Wall and Siegmund Duality Relations for Birth and Death Chains with Reflecting Barrier

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Abstract

For a birth and death chain on the nonnegative integers with birth and death probabilities p_i and $q_i \equiv 1 - p_i$ and reflecting barrier at 0, it is shown that the right tails of the probability of the first return from state 0 to state 0 are simple transition probabilities of a dual birth and death chain obtained by switching p_i and q_i. Combinatorial and analytical proofs are presented. Extensions and relations to other concepts of duality in the literature are discussed. [1]

1 Introduction

Let $\mathbf{P}_0$ govern a reflecting birth and death chain (X_n) on the nonnegative integers $\{0, 1, 2, \ldots\}$ started at $X_0 = 0$ with one-step upward (birth) transition probabilities p_i and one-step

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downward (death) transition probabilities q_i, where $q_0 = 0$ and $p_i + q_i = 1$ for all $i \geq 0$. Define $T_0 := \inf\{n > 0 : X_n = 0\}$ as the time of first return. Similarly, let $\tilde{\mathbf{P}}_0$ govern (X_n) as a corresponding *dual* birth and death chain with transition probabilities defined by switching p_i and q_i for $i \geq 1$. In this paper we will show by different methods that the $\mathbf{P}_0$ distribution of T_0 is related to the ordinary transition probabilities of $\tilde{\mathbf{P}}_0$.

Theorem 1.1 *With the notation of the previous paragraph, the probabilities* $\mathbf{P}_0$ *and* $\tilde{\mathbf{P}}_0$ *satisfy*

$$\mathbf{P}_0(T_0 = 2n) = \tilde{\mathbf{P}}_0(X_{2n-2} = 0) - \tilde{\mathbf{P}}_0(X_{2n} = 0) \qquad (n = 1, 2, 3, \ldots), \tag{1}$$

or equivalently

$$\mathbf{P}_0(2n < T_0 \leq \infty) = \tilde{\mathbf{P}}_0(X_{2n} = 0) \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

Elementary arguments allow this formula to be reexpressed in a number of different ways. Let $L_m := \max\{k : 0 \leq k \leq m, X_k = 0\}$. By the Markov property of $\mathbf{P}_0$

$$\mathbf{P}_0(L_{2n} = k) = \mathbf{P}_0(X_k = 0)\ \mathbf{P}_0(T_0 > 2n - k) \qquad (0 \leq k \leq 2n), \tag{3}$$

which allows (2) to be generalized to

$$\tilde{\mathbf{P}}_0(L_{2n} = k) = \mathbf{P}_0(L_{2n} = 2n - k) \qquad (0 \leq k \leq 2n). \tag{4}$$

In other words: for each n

$$\textit{the } \tilde{\mathbf{P}}_0 \textit{ distribution of } L_{2n} \textit{ equals the } \mathbf{P}_0 \textit{ distribution of } 2n - L_{2n}. \tag{5}$$

If $p_i = 1/2$ for all i, then $\mathbf{P}_0 = \tilde{\mathbf{P}}_0$ governs (X_n) as if $X_n = |S_n|$ for a simple symmetric random walk (S_n) on the integers started at $S_0 = 0$. Then (2) reduces to a well-known relation for this random walk which Feller [7] derives from the reflection principle, and (5) reduces to the symmetry about n of the discrete arcsine distribution of L_{2n}.

Section 2 shows how these identities amount to an identity of Wall [26] for continued fraction expansions of corresponding generating functions, whose probabilistic interpretation was indicated by Gerl [9]. A second proof is based on a path bijection argument and presented in Section 3. Section 4 compares this duality relation with a general form of duality between reflecting and absorbing barrier processes introduced by Siegmund [23]. The connection between the two notions of duality is exploited to yield a third proof of Theorem 1.1 and extensions thereof. In Section 5 we consider the duality (1) for birth and death chains on the nonnegative integers with holding probabilities which may be strictly positive. According to the spectral theory of Karlin and McGregor [14] for birth and death chains on the nonnegative integers, for $\mathbf{Q}_0$ governing such a chain, $\mathbf{Q}_0(X_{2n} = 0)$ is the nth moment of a probability distribution on the interval $[-1, 1]$. These moments are related to the canonical moments of the spectral measure and to continued fractions via its Stieltjes transform. We derive an integral representation for the first return probabilities which is used to prove (2) and to show that the existence of $\tilde{\mathbf{Q}}_0$ governing a dual chain such that an analog of (1) holds is equivalent to the assumption that all holding probabilities of $\mathbf{Q}_0$ vanish. If this property is satisfied we characterize for a given $\mathbf{Q}_0$ all probabilities $\tilde{\mathbf{Q}}_0$ such that (1) is satisfied. A final Section 6 contains miscellaneous further remarks.

