Theorems 4.1 and 4.4 to the asymptotic study of

$$\int_0^t ds(Kf)(Z_s),$$

where $f\colon \mathbb{C} \to \mathbb{C}$ is a bounded Borel function with compact support and

$$Kf(z) = \text{principal value of } \int \underline{d\xi} f(\xi)\frac{k((z-\xi)')}{|z-\xi|^2}$$

with $k\colon V \to \mathbb{C}$ a holomorphic function defined on a neighborhood $V$ of the unit disc, such that

$$k_C = 0. \tag{4.o}$$

For the existence of $Kf$, see Stein [(1970), Theorem 3, page 39]. In the particular case $k(z) = z$, $Kf = Rf$ is the (complex) Riesz transform of $f$.

THEOREM 4.8. *Assume that the above hypotheses on $k$ and $f$ are satisfied. Let* $f_{\mathbb{C}} = \int \underline{d\xi} f(\xi)$. *Then*

$$\frac{2}{\log t}\int_0^t ds\,Kf(Z_s) \xrightarrow[t\to\infty]{d} \sqrt{2}\, f_{\mathbb{C}}\|k\|_*\int_0^\sigma d\gamma_s 1_{(\beta_s > 0)}, \tag{4.p}$$

*where $\gamma$ is a complex Brownian motion independent of $\beta$ and $\sigma = \inf\{s\colon \beta_s = 1\}$.*

REMARKS. (i) Our motivation for this theorem comes from the study undertaken by Yamada (1986), who shows

$$\left(\frac{1}{\sqrt{\lambda}}\int_0^{\lambda t} ds\, Hf(B_s),\ t\geq 0\right) \xrightarrow[\lambda\to\infty]{d} \left(f_{\mathbb{R}} H_t,\ t\geq 0\right),$$

where $(B_t)$ is now a one-dimensional Brownian motion, $f\colon \mathbb{R}\to\mathbb{R}$ a bounded Borel function with compact support, $f_{\mathbb{R}}$ is the Lebesgue integral of $f$ over $\mathbb{R}$, $Hf$ the Hilbert transform of $f$ and

$$H_t = \lim_{\varepsilon\to 0}\int_0^t \frac{ds}{B_s} 1_{(|B_s|\geq\varepsilon)}.$$

(ii) In comparison with Theorem 4.1, only the "large" component featured in the limit (4.m) is present in (4.p). This may be explained heuristically by the smoothing out of singularities at finite distance by the kernel $K$.

PROOF OF THEOREM 4.8. (i) We only need to show that, for $r$ large enough and $z_*$ such that $|z_*| < r$, $z_*\neq z_0$, we have

$$\frac{1}{\log t}\int_0^t ds\, Kf(Z_s) \underset{t\to\infty}{\sim} \frac{f_{\mathbb{C}}}{\log t}\int_0^t ds\, 1_{(|Z_s - z_*|>r)} \frac{k((Z_s - z_*)')}{|Z_s - z_*|^2}. \tag{4.q}$$

Indeed, once (4.q) is proved, then (4.p) follows from Theorem 4.1.

(ii) In order that (4.q) be satisfied, it is sufficient that the function of $z$,

$$F_{r,z_*}(z) = 1_{(|z-z_*|<r)} Kf(z) + 1_{(|z-z_*|<r)}\left(Kf(z) - f_{\mathbb{C}}\frac{k((z-z_*)')}{|z-z_*|^2}\right),$$

belong to $L^1(\mathbb{C}, \underline{dz})$ and that its integral with respect to Lebesgue measure be 0.

(iii) We first show that $F_{r,z_*}\in L^1(\mathbb{C}, \underline{dz})$. First, the function $z\to 1_{(|z-z_*|<r)}Kf(z)$ belongs to $L^2(\mathbb{C}, \underline{dz})$, hence to $L^1(\mathbb{C}, \underline{dz})$. Second, we have

$$\begin{aligned}&\int_{|z-z_*|>r} \underline{dz}\left|Kf(z) - f_{\mathbb{C}}\frac{k((z-z_*)')}{|z-z_*|^2}\right| \\ &\qquad\leq \int_{|z-z_*|>r}\underline{dz}\int \underline{d\xi}\,|f(\xi)|\left|\frac{k((z-z_*)')}{|z-\xi|^2} - \frac{k((z-z_*)')}{|z-z_*|^2}\right|.\end{aligned}$$

For clarity, we write $k_\xi(z) = k((z-\xi)')$. Then, we have

$$\left|\frac{k_\xi(z)}{|z-\xi|^2} - \frac{k_{z_*}(z)}{|z-z_*|^2}\right| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \frac{|k_\xi(z) - k_{z_*}(z)|}{|z-\xi|^2}$$

and

$$\mathrm{II} = \left|k_{z_*}(z)\right| \frac{|\xi - z_*|(|z - z_*| + |z - \xi|)}{|z - \xi|^2 |z - z_*|^2}.$$