2 A proof based on Wall's identity

It is well known and simple to show that the generating functions

$$P_0(z) := \sum_{n=0}^{\infty} \mathbf{P}_0(X_{2n} = 0) z^n; \quad F_0(z) := \sum_{n=1}^{\infty} \mathbf{P}_0(T_0 = 2n) z^n \tag{6}$$

are related by

$$F_0(z) = 1 - \frac{1}{P_0(z)}. \tag{7}$$

I. J. Good [10] used (7) and a simple recursive argument to obtain expressions for $P_0(z)$ and $F_0(z)$ as continued fractions. In particular, Good obtained

$$P_0(z) = \frac{1\ |}{|\ 1} \ - \ \frac{q_1 z\ |}{|\ \ 1} \ - \ \frac{p_1 q_2 z|}{|\ \ 1} \ - \ \frac{p_2 q_3 z|}{|\ \ 1} \ - \ \cdots, \tag{8}$$

where the continued fraction converges at least for $|z| < 1$. Continued fractions of this form were considered already by Wall [25, 26], who showed that for $\tilde{P}_0(z)$ defined by (8) with $\tilde{p}_i = q_i$ instead of p_i and $\tilde{q}_i = p_i$ instead of q_i, there is the identity [(75.3) in [26]]

$$P_0(z)\tilde{P}_0(z) = \frac{1}{1-z}, \quad |z| < 1. \tag{9}$$

Or, in terms of $F_0(z)$ and $\tilde{F}_0(z) = 1 - 1/\tilde{P}_0(z)$ [see (75.6) in [26]],

$$\tilde{F}_0(z) = \frac{F_0(z) - z}{F_0(z) - 1}. \tag{10}$$

As observed by Gerl [9], the above formulae express the generating functions for the dual chain

$$\tilde{P}_0(z) := \sum_n \tilde{\mathbf{P}}_0(X_{2n} = 0) z^n; \quad \tilde{F}_0(z) := \sum_n \tilde{\mathbf{P}}_0(T_0 = 2n) z^n \tag{11}$$

in terms of those of the original chain. From these relations the duality (1) in Theorem 1.1 follows easily by using (7) and comparing coefficients in the corresponding power series of $F_0(z)$ and $\tilde{P}_0(z)$.

The above argument shows that the probabilistic identity (1) is easily derived from Wall's identity (9), and vice versa. For completeness, we now recall Good's proof of (8) and Wall's proof of (9). For a proof of the identity (8), for $n \geq 1$ let

$$u_r^{(2n)} := \mathbf{P}_r(X_{2n} = r \text{ and } X_k > r \text{ for } k = 1, \ldots, 2n-1)$$

denote the $\mathbf{P}$-probability that, started at the point r, the process has a birth as its first transition and returns to r for the first time at step number $2n$. Similarly, for $n \geq 0$ define

$$v_r^{(2n)} := \mathbf{P}_r(X_{2n} = r \text{ and } X_k \geq r \text{ for } k = 1, \ldots, 2n-1)$$

and consider the corresponding generating functions

$$F_r(z) := \sum_{n=1}^{\infty} u_r^{(2n)} z^n \ ; \qquad P_r(z) := \sum_{n=0}^{\infty} v_r^{(2n)} z^n .$$

[Note that when $r = 0$ the notation is consistent with (6).] From $u_r^{(2n)} = p_r v_{r+1}^{(2n-2)} q_{r+1}$ we obtain

$$F_r(z) = p_r q_{r+1} z P_{r+1}(z) = \frac{p_r q_{r+1} z}{1 - F_{r+1}(z)}$$

and iterating this relation gives

$$F_0(z) = \frac{q_1 z|}{|\ 1} - \frac{p_1 q_2 z|}{|\ \ 1} - \frac{p_2 q_3 z|}{|\ \ 1} - \cdots \ .$$

(Note that the convergence of this continued fraction for $|z| < 1$ follows from Wall [26], Theorem 54.1.) Now (8) follows from (7).

For a proof of (9), let

$$\frac{A_n(z)}{B_n(z)} \ = \ \frac{1\ |}{|\ 1} \ + \ \frac{q_1 z\ |}{|\ \ 1} \ + \frac{p_1 q_2 z|}{|\ \ 1} \ + \ \cdots \ + \ \frac{p_{n-2} q_{n-1} z|}{|\ \ \ 1}$$

and

$$\frac{C_n(z)}{D_n(z)} \ = \ \frac{1\ |}{|\ 1} \ + \ \frac{p_1 z\ |}{|\ \ 1} \ + \ \frac{q_1 p_2 z|}{|\ \ 1} \ + \ \cdots \ + \ \frac{q_{n-2} p_{n-1} z|}{|\ \ \ 1}$$

denote the nth approximant for the generating function $P_0(-z)$ and $\tilde{P}_0(-z)$; here A_n, B_n, C_n, and D_n are polynomials. Using the recurrence relations [see (1.4) of [26]]

$$\begin{aligned} A_{n+1}(z) &= A_n(z) + p_{n-1} q_n z A_{n-1}(z), \quad A_1(z) = 1, \quad A_0(z) = 0 \\ D_{n+1}(z) &= D_n(z) + q_{n-1} p_n z D_{n-1}(z), \quad D_1(z) = 1, \quad D_0(z) = 1, \end{aligned}$$

we readily derive (by induction)

$$(1 + z) A_n(z) = q_{n-1} z D_{n-1}(z) + D_n(z), \quad n \geq 2$$

and similarly

$$B_n(z) = q_{n-1} z C_{n-1}(z) + C_n(z), \quad n \geq 1.$$

Combining these two equations yields

$$\begin{aligned} (1 + z) A_n(z) C_{n-1}(z) - B_n(z) D_{n-1}(z) &= D_n(z) C_{n-1}(z) - D_{n-1}(z) C_n(z) \\ &= k_n z^{n-1} \end{aligned}$$

for $n \geq 1$, where $k_n := (-1)^n p_{n-1} \prod_{j=1}^{n-2} (p_j q_j)$ for $n \geq 2$ and $k_1 := 1$, and where the last line of the display follows from [26], p. 16. By the Euler–Minding formulas (see Perron [20], p. 5), for $n \geq 1$ the polynomials B_n and D_n have degree at most $n - 1$, have positive coefficients, and satisfy $B_n(0) = D_n(0) = 1$. Thus, for all $n \geq 2$, we obtain for all $z > 0$

$$\left| \frac{A_n(z)}{B_n(z)} \frac{C_{n-1}(z)}{D_{n-1}(z)} \ - \ \frac{1}{1+z} \right| \ = \ \left| \frac{k_n z^{n-1}}{(1+z) B_n(z) D_{n-1}(z)} \right| \ \leq \ |z|^{n-1},$$

which implies for all $0 < z < 1$ that

$$P_0(-z)\tilde{P}_0(-z) \;=\; \lim_{n\to\infty} \frac{A_n(z)}{B_n(z)} \;\lim_{n\to\infty} \frac{C_{n-1}(z)}{D_{n-1}(z)} \;=\; \frac{1}{1+z}. \tag{12}$$

Now both sides of (12) are analytic functions in the unit disk $\{z : |z| < 1\}$ and the relation (9) follows.

3 Combinatorial proof of the duality relation

The assertion of Theorem 1.1 will be proved in the form (2) with the roles of $\mathbf{P}_0$ and $\tilde{\mathbf{P}}_0$ interchanged. For fixed n each side of (2) is a polynomial in $q_1, \ldots, q_n$:

$$\mathbf{P}_0(X_{2n} = 0) = F(q_1, \ldots, q_n),$$

say, is the sum over all paths from $(0,0)$ to $(2n,0)$ of products of conditional probabilities along paths in a diagram, call it $D(F,n)$, as in the middle left of Figure 1 for $n = 3$. Similarly

$$\tilde{\mathbf{P}}_0(T_0 > 2n) = G(q_1, \ldots, q_n),$$

say, is a sum of products over paths starting at $(0,0)$ and ending at any of the n points $(n+1, n+1), (n+2, n), \ldots, (2n, 2)$, in a diagram, call it $D(G,n)$, as shown in the top right of Figure 1. The problem is to show $D(F,n) \cong D(G,n)$, where for two diagrams D and D', each defined by a collection of directed segments with associated factors (e.g., 1, q_i, or $p_i = 1 - q_i$ for some i), and each inducing a polynomial in $(q_1, q_2, \ldots)$ by a sum of products over paths, $D \cong D'$ means the two polynomials are identical. Let $D'(G,n)$ denote the diagram with the same shape as $D(F,n)$, as in the middle right panel of Figure 1 for $n = 3$, obtained from $D(G,n)$ as follows: Move the initial path segment with a factor of 1 from the left end of $D(G,n)$ to the right end (where it then slopes downward instead of upward), insert $n-1$ additional segments with factors of 1 to join up the loose ends on the downsloping right side of $D(G,n)$, and finally reverse the direction of paths. There is an obvious one-to-one product-preserving correspondence between paths in $D(G,n)$ and $D'(G,n)$, so $D(G,n) \cong D'(G,n)$. For general n the diagram $D'(G,n)$ has the same set of paths as in $D(F,n)$, but with the factors in $D(F,n)$ modified as follows:

- on segment $(i,i) \to (i+1, i+1)$, $1 \le i \le n-1$, replace p_i by 1;
- elsewhere replace p_i by p_{i+2};
- on segment $(2k, 0) \to (2k+1, 1)$, $1 \le k \le n-1$, replace 1 by p_2;