Let $A$ be such that $\operatorname{supp}(f) \subset \{\xi\colon |\xi| \le A\}$. Then, we have

$$\mathrm{II} \le \kappa \frac{(A + |z_*|)(2|z| + A + |z_*|)}{(|z| - A)^2 (|z| - |z_*|)^2}$$

and

$$\begin{aligned} \mathrm{I} &\le \kappa' \frac{1}{|z - \xi|^2} \left| \frac{z - \xi}{|z - \xi|} - \frac{z - z_*}{|z - z_*|} \right| \\ &\le \kappa' \frac{|z|\,|\xi - z_*| + |\xi|\,|z - z_*|}{|z - \xi|^3 |z - z_*|} \\ &\le \kappa' \frac{|z|(A + |z_*|) + A(|z| + |z_*|)}{(|z| - A)^3 (|z| - |z_*|)}, \end{aligned}$$

where $\kappa = \sup_{|z|=1} |k(z)|$ and $\kappa' = \sup_{|z|=1} |k'(z)|$. It is now immediate from these estimates that

$$\int_{|z - z_*| > r} \underline{dz} \left| Kf(z) - f_{\mathbb{C}} \frac{k((z - z_*)')}{|z - z_*|^2} \right| < \infty.$$

(iv) We now show

(4.r) $$\int F_{r, z_*}(z) = 0.$$

From the dominated convergence theorem, we need only prove

(4.r′) $$\lim_{M \to \infty} \int_{|z - z_*| \le M} F_{r, z_*}(z)\, \underline{dz} = 0.$$

Now, for any $M > r$, we have, using (4.o),

$$\int_{|z - z_*| \le M} \underline{dz}\, F_{r, z_*}(z) = \int_{|z - z_*| \le M} \underline{dz}\, Kf(z)$$

and, in fact, we shall prove

(4.s) $$\int_{|z - z_*| \le M} \underline{dz}\, Kf(z) = O\left(\frac{1}{M}\right) \quad \text{as } M \to \infty.$$

We may as well assume that $z_* = 0$, which is done by translating the function $f$ and changing $z$ into $z - z_*$.

Consider $M$ as fixed for the moment. Then

$$\begin{aligned} \int_{|z| \le M} \underline{dz}\, Kf(z) &= \lim_{\varepsilon \to 0} \int_{|z| \le M} \underline{dz} \int_{|z - \xi| \ge \varepsilon} \underline{d\xi}\, f(\xi) \frac{k((z - \xi)')}{|z - \xi|^2} \\ &= \lim_{\varepsilon \to 0} \int \underline{d\xi}\, f(\xi) H_{\varepsilon, M}(\xi), \end{aligned}$$

where

$$H_{\varepsilon, M}(\xi) = \int_{|z-\xi|\geq\varepsilon} \underline{dz}\, \frac{k((z-\xi)')}{|z-\xi|^2} 1_{(|z|\leq M)}.$$

Now, the trick is that we also have

$$H_{\varepsilon, M}(\xi) = \int_{|z-\xi|\geq\varepsilon} \underline{dz}\, \frac{k((z-\xi)')}{|z-\xi|^2} \left(1_{(|z|\leq M)} - 1_{(|z-\xi|\leq M)}\right).$$

We introduce a fixed $A > 0$ such that $\operatorname{supp}(f) \subset \{z\colon |z| \leq A\}$. We now remark that, for $|\xi| \leq A$,

$$\left|1_{(|z|\leq M)} - 1_{(|z-\xi|\leq M)}\right| \leq 1_{(M\leq|z-\xi|\leq M+|\xi|)} + 1_{(M-|\xi|\leq|z-\xi|\leq M)}.$$

Consequently, we have, for $|\xi| \leq A$,

$$|H_{\varepsilon, M}(\xi)| \leq \left(\frac{\pi}{M^2}\left[(M+A)^2 - M^2\right] + \frac{\pi}{(M-A)^2}\left[M^2 - (M-a)^2\right]\right) \sup_{|z|=1} |k(z)|.$$

The right-hand side does not depend either on $\varepsilon$ or $\xi$, so that we have finally shown (4.s). □

**5. Occupation times of circles.** Let

$$A(r,t) = \int_0^t 1(R_s \leq r)\, ds, \quad \text{where } R_s = |Z_s|,$$

be the occupation time of the circle of radius $r$ centered at 0, up to time $t$ by the Brownian motion $Z$ starting at $z_0 \neq 0$. The Kallianpur–Robbins law (1.a)* describes the asymptotic distribution of $A(r,t)$ as $t \to \infty$ for each fixed $r$. But a more interesting result is obtained by letting $r$ vary as a function of $t$. Put $t = e^{2h}$ as usual. Anticipating that $A(\cdot, e^{2h})$ will behave like $A(\cdot, T(e^h))$, where $T(r) = \inf\{t\colon R_t = r\}$, consider that by the occupation density formula for local time (A.7)*,

$$A(r(h), T(e^h)) = \int_0^{r(h)} L(R, r, T(e^h))\, dr,$$

where by (A.8)* and (B.2)*,

$$\begin{aligned} L(R, r, T(e^h)) &= rL(\log R, \log r, T(e^h)) \\ &= rL(\beta, \log r, \sigma_h) \\ &= hrL(\beta^{(h)}, (\log r)/h, \sigma(\beta^{(h)})). \end{aligned}$$

This suggests taking $r(h) = e^{ah}$ to obtain

$$\begin{aligned} A(e^{ah}, T(e^a)) &= \int_0^{e^{ah}} hrL(\beta^{(h)}, (\log r)/h, \sigma(\beta^{(h)}))\, dr \\ &= \tfrac{1}{2}he^{2ha}\int_0^{\infty} L(\beta^{(h)}, a-y, \sigma(\beta^{(h)}))2he^{-2hy}\, dy. \end{aligned}$$

The continuity properties of Brownian local time show that the supremum over all $a$, of the difference between $L(\beta^{(h)}, a, \sigma(\beta^{(h)}))$ and the integral in this last expression, tends to 0 in probability as $h \to \infty$. That is to say, the process

$$\left(2A\left(e^{ah}, T(e^{h})\right)/he^{2ha},\ -\infty < a < \infty\right)$$

viewed as a random element in the space $C[(-\infty, \infty), \mathbb{R}]$, with the topology of uniform convergence, has log scaling limit the Brownian local time process

$$(L(\beta, a, \sigma),\ -\infty < a < \infty). \tag{5.a}$$

According to the Ray–Knight theorem, the distribution of this process may be described as follows. Let $X(v) = L(\beta, 1 - v, \sigma)$. Then, the process $X$ is an inhomogeneous Markov process, homogeneous on each of the intervals $(-\infty, 0]$, $[0,1]$ and $[1, \infty)$, with $X(v) = 0$, $v \le 0$, $(X(v),\ 0 \le v \le 1)$ *the square of a two-dimensional Bessel process* and $(X(v),\ 1 \le v < \infty)$ *the square of a zero-dimensional Bessel process*. [See, e.g., Walsh (1978).]

Transforming in the usual way from time $T(e^h)$ to time $e^{2h}$ and putting $h = \frac{1}{2}\log t$, we obtain the following log scaling law.

THEOREM 5.1. *For a Brownian motion $Z$ starting at $z_0 \neq 0$, and $A(r, t)$ the occupation time of the circle of radius $r$ by $Z$ up to time $t$, as $t \to \infty$, the process*

$$\left\{\frac{4A(t^{a/2}, t)}{t^a \log t},\ -\infty < a < \infty\right\} \tag{5.b}$$

*converges in distribution in the space of continuous functions with compact support, with the topology of uniform convergence, to the Markov process*

$$\{X(1 - a),\ -\infty < a < \infty\} \tag{5.c}$$

*described above.*

REMARKS. (i) For each $t > 0$, the process in (5.b) is strictly positive over the random interval

$$I_t < a < J_t$$

and otherwise identically zero, where

$$I_t = \inf_{0 \le s \le t} \log R_s/\log t, \qquad J_t = \sup_{0 \le s \le t} \log R_s/\log t.$$

The same is true for the limiting process (5.c), for

$$I = \inf_{0 \le u \le \sigma} \beta_u, \qquad J = 1.$$

According to (8.d2)*, $I_t$ converges in distribution to $I$, and $J_t$ to $J$, which strengthens still further the already strong mode of convergence.

(ii) The theorem suggests that the functional

$$G_a(t) = 2A(t^{a/2}, t)/t^a$$

must be logarithmically attracted to some process $\gamma_a$. After writing

$$G_a(t) = \Gamma_a(U_t, \zeta),$$

the process $\gamma_a$ can be calculated as in (8.t)* as

$$\gamma_a(u, \zeta) = \lim_{h \to \infty} \Gamma_a^{(h)}\big(u, \zeta^{(1/h)}\big).$$

After some calculation, it emerges that

$$\gamma_a(u, \zeta) = L\big(\beta, a\bar{\beta}_u, u\big), \tag{5.d}$$

where $\bar{\beta}_u = \sup_{0 \le s \le u} \beta_s$. The convergence is uniform on compacts, as required for Theorem 8.4*. The appearance of the factor $\bar{\beta}_u$ in (5.d) is explained by the necessity for $\gamma_a$ to commute with Brownian scaling. (See Proposition 2.1.) Interestingly, the factor $\bar{\beta}_u$ is suppressed at the time $u = \sigma$ which is relevant to the asymptotics of $G_a(t)$ for fixed times $t$. But this factor appears in the asymptotics of $G_a(T_h)$ as $h \to \infty$ for any family of times $T_h$ in Table 8.2*, whose asymptotic time is not $\sigma$. The limiting process as $a$ varies seems then to be rather hard to describe explicitly. Some related questions are taken up in Le Gall and Yor (1988).

(iii) The previous remark and Theorem 8.2* give a result for occupation times $A_j(r, t)$ of circles of radius $r$ centered at points $z_j$, $j = 1, \dots, n$, distinct from the starting point $z_0$. The limit processes $X_j(v)$ are then given by

$$X_j(v) = L\big(\beta^j, 1 - v, \sigma(\beta^j)\big),$$

where the $\beta^j$ are the real parts of the linked asymptotic complex Brownian motions $\zeta^j$, denoted $\zeta^{j,\infty}$ in Theorem 6.2*. From the description of the $\zeta^j$ in that theorem, the process $X_j(v)$ are identical to a common process $X_+(v)$ for $v \le 1$, and move conditionally independently given their common value $X_+(1)$ for $v \ge 1$. The value $X_+(1)$ is identical to $\Lambda$, the asymptotic local time variable governing the Kallianpur–Robbins law (1.a)*.