Note that the factors q_i on down-segments appear in the same places in both $D(F,n)$ and $D'(G,n)$. Only the factors on up-segments are different. In either diagram $D(F,n)$ or $D'(G,n)$, call a path through the diagram from $(0,0)$ to $(2n,0)$, combined with a choice of 1 or $-q_i$ for each up-segment with a factor p_i, an *expanded path*. The set of all expanded

paths in $D(F,n)$ defines an *expanded diagram*, call it $D^*(F,n)$, in which each up-segment in $D(F,n)$ labelled by a p_i is replaced by two distinct up-segments labelled 1 and $-q_i$. Let $D^*(G,n)$ be derived similarly from $D'(G,n)$. By construction $D(F,n) \cong D^*(F,n)$ and $D(G,n) \cong D'(G,n) \cong D^*(G,n)$. The argument is completed by the following proposition, which shows $D^*(F,n) \cong D^*(G,n)$.

In either diagram $D^*(F,n)$ or $D^*(G,n)$, call a sequence of $2n$ factors derived from an expanded path a *selection* from the expanded diagram. Note that the selection determines the expanded path, since each factor 1 or $-q_i$ indicates an up-segment, and each q_i a down-segment.

Proposition 3.1 *For each selection from either $D^*(F,n)$ or $D^*(G,n)$, there is a unique corresponding selection from the other diagram with the same sequence of factors modulo signs, and such corresponding selections have identical products.*

Granted the correspondence between selections modulo signs, the equality of corresponding products follows easily: corresponding selections have the same number of factors 1, hence the same number of $-$ factors, because for every selection in either diagram, the number of 1s plus the number of $-$ factors equals n.

For $m \le n$ the expanded diagram $D^*(F,m)$ is obtained in an obvious way by restriction of the paths in $D^*(F,n)$ to the region $\{(i,k) : i+k \le 2m\}$. The diagrams $D^*(G,m)$ and $D^*(G,n)$ are similarly related. This can be seen for $m = 1$ or 2 and $n = 3$ in Figure 1.

The correspondence between $D^*(F,1)$ and $D^*(G,1)$ is trivial, since these two diagrams are identical: $F(q_1) = G(q_1) = q_1$. The correspondence between selections in $D^*(F,2)$ and $D^*(G,2)$ can be seen from Figure 1 to be as follows: there are 3 selections in each expanded diagram with corresponding products:

Product from $D^*(F,2)$		Product from $D^*(G,2)$
$(1)(q_1)(1)(q_1)$	$=$	$(1)(q_1)(1)(q_1)$
$(1)(1)(q_2)(q_1)$	$=$	$(1)(1)(q_2)(q_1)$
$(1)(-q_1)(q_2)(q_1)$	$=$	$(1)(q_1)(-q_2)(q_1)$

Thus $F(q_1,q_2) = G(q_1,q_2) = q_1^2 + q_2 q_1 - q_1^2 q_2$.

For $n = 3$ it can be seen from Figure 1 that there are 13 selections in each expanded diagram with corresponding products:

Product from $D^*(F,3)$		Product from $D^*(G,3)$
$(1)(1)(1)(q_3)(q_2)(q_1)$	$=$	$(1)(1)(1)(q_3)(q_2)(q_1)$
$(1)(1)(-q_2)(q_3)(q_2)(q_1)$	$=$	$(1)(1)(q_2)(-q_3)(q_2)(q_1)$
$(1)(1)(q_2)(1)(q_2)(q_1)$	$=$	$(1)(1)(q_2)(1)(q_2)(q_1)$
$(1)(1)(q_2)(-q_1)(q_2)(q_1)$	$=$	$(1)(1)(q_2)(q_1)(-q_2)(q_1)$
$(1)(1)(q_2)(q_1)(1)(q_1)$	$=$	$(1)(1)(q_2)(q_1)(1)(q_1)$
$(1)(-q_1)(q_2)(1)(q_2)(q_1)$	$=$	$(1)(q_1)(-q_2)(1)(q_2)(q_1)$
$(1)(-q_1)(q_2)(-q_1)(q_2)(q_1)$	$=$	$(1)(q_1)(-q_2)(q_1)(-q_2)(q_1)$

$$
\begin{aligned}
(1)(-q_1)(-q_2)(q_3)(q_2)(q_1) &= (1)(q_1)(-q_2)(-q_3)(q_2)(q_1)\\
(1)(-q_1)(1)(q_3)(q_2)(q_1) &= (1)(q_1)(1)(-q_3)(q_2)(q_1)\\
(1)(-q_1)(-q_2)(q_3)(q_2)(q_1) &= (1)(q_1)(-q_2)(-q_3)(q_2)(q_1)\\
(1)(q_1)(1)(q_1)(1)(q_1) &= (1)(q_1)(1)(q_1)(1)(q_1)\\
(1)(q_1)(1)(1)(q_2)(q_1) &= (1)(q_1)(1)(1)(q_2)(q_1)\\
(1)(q_1)(1)(-q_1)(q_2)(q_1) &= (1)(q_1)(1)(q_1)(-q_2)(q_1)
\end{aligned}
$$

The sum of these 13 products is $F(q_1, q_2, q_3) = G(q_1, q_2, q_3)$.