(iv) If we let

$$\Phi_j(r, t) = \int_0^t d\Phi_j(s)\, 1(|Z_s - z_j| \le r)\, ds, \qquad -\infty < r < \infty,$$

a similar argument shows that the above mentioned convergence holds jointly with that of the processes

$$\big(2\Phi_j(t^{a/2}, t)/\log t,\ -\infty < a < \infty,\ j = 1, \dots, n\big)$$

which converge in the same sense to

$$\left(\int_0^{\sigma_j} d\theta_s^j\, 1\big(\theta_s^j \le a\big),\ -\infty < a < \infty,\ j = 1, \dots, n\right),$$

where $\sigma_j = \sigma(\beta^j)$. For $a = 1$ and $0$ this includes the previous results for big and small windings. If we let $a = 1 - v$ as before and write

$$\phi_j(v) = \int_0^{\sigma_j} d\theta_s^j\, 1\big(\beta_s^j \le 1 - v\big),$$

then it can be shown that

$$\begin{aligned}\phi_j(v) &= 0, \qquad v \le 0\\ &= B_+\left(\int_0^v X_+(u)\,du\right), \qquad 0 \le v \le 1\\ &= B_+\left(\int_0^1 X_+(u)\,du\right) + B_j\left(\int_1^v X_j(u)\,du\right), \qquad v \ge 1,\end{aligned}$$

where the processes $X_+$ and $X_j$ were described above and $B_+, B_1, \dots, B_n$ are $n$ independent Brownian motions independent also of the processes $X_+$ and $X_j$.

**6. Asymptotic theorem for square integrable martingale additive functionals.** Kasahara and Kotani (1979)* show that if $f\colon \mathbb{C} \to \mathbb{R}$ is a bounded Borel function such that

$$\text{(6.a)}\qquad \begin{aligned}&\int \underline{dz}\,|z|^\alpha |f(z)| < \infty \quad \text{for some } \alpha > 0,\\ &\int \underline{dz}\, f(z) = 0,\end{aligned}$$

then

$$\text{(6.a}'\text{)}\qquad (\log t)^{-1/2}\int_0^t ds\, f(Z_s) \text{ converges in distribution as } t \to \infty.$$

Messulam and Yor (1982)* prove that if $u$ and $v$ are bounded Borel functions from $\mathbb{C}$ to $\mathbb{R}$, and

$$\text{(6.b)}\qquad \int \underline{dz}\,\left(u^2(z) + v^2(z)\right) < \infty,$$

then

$$\text{(6.b}'\text{)}\qquad (\log t)^{-1/2} M_t^{u,v} \overset{\text{def}}{=} (\log t)^{-1/2}\int_0^t \left(u(Z_s)\,dX_s + v(Z_s)\,dY_s\right)$$

converges in distribution as $t \to \infty$.

We first remark that the two results are closely connected. More precisely, the limit in law for (6.a′) can be obtained as a consequence of (6.b′). Indeed, recall that if $g(x) \equiv (1/\pi)\log|x|$, then $\frac{1}{2}\Delta g(x) = \delta_0(x)$, in the sense of Schwartz distributions. Therefore, if $F \overset{\text{def}}{=} f * g$, we obtain $\frac{1}{2}\Delta F = f$ and Itô's formula gives

$$\text{(6.c)}\qquad F(Z_t) = F(Z_0) + \int_0^t (\nabla F(Z_s), dZ_s) + \int_0^t f(Z_s)\,ds.$$

Replacing $z$ by $(z - z_0)$, we may assume $z_0 = 0$. Now the hypothesis $\int dx\, f(x) =$

0 implies

$$F(Z_t) \stackrel{d}{=} \frac{1}{\pi} \int \underline{dz}\, f(z) \log\left| Z_1 - \frac{z}{\sqrt{t}} \right| \xrightarrow[t\to\infty]{P} 0.$$

Hence, we deduce from (6.c) that

$$\text{(6.d)} \qquad \frac{1}{(\log t)^{1/2}} \left( \int_0^t ds\, f(Z_s) + \int_0^t (\nabla F(Z_s), dZ_s) \right) \xrightarrow[t\to\infty]{P} 0.$$

The following theorem gives the asymptotic distribution of the stochastic integral featured in (6.d), hence also of the Riemann integral featured in (6.d).

THEOREM 6.1. *Let $u, v\colon \mathbb{C} \to \mathbb{R}$ be two bounded Borel functions such that $\int \underline{dz}\,(u^2 + v^2)(z) < \infty$. Then, as $t \to \infty$,*

$$\left( \frac{2}{\log t} \right)^{1/2} M_t^{u,v}$$

*converges in distribution to*

$$\Lambda^{1/2}\{\eta(u) + \chi(v)\},$$

*where $\Lambda$, $\eta$ and $\chi$ are independent, $\Lambda$ has the same meaning as in Theorem 4.1 and $\eta$ and $\chi$ are two independent Gaussian measures on $\mathbb{R}^2$, with intensity $\underline{dz}/2\pi$. Moreover, this limit in law holds jointly with all limits in law already encountered in the present paper, and $\eta$ and $\chi$ are independent from the vectors $\zeta$, $\Gamma = (\Gamma_-^j;\ 1 \le j \le n)$ and $\Gamma_+$ featured in the limit laws stated in Theorems 1.1 and 4.1.*

PROOF. (i) By linearity, it is sufficient to show that, for a given pair of functions $u, v$ which satisfy the above hypotheses, the family of variables $(2/\log t)^{1/2} M_t^{u,v}$ converges in law, as $t \to \infty$, toward

$$\frac{1}{\sqrt{2\pi}} \left\| (u^2 + v^2)^{1/2} \right\|_{L^2(\mathbb{C})} \delta_\Lambda,$$

where $\delta$ is a one-dimensional Brownian motion which is independent of the vectors $\zeta$, $\Gamma_-$ and $\Gamma_+$.

(ii) Call $(\mu_t^{u,v};\ t \ge 0)$ the real-valued Brownian motion such that

$$M_t^{u,v} = \mu^{u,v}_{\langle M^{u,v}\rangle_t}, \qquad t \ge 0.$$

Thanks to the Kallianpur–Robbins law (1.a)*, we know that

$$\frac{2}{\log t} \langle M^{u,v} \rangle_t \xrightarrow{d} \frac{1}{2\pi} \left( \int \underline{dz}\,(u^2 + v^2)(z) \right) \Lambda$$

so that it now suffices to show

$$\text{(6.e)} \qquad \left( \xi^{\alpha;h};\ \alpha \in A;\ \mu^{u,v;h^{1/2}} \right) \xrightarrow[h\to\infty]{d} (\delta^\alpha;\ \alpha \in A;\ \nu),$$

where:
$A$ is the finite set of conformal martingales $(N^\alpha;\ \alpha \in A)$ of the form

$$\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(Z_s \in D_-^j)}, \qquad 1 \le j \le n,$$

$$\int_0^t \frac{dZ_s}{Z_s - z_1} 1_{(Z_s \in D_+^1)},$$

$$\int_0^t \frac{dZ_s}{Z_s - z_j} 1_{(Z_s \in D_-^j)} f_j(e^{i\Phi_s^j}), \qquad 1 \le j \le n, \text{ where } (f_j)_C = 0,$$

$$\int_0^t \frac{dZ_s}{Z_s - z_1} 1_{(Z_s \in D_+^1)} g(e^{i\Phi_s^1}), \quad \text{where } g_C = 0;$$

for each $\alpha \in A$, $\xi^\alpha$ is the complex Brownian motion associated to $N^\alpha$;
for each $\alpha \in A$, $\delta^\alpha$ is a complex Brownian motion, with the joint distribution of $(\delta_\alpha;\ \alpha \in A)$ determined by Theorems 1.1 and 4.1;
$\nu$ is a real-valued Brownian motion independent of $(\delta^\alpha;\ \alpha \in A)$.

(iii) To prove (6.e), we shall apply the results of the Appendix to the family of martingales

$$\frac{1}{h} N \quad \text{and} \quad \frac{1}{h^{1/2}} M \quad \text{as } h \to \infty,$$

where we have dropped the superscripts $\alpha$, $u$ and $v$. With the help of the Appendix, what we have to prove is that

$$\frac{1}{h^{3/2}} \int_0^{\langle N \rangle_{h^2}^{-1}} |d\langle N, M \rangle_s| \xrightarrow[h \to \infty]{P} 0,$$

which is easily seen to be equivalent to

$$\text{(6.f)} \qquad \frac{1}{\langle N \rangle_t^{3/4}} \int_0^t |d\langle N, M \rangle_s| \xrightarrow[t \to \infty]{P} 0.$$

Since we know that $\langle N \rangle_t / (\log t)^2$ converges in distribution, as $t \to \infty$, toward a strictly positive random variable, (6.f) is equivalent to

$$\text{(6.g)} \qquad \frac{1}{(\log t)^{3/2}} \int_0^t |d\langle N, M \rangle_s| \xrightarrow[t \to \infty]{P} 0.$$

(iv) For simplicity, we may assume that $z_j = 0$, so that we obtain, in all cases,

$$|d\langle M, N \rangle_t| \le \frac{1}{|Z_t|} (|u| + |v|)(Z_t)\, dt \overset{\text{def}}{=} \frac{1}{|Z_t|} w(Z_t)\, dt,$$

where $w(z) = (|u| + |v|)(z)$. Hence, it suffices to show

$$\text{(6.h)} \qquad \frac{1}{(\log t)^{3/2}} E\left[\int_0^t \frac{ds}{|Z_s|} w(Z_s)\right] \xrightarrow[t \to \infty]{} 0.$$

This is an immediate consequence of the following proposition, which is a close relative of Proposition 4.2. □

**PROPOSITION** 6.2. *Let* $w\colon \mathbb{C} \to \mathbb{R}_+$ *be a locally bounded Borel function. Then, there exists a universal constant c such that*

$$\limsup_{t\to\infty} \frac{1}{(\log t)^{3/2}} E\left[\int_0^t \frac{ds}{|Z_s|} w(Z_s)\right] \le c \lim_{r\to\infty} \left(\int_{(|z|\ge r)} \underline{dz}\, w^2(z)\right)^{1/2}.$$

*In particular, if* $\int \underline{dz}\, w^2(z) < \infty$, *then*

$$\lim_{t\to\infty} \frac{1}{(\log t)^{3/2}} E\left[\int_0^t \frac{ds}{|Z_s|} w(Z_s)\right] = 0.$$

**PROOF.** Thanks to the Kallianpur–Robbins law (1.a)*, we may restrict attention to

$$\mathcal{J}_t(w) = E\left[\int_0^t \frac{ds}{|Z_s|} 1_{(|Z_s|\ge R)} w(Z_s)\right] \quad \text{with } R > |z_0|.$$

Then, using the same notation as in the proof of Proposition 4.2, we have, for any $R' > R$,

$$\begin{aligned}
\mathcal{J}_t(w) &= \int_R^\infty dr \int_0^{2\pi} d\theta\, w(re^{i\theta}) \Delta\left(\frac{|re^{i\theta} - z_0|^2}{2t}\right) \\
&\le \int_R^\infty dr \int_0^{2\pi} d\theta\, w(re^{i\theta}) \Delta\left(\frac{(r - |z_0|)^2}{2t}\right) \\
&\le \int_R^{R'} dr \int_0^{2\pi} d\theta\, w(re^{i\theta}) \Delta\left(\frac{(R - |z_0|)^2}{2t}\right) \\
&\quad + \left(\int_{|z|\ge R'} \underline{dz}\, w^2(z)\right)^{1/2} \left(\int_{R'}^\infty \frac{dr}{r} 2\pi \Delta^2\left(\frac{(r - |z_0|)^2}{2t}\right)\right)^{1/2}.
\end{aligned}$$