Proof of the correspondence for general n. Assume, for simplicity of notation but without loss of generality, that $0 < q_i < 1$ for every i, and that the q_i are all distinct. This allows a selection to be treated as a sequence of real numbers rather than a sequence of symbolic factors. Let $(r_1, \ldots, r_{2n})$ denote a selection from $D^*(F, n)$ and let $(x_0, x_1, \ldots, x_{2n})$ be the corresponding lattice path. Note that $(x_0, x_1, \ldots, x_{2n})$ is a path in $D(F, n)$ rather than an extended path in $D^*(F, n)$. So $(r_1, \ldots, r_{2n})$ determines $(x_0, x_1, \ldots, x_{2n})$ but not vice versa. To be precise, $(x_0, x_1, \ldots, x_{2n})$ is a sequence of non-negative integers with $x_0 = x_{2n} = 0$ and $(r_1, \ldots, r_{2n})$ is a sequence of real numbers such that each r_k equals $1, q_x$, or $-q_x$ for some $1 \le x \le n$, and for each $0 \le k \le 2n-1$ we have

$$
r_{k+1} = \begin{cases} 1 & \text{if } x_k = 0 \\ 1, -q_{x_k}, \text{ or } q_{x_k} & \text{if } x_k > 0 \text{ and } k + x_k < 2n \\ q_{x_k} & \text{if } k + x_k = 2n. \end{cases} \tag{13}
$$

The path (x_k) can be recovered inductively from (r_k) by

$$
x_{k+1} = \begin{cases} x_k + 1 & \text{if } r_{k+1} = 1 \text{ or } -q_{x_k} \\ x_k - 1 & \text{if } r_{k+1} = q_{x_k}. \end{cases} \tag{14}
$$

Similarly, a sequence $(s_1, \ldots, s_{2n})$ is a selection from $D^*(G, n)$ associated with the lattice path $(y_0, y_1, \ldots, y_{2n})$ in $D(G, n)$ if and only if $y_0 = y_{2n} = 0, y_m \ge 0$ for $1 \le m \le 2n$, each s_k equals $1, q_y$, or $-q_y$ for some $1 \le y \le n$, $s_1 = 1$, and for each $1 \le k \le 2n-1$,

$$
s_{k+1} = \begin{cases} 1 \text{ or } q_{y_k} & \text{if } 1 \le k \le n-1 \text{ and } y_k = k \\ 1, -q_{y_k+2}, \text{ or } q_{y_k} & \text{if } 0 < y_k < k \text{ and } k + y_k < 2n \\ 1 \text{ or } -q_2 & \text{if } 2 \le k \le 2n-2 \text{ and } y_k = 0 \\ q_{y_k} & \text{if } k + y_k = 2n. \end{cases} \tag{15}
$$

The path (y_k) can be recovered inductively from (s_k) by

$$
y_{k+1} = \begin{cases} y_k + 1 & \text{if } s_{k+1} = 1 \text{ or } -q_{y_k+2} \\ y_k - 1 & \text{if } s_{k+1} = q_{y_k}. \end{cases} \tag{16}
$$

It is claimed that

(i) for each selection $(r_1, \ldots, r_{2n})$ in $D^*(F, n)$, there exists a unique selection $(s_1, \ldots, s_{2n})$ in $D^*(G, n)$ such that $|s_k| = |r_k|$ for every $1 \le k \le 2n$;

(ii) each selection $(s_1, \ldots, s_{2n})$ in $D^*(G,n)$ can be so obtained from some selection $(r_1, \ldots, r_{2n})$ in $D^*(F,n)$;

(iii) if (x_k) and (y_k) are the lattice paths associated with such (r_k) and (s_k) then $x_k - y_k$ equals either 0 or 2 for every $0 \le k \le 2n$.

Claims (i) and (iii) will be established by induction over k for $1 \le k \le 2n$. Note that in (i) the only issue is existence: uniqueness is obvious from (15) because in every case the options for s_{k+1} have distinct absolute values.

Claim (ii) is shown by a similar induction in the reverse direction, decrementing k from $2n$ to 1.