As already noted in the proof of Proposition 4.2,

$$\frac{1}{\log t} \Delta\left(\frac{(R - |z_0|)^2}{t}\right) \xrightarrow[t\to\infty]{} \frac{1}{2\pi}$$

and it is easily seen that

$$\frac{1}{(\log t)^3} \int_{R'}^\infty \frac{dr}{r} (2\pi)\, \Delta^2\left(\frac{(r - |z_0|)^2}{2t}\right) \text{ converges as } t \to \infty,$$

so we obtain

$$\limsup_{t\to\infty} \frac{1}{(\log t)^{3/2}} \mathcal{J}_t(w) \le c\left(\int_{|z|\ge R'} \underline{dz}\, w^2(z)\right)^{1/2}.$$

The proof of the inequality stated in the proposition is now completed by letting $R'$ tend to $\infty$. □

In fact, Proposition 6.2 appears as a special case ($p = 2$) of the following set of inequalities (6.i) indexed by $p \in (1, \infty)$, while Proposition 4.2 is the limit case $p = 1$. The only change to be made in the proof of Proposition 6.2 in order to prove (6.i) is the replacement of the Cauchy–Schwarz inequality by Hölder's inequality.

PROPOSITION 6.3. *Let $w\colon \mathbb{C} \to \mathbb{R}_+$ be a bounded Borel function and let $p, q$ satisfy $1/p + 1/q = 1$, with $p \in (1, \infty)$. Then there exists a universal constant $c_p$ such that*

$$\text{(6.i)}\quad \limsup_{t\to\infty} \frac{1}{(\log t)^{1+1/p}} E\left(\int_0^t \frac{ds}{|Z_s|^{2/p}} w(Z_s)\right) \le c_p \lim_{r\to\infty}\left(\int_{|z|\ge r} \underline{dz}\, w^q(z)\right)^{1/q}.$$

*In particular, if $\int \underline{dz}\, w^q(z) < \infty$, then the limit of the left side of* (6.i) *as $t \to \infty$ is* 0.

Apart from those examples considered already, we do not know any interesting applications of these inequalities, e.g., to prove asymptotic independence, because we do not know how to get limits in law for additive functions with normalization by $(\log t)^\alpha$ except for $\alpha = \frac{1}{2}$, 1 or 2.

*Application of Theorem* 6.1 *to winding numbers in annuli.* Consider again the winding processes $\Phi_j$ for a finite number of distinct points $z_j$, $1 \le j \le n$, distinct also from the starting point $z_0$ of the complex Brownian motion $Z$.

THEOREM 6.4 [Messulam and Yor (1982)*, Theorem 4.3]. (i) *For each $j$, there exists a jointly continuous version of the family of variables*

$$M^j(t, a) \overset{\text{def}}{=} \int_0^t 1_{(a \le |Z_s - z_j| \le 1)}\, d\Phi_s^j; \qquad t \ge 0,\ a \in (0, 1].$$

(ii) *As $t \to \infty$, the $n$-tuple of $C(0,1]$-valued random variables*

$$\left(\left(\frac{2}{\log t}\right)^{1/2} M^j(t, a),\ a \in (0, 1)];\ 1 \le j \le n\right)$$

*converges in distribution toward*

$$\Lambda^{1/2}\left(\alpha^j_{-\log a},\ a \in (0, 1];\ 1 \le j \le n\right),$$

*where $\Lambda$ is as defined in Theorem* 4.1 *and $(\alpha_t^j;\ 1 \le j \le n,\ t \ge 0)$ is a Gaussian* *process independent of* $\Lambda$ *with covariance determined by the identity*

$$(6.\mathrm{j})\qquad E\left(\alpha^i_{-\log a}\alpha^j_{-\log b}\right) = \frac{1}{2\pi}\int_{A(i,a,b,j)} \underline{dz}\,\frac{(z-z_i)\cdot(z-z_j)}{|z-z_i|^2|z-z_j|^2},$$

*where*

$$A(i,a,b,j) = \{a \le |z-z_i| \le 1\} \cap \{b \le |z-z_j| \le 1\}$$

*and* $u \cdot v$ *is the scalar product in* $\mathbb{R}^2$ *of* $u$ *and* $v$. *In particular, for each* $j$, $(\alpha^j_t,\ t \ge 0)$ *is a standard Brownian motion and* $\alpha_i$ *and* $\alpha_j$ *are independent if* $|z_i - z_j| \ge 2$.

REMARK. We prove this result here, since in Messulam and Yor (1982)* the proof of the first assertion was skipped, while the proof of tightness given there is in error. The last line of that paper appealed to the finiteness of $E(\sigma^{p/2})$ for a $p > 1$, where $\sigma = \inf\{t\colon \beta_t = 1\}$. Of course, this is wrong. As is well known, $E(\sigma^{p/2}) < \infty$ iff $p < 1$.