Fix an arbitrary selection (r_k) with path (x_k) in $D^*(F,n)$, and make the following:

Inductive Hypothesis for (i) and (iii): *There is a unique possible sequence $(s_1, \ldots, s_k)$ for the first k factors of a selection in $D^*(G,n)$ with $|s_j| = |r_j|$ for every $1 \le j \le k$, and if $(y_1, \ldots, y_k)$ is the initial portion of the lattice path associated with $(s_1, \ldots, s_k)$ via (16), then $x_k - y_k \in \{0, 2\}$.*

This is trivial for $k = 1$. Assuming the inductive hypothesis for k, consider how s_{k+1} is determined by x_k, y_k, and r_{k+1} via (13) and (15). There are six possible cases, according to whether $x_k - y_k$ equals 0 or 2, and whether r_{k+1} equals 1, q_{x_k}, or $-q_{x_k}$. Each case is described by a row in the following table. In each row the assumed values of $x_k - y_k$ and r_{k+1} are given in the first and second columns. These values, combined with the basic rules (14), (15), (16), and $|s_{k+1}| = |r_{k+1}|$, then force the values of $x_{k+1} - x_k$, s_{k+1}, $y_{k+1} - y_k$, and $x_{k+1} - y_{k+1}$ as indicated. In each case, there is a unique possible choice of s_{k+1} according to the rules of $D^*(G,n)$ such that $|s_{k+1}| = |r_{k+1}|$, and this choice of s_{k+1} implies $x_{k+1} - y_{k+1} \in \{0, 2\}$. This is the inductive hypothesis with $k+1$ instead of k.

$x_k - y_k$	r_{k+1}	$x_{k+1} - x_k$	s_{k+1}	$y_{k+1} - y_k$	$x_{k+1} - y_{k+1}$
0	1	1	1	1	0
2	1	1	1	1	2
0	q_{x_k}	-1	q_{y_k}	-1	0
2	q_{x_k}	-1	$-q_{y_k+2}$	1	0
0	$-q_{x_k}$	1	q_{y_k}	-1	2
2	$-q_{x_k}$	1	$-q_{y_k+2}$	1	2

To check a row of the table, it must be verified that the value of s_{k+1} indicated in the 4th column with $|s_{k+1}| = |r_{k+1}|$ is actually available according to the rules of $D^*(G,n)$. (As mentioned before, uniqueness of such an s_{k+1} is obvious). The increments $x_{k+1} - x_k$ and $y_{k+1} - y_k$ are then forced by (14) and (16), respectively, and $x_{k+1} - y_{k+1}$ is found from

$$x_{k+1} - y_{k+1} = (x_k - y_k) + (x_{k+1} - x_k) - (y_{k+1} - y_k).$$

The cases work out as follows, according to the values of $(x_k - y_k, r_{k+1})$:

- $(0,1)$ or $(2,1)$. That is, $r_{k+1} = 1$, which implies $k + x_k < 2n$, hence $k + y_k < 2n$. So from (15), 1 is available as a choice for s_{k+1}.

- $(0, q_{x_k})$. The availability of the choice $r_{k+1} = q_{x_k}$ implies $y_k = x_k > 0$. The set of possible choices for s_{k+1} is then

$$\text{either } \{1, q_{y_k}\},\ \{1, -q_{y_k+2}, q_{y_k}\}, \text{ or } \{q_{y_k}\},$$

  depending on (k, y_k). Whatever this set of choices, there is a choice with $|s_{k+1}| = |r_{k+1}|$, namely, $s_{k+1} = q_{y_k}$.

- $(2, q_{x_k})$. Since $x_k \le k$ and $k + x_k \le 2n$ for all paths (x_k) in $D(F, n)$, we have $y_k < k$ and $k + y_k < 2n$. The set of possible choices for s_{k+1} is therefore

$$\text{either } \{1, -q_{y_k+2}, q_{y_k}\} \text{ if } y_k > 0, \text{ or } \{1, -q_{y_k+2}\} \text{ if } y_k = 0.$$

  Either way, there is a choice with $|s_{k+1}| = |r_{k+1}|$, namely, $s_{k+1} = -q_{y_k+2} = -q_{x_k}$.

- $(0, -q_{x_k})$. The choice of $r_{k+1} = -q_{x_k}$ implies $x_k > 0$ and $k + x_k < 2n$. The set of possible choices for s_{k+1} is therefore

$$\text{either } \{1, q_{x_k}\} \text{ if } y_k = k, \text{ or } \{1, -q_{x_k+2}, q_{x_k}\} \text{ if } y_k < k.$$

  Either way, the choice with $|s_{k+1}| = |r_{k+1}|$ is $s_{k+1} = q_{x_k}$.

- $(2, -q_{x_k})$. Then $y_k < k$ and $y_k + k < 2n$. The set of possible choices for s_{k+1} is

$$\text{either } \{1, -q_{x_k}\} \text{ if } y_k = 0, \text{ or } \{1, -q_{x_k}, q_{x_{k-2}}\} \text{ if } y_k > 0.$$

  Either way, the choice with $|s_{k+1}| = |r_{k+1}|$ is $s_{k+1} = -q_{x_k}$. ■

Remark 3.2 As a check on the above correspondence, it can be seen by direct calculation that in either $D^*(F, n)$ or $D^*(G, n)$ the number of expanded paths is

$$\sum_{u=0}^{n-1} \frac{n-u}{n+u} \binom{n+u}{n} 2^u \tag{17}$$

The coefficient of 2^u represents the number of different paths in $D(F, n)$, or in $D(G, n)$, that have u segments with factors of the form p_i for some i. Each such path contributes 2^u selections in the expanded diagram. In $D(G, n)$ the coefficient can be evaluated using the ballot theorem. In $D(F, n)$ the same coefficient is found using Theorem 4 in Section III.7 of Feller [7].