PROOF OF THEOREM 6.4. It is natural to break the proof into three parts:

1. The joint continuity 6.4(i).
2. Convergence of finite-dimensional distributions in 6.4(ii). This is an immediate application of Theorem 6.1.
3. For each $j$ and each $\varepsilon \in (0,1)$, tightness of the laws of

$$\left(\frac{1}{(\log t)^{1/2}} M^j(t,a),\ a \in [\varepsilon, 1]\right) \quad \text{for } t \ge 2, \text{say}.$$

Both 1 and 3 will now be established as consequences of Kolmogorov's lemma. To do so, it suffices to show that for each $\varepsilon \in (0,1)$ there exist $p > 0$, $\delta > 0$ and a constant $c$ such that for $\varepsilon < a < b < 1$,

$$\sup_{t\ge 2} \frac{1}{(\log t)^{p/2}} E\left(\sup_{s\le t} \left|M^j(s,a) - M^j(s,b)\right|^p\right) \le c|a-b|^{1+\delta}.$$

Using the Burkholder–Davis–Gundy inequalities, it suffices to show

$$(6.\mathrm{k})\qquad \sup_{t\ge 2} \frac{1}{(\log t)^{p/2}} E\left[\left(\int_0^t \frac{ds}{|Z_s - z_j|^2} 1_{(a \le |Z_s - z_j| \le b)}\right)^{p/2}\right] \le c|a-b|^{1+\delta},$$

where $c$ changes from line to line. This is an immediate consequence of the following estimate:
*For each* $n = 1, 2, \ldots$ *and* $R > 0$, *there exists a constant* $C_{n,R}$ *such that for every Borel function* $f\colon \mathbb{C} \to \mathbb{R}_+$ *with support in* $\{z\colon |z| \le R\}$,

$$(6.\mathrm{l})\qquad \sup_{t\ge 2} E\left[\left(\frac{1}{\log t}\int_0^t ds\, f(B_s)\right)^n\right] \le C_{n,R}\left(\int \underline{dx}\, f^2(x)\right)^{n/2}.$$

Consider the case $n = 2$. By the Markov property and using the notation (2.h),

$$E\left[\left(\int_0^t ds\, f(B_s)\right)^2\right] \le 2E\left[\int_0^t ds\, f(B_s)\int_{|y|\le 2R} \underline{dy}\, f(B_s+y)\,\Delta(|y|^2/t)\right]$$

$$\le 2E\left[\int_0^t ds\, f(B_s)\right]\left(\int \underline{dy}\, f^2(y)\right)^{1/2}\left(\int_{|y|\le 2R} \underline{dy}\,\Delta^2(|y|^2/t)\right)^{1/2}$$

$$\le C_r(\log t)E\left[\int_0^t ds\, f(B_s)\right]\left(\int \underline{dy}\, f^2(y)\right)^{1/2},$$

and the same estimate leads to (6.1) for $n = 2$, and finally for each $n$ by repeated application of the Markov property. □

## APPENDIX

**An asymptotic version of Knight's theorem on continuous orthogonal martingales.**

*Introduction.* Let $(M^n)$ and $(N^n)$ be two sequences of continuous local martingales defined over a right continuous complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge 0}, P)$ and such that for every $n$

$$M_0^n = N_0^n = 0 \quad \text{and} \quad \langle M^n\rangle_\infty = \langle N^n\rangle_\infty = \infty.$$

Let $\mu_t^n \equiv \inf\{u\colon \langle M^n\rangle_u > t\}$ and $\nu_t^n \equiv \inf\{u\colon \langle N^n\rangle_u > t\}$ be the right continuous inverses of the increasing processes associated respectively with $M^n$ and $N^n$. According to Dambis (1965)* and Dubins and Schwartz (1965)*, $B_t^n \equiv M^n(\mu_t^n)$ $(t \ge 0)$ and $C_t^n \equiv N^n(\nu_t^n)$ $(t\ge 0)$ are real-valued Brownian motions.

The aim of this Appendix is to refine criteria depending on $\langle M^n, N^n\rangle$, $\langle M^n\rangle$ and $\langle N^n\rangle$ and stated in AL* and Le Gall and Yor (1986)*, which ensure the convergence in distribution of $(B^n, C^n)$, viewed as continuous $\mathbb{R}^2$-valued processes, toward either $(\beta,\gamma)$, or $(\beta,\beta)$, where $\beta$ and $\gamma$ are two real-valued independent Brownian motions. In the first case, we say that $B^n$ and $C^n$ are asymptotically independent, while in the second case, we say that they are asymptotically identical.

The criteria obtained in this appendix apply not only to the asymptotics of winding numbers and connected questions, but also to many studies of limits in law such as are to be found in Papanicolaou, Stroock and Varadhan (1977)*.

*Asymptotically independent Brownian motions.* Our main result is the following.

THEOREM A1. *If, for every $t$,*

$$\lim_{n\to\infty}\langle M^n, N^n\rangle_{\mu_t^n} = \lim_{n\to\infty}\langle M^n, N^n\rangle_{\nu_t^n} = 0$$

*in probability, then $B^n$ and $C^n$ are asymptotically independent.*

PROOF. 1. The laws of the one-dimensional processes $B^n$ and $C^n$ are all equal to the one-dimensional Wiener measure. Therefore, the laws of the sequence $(B^n, C^n)$ of $\mathbb{R}^2$-valued continuous processes are weakly relatively compact and it remains to prove that the finite-dimensional marginals of $(B^n, C^n)$ converge weakly toward the corresponding marginals to a two-dimensional Brownian motion.