4 Wall duality and Siegmund duality

4.1 Siegmund duality

It is instructive to compare the duality relation in Theorem 1.1 between two reflecting birth and death chains, call it *Wall duality*, with a general form of duality between reflecting

and absorbing barrier processes introduced by Siegmund [23], and studied also by Cox and Rösler [3], Clifford and Sudbury [2], and Diaconis and Fill [6]. Let $(\mathbf{Q}_x)$, indexed by initial state $x \geq 0$, be a family of distributions governing (X_n) as a Markov chain with state space $\{0, 1, 2, \ldots\}$. Another such family of distributions $(\hat{\mathbf{Q}}_x)$ is said to be *dual to* $(\mathbf{Q}_x)$ *in Siegmund's sense* if, for all states y and z,

$$\mathbf{Q}_y(X_n \leq z) = \hat{\mathbf{Q}}_z(X_n \geq y). \tag{18}$$

for each $n \geq 0$. Two necessary conditions for the existence of a Siegmund dual follow immediately. First, taking $y = 0$ shows that the identity (18) can hold only if $\mathbf{Q}_0$ makes 0 an absorbing state. Second, the existence of a Siegmund dual implies that the left side of (18) is a nonincreasing function of y; using terminology introduced by Daley [4], this says that $(\mathbf{Q}_x)$ must be a *stochastically monotone* chain. Siegmund showed that these two conditions are also sufficient, and, by repeated application of the Chapman–Kolmogorov equations, that if (18) holds for $n = 1$ then it holds also for every $n \geq 0$.

Now let $(\mathbf{Q}_x)$ govern an absorbing ($r_0 = 1$) birth and death chain with birth, death, and hold probabilities p_x, q_x, and r_x, respectively. It is known and easily checked that a Siegmund dual $(\hat{\mathbf{Q}}_x)$ exists if and only if

$$p_x + q_{x+1} \leq 1 \text{ for all } x \geq 1 \tag{19}$$

in which case the Siegmund dual $(\hat{\mathbf{Q}}_x)$ is uniquely determined as a partially reflecting birth and death chain with one-step probabilities

$$\hat{q}_x = p_x, \quad \hat{r}_x = 1 - (p_x + q_{x+1}), \quad \hat{p}_x = q_{x+1} \quad (x \geq 0).$$

The constraint (19) is usually regarded as an artifact of discrete time, as both continuous-time birth and death processes and diffusions with absorption at 0 have Siegmund duals without such constraints. Also, as noted in Section 5 of Cox and Rösler [3], when $r_x = 0$ for all x one can work around this constraint by watching the chain only every two steps. We use this device in Section 4.2 to connect Wall duality to Siegmund duality.

Assuming as above that $(\mathbf{Q}_x)$ admits a Siegmund dual $(\hat{\mathbf{Q}}_x)$, the basic identity (18) implies that for each $n \geq 0$

$$\mathbf{Q}_1(T_0 > n) = \mathbf{Q}_1(X_n \geq 1) = 1 - \mathbf{Q}_1(X_n \leq 0) = 1 - \hat{\mathbf{Q}}_0(X_n \geq 1) = \hat{\mathbf{Q}}_0(X_n = 0). \tag{20}$$

Consider now a family of probabilities $\mathbf{P} := (\mathbf{P}_x)$ governing a birth and death chain on $\{0, 1, 2, \ldots\}$ with birth, death, and hold probabilities p_x, q_x, and r_x which are arbitrary subject to (19) for $x \geq 1$, but not subject to the absorbtion condition $r_0 = 1$ imposed on $\mathbf{Q} := (\mathbf{Q}_x)$. Then

$$\mathbf{P}_1(T_0 > n) = \mathbf{Q}_1(T_0 > n) = \hat{\mathbf{Q}}_0(X_n = 0) \qquad (n \geq 0) \tag{21}$$

where $\hat{\mathbf{Q}}$ is the Siegmund dual of $\mathbf{Q}$, with $\mathbf{Q}$ obtained from $\mathbf{P}$ by transforming 0 into an absorbing barrier. It then follows that, for any $n \geq 1$,

$$\mathbf{P}_0(T_0 > n) = p_0 \mathbf{P}_1(T_0 > n - 1) = p_0 \hat{\mathbf{Q}}_0(X_{n-1} = 0). \tag{22}$$