2. Let $0 = t_0 < t_1 < \cdots < t_p = t$ and consider real numbers $f_1, \dots, f_{p-1}$ and $g_1, \dots, g_{p-1}$. We set

$$f = \sum_j f_j 1_{(t_j, t_{j+1}]}, \qquad B^n(f) = \sum_j f_j\left(B^n_{t_{j+1}} - B^n_{t_j}\right),$$

$$g = \sum_j g_j 1_{(t_j, t_{j+1}]}, \qquad C^n(g) = \sum_j g_j\left(C^n_{t_{j+1}} - C^n_{t_j}\right).$$

Next, observe that if we set

$$U^n_s = \int_0^s f(\langle M^n\rangle_u)\, dM^n_u \quad \text{and} \quad V^n_s = \int_0^s g(\langle N^n\rangle_u)\, dN^n_u,$$

then

$$B^n(f) = U^n_\infty \quad \text{and} \quad C^n(g) = V^n_\infty.$$

Therefore, the identity

$$E\left[\exp\left\{i(U^n_\infty + V^n_\infty) + \tfrac{1}{2}\langle U^n + V^n\rangle_\infty\right\}\right] = 1$$

yields

$$\text{(7.a)} \qquad E\left[\{\exp i(B^n(f) + C^n(g))\}H^n\right] = \exp - \tfrac{1}{2}\int (f^2 + g^2)(t)\, dt,$$

where

$$H^n = \exp \int_0^\infty f(\langle M^n\rangle_s) g(\langle N^n\rangle_s)\, d\langle M^n, N^n\rangle_s.$$

3. We now remark that, on one hand, the estimate

$$H^n \le \exp(\|f\|_2 \|g\|_2)$$

follows from the Kunita–Watanabe inequality and, on the other hand, since

$$H^n = \exp\left(\sum_{j,k} f_j g_k\left(\langle M^n, N^n\rangle_{\mu^n_{t_{j+1}} \wedge \nu^n_{t_{k+1}}} - \langle M^n, N^n\rangle_{\mu^n_{t_j} \vee \nu^n_{t_k}}\right) 1_{(\mu^n_{t_j} \vee \nu^n_{t_k} < \mu^n_{t_{j+1}} \wedge \nu^n_{t_{k+1}})}\right),$$

the hypothesis clearly implies that $H^n$ converges to 1 in probability, hence in $L^1$, by application of the dominated convergence theorem. Looking back at (7.a), we find that

$$\lim_{n\to\infty} E\left[\exp i(B^n(f) + C^n(g))\right] = \exp - \tfrac{1}{2}\int (f^2 + g^2)(t)\, dt,$$

which is the desired result. □

Of particular interest to us is the case when $M_t^n = (1/\sqrt{n})M_t$ and $N_t^n = (1/\sqrt{n})N_t$, since then the one-dimensional Brownian motions $B^n$ and $C^n$ are obtained from $B$ and $C$ by the Brownian scaling operations

$$B_t^n = \frac{1}{\sqrt{n}} B_{nt} \quad \text{and} \quad C_t^n = \frac{1}{\sqrt{n}} C_{nt}.$$

We then obtain the following

COROLLARY. *If $M$ and $N$ are such that*

$$\lim_{t\to\infty} \frac{\langle M, N\rangle_t}{\langle M\rangle_t} = \lim_{t\to\infty} \langle M, N\rangle_t / \langle N\rangle_t = 0 \quad a.s., \tag{7.b}$$

*then $B^n$ and $C^n$ are asymptotically independent.*

PROOF. We remark that $\mu_t^n = \mu(nt)$ and $\nu_t^n = \nu(nt)$, so that (2.b) gives for every $t > 0$,

$$\langle M^n, N^n\rangle_{\mu_t^n} = \frac{1}{n}\langle M, N\rangle_{\mu(nt)} \xrightarrow[n\to\infty]{\text{a.s.}} 0,$$

and likewise for $\nu$ instead of $\mu$. The conclusion now follows from Theorem A1. □

*Asymptotically identical Brownian motions.* We now present analogues of Theorem A1 and its corollary in the case when $B^n$ and $C^n$ are asymptotically independent; however, the contents of these are the same as in AL* and Le Gall and Yor (1986)*, to which we refer the reader for proofs.

THEOREM A2. *If for every $t$,*

$$\lim_{n\to\infty} \langle M^n - N^n\rangle_{\mu_t^n} = \lim_{n\to\infty} \langle M^n - N^n\rangle_{\nu_t^n} = 0$$

*in probability, then $B^n$ and $C^n$ are asymptotically identical.*

In the case when $M_t^n = (1/\sqrt{n})M_t$ and $N_t^n = (1/\sqrt{n})N_t$, we obtain the following:

COROLLARY. *If $M$ and $N$ are such that*

$$\lim_{t\to\infty} \langle M - N\rangle_t / \langle M\rangle_t = \lim_{t\to\infty} \langle M - N\rangle_t / \langle N\rangle_t = 0 \quad a.s.$$

*then $B^n$ and $C^n$ are asymptotically identical.*

*Corrections to Pitman and Yor* (1986). In Theorem 8.2 (page 764) replace $\Psi_j(\zeta^{j,\infty})$ by $\psi_j(\zeta^{j,\infty})$. In Table 2 (page 767), line (4): replace $v/\|A\|$ by $2\pi v/\|A\|$.

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