4.2 The connection

In this section we use the notation birth_x, death_x, and hold_x for the transition probabilities of a generic birth and death chain on $\{0,1,2,\ldots\}$. As in Sections 1 through 3, let $\mathbf{P} := (\mathbf{P}_x)$ be *any* birth and death chain on $\{0,1,2,\ldots\}$ *with no holding* and $\text{birth}_x = p_x$ and $\text{death}_x = q_x$ (implying $p_0 = 1$ and $q_0 = 0$). Watching this chain every two steps leads to consideration of the family of probabilities $\mathbf{R} = (\mathbf{R}_x)$ governing a birth and death chain on $\{0,1,2,\ldots\}$ with transition probabilities

$$\begin{aligned} \text{death}_x &= q_{2x-1}q_{2x}, \quad \text{birth}_x = p_{2x}p_{2x+1}, \\ \text{hold}_x &= 1 - (q_{2x-1}q_{2x} + p_{2x}p_{2x+1}) = p_{2x-1}q_{2x} + p_{2x}q_{2x+1} \end{aligned}$$

if $x \neq 0$ and

$$\text{death}_0 = 0, \quad \text{birth}_0 = p_0p_1 = p_1, \quad \text{hold}_0 = 1 - p_1 = q_1.$$

Clearly,

$$\mathbf{R}_x(X_n = y) = \mathbf{P}_{2x}(X_{2n} = 2y) \qquad (x, y = 0,1,2,\ldots) \tag{23}$$

Since for $\mathbf{R}$ we evidently have

$$\text{birth}_x + \text{death}_{x+1} = p_{2x}p_{2x+1} + q_{2x+1}q_{2x+2} \leq 1,$$

equation (22) can be applied to $\mathbf{R}$: for $n \geq 1$,

$$\mathbf{P}_0(T_0 > 2n) = \mathbf{R}_0(T_0 > n) = (p_0p_1)\hat{\mathbf{R}}_0(X_{n-1} = 0) = \hat{\mathbf{R}}_0(X_{n-1} = 0)p_1, \tag{24}$$

where $\hat{\mathbf{R}}$ is the Siegmund dual of ($\mathbf{R}$, with 0 changed to absorbing barrier). But $\hat{\mathbf{R}}$ is a birth and death chain with transition probabilities

$$\begin{aligned} \text{death}_x &= p_{2x}p_{2x+1}, \quad \text{birth}_x = q_{2x+1}q_{2x+2}, \\ \text{hold}_x &= 1 - (p_{2x}p_{2x+1} + q_{2x+1}q_{2x+2}) = q_{2x}p_{2x+1} + q_{2x+1}p_{2x+2} \end{aligned}$$

if $x \neq 0$ and

$$\text{death}_0 = 0, \quad \text{birth}_0 = q_1q_2, \quad \text{hold}_0 = 1 - q_1q_2 = p_1 + q_1p_2.$$

From this it is apparent that, for all x and y,

$$\hat{\mathbf{R}}_x(X_n = y) = \tilde{\mathbf{P}}_{2x+1}(X_{2n} = 2y + 1), \tag{25}$$

where $\tilde{\mathbf{P}}$ is defined as the law governing a birth and death chain with

$$\text{death}_x = p_x, \quad \text{birth}_x = q_x, \quad \text{hold}_x = 0$$

if $x \neq 0$ and

$$\text{death}_0 = 0, \quad \text{birth}_0 = 1, \quad \text{hold}_0 = 0.$$

In particular, for $n \geq 0$,

$$\hat{\mathbf{R}}_0(X_n = 0) = \tilde{\mathbf{P}}_1(X_{2n} = 1),$$

and so by (24) we have, for $n \geq 1$,

$$\mathbf{P}_0(T_0 > 2n) = \tilde{\mathbf{P}}_1(X_{2n-2} = 1)p_1 = \tilde{\mathbf{P}}_1(X_{2n-1} = 0) = \tilde{\mathbf{P}}_0(X_{2n} = 0).$$

Since the result is plainly true also for $n = 0$, we have established the *Wall duality* relation (2).

4.3 More general results

The connection between Wall duality and Siegmund duality can be exploited to derive more general results than Theorem 1.1 for chains $\mathbf{P}$ and $\tilde{\mathbf{P}}$ in Wall duality. The following theorem allows for an arbitrary initial state for $\mathbf{P}$ and yields full information in terms of the $\tilde{\mathbf{P}}$ transition probabilities about both the distribution of T_0 and the conditional distribution of X_ν given no visit to the origin by time ν.

Theorem 4.1 *Let* $\mathbf{P}$ *and* $\tilde{\mathbf{P}}$ *govern Wall dual chains. For* $\nu \geq 0$ *and* $x \geq 0$ *and* $y \geq 1$ *with* $x + y \equiv \nu \pmod 2$,

$$\mathbf{P}_x(T_0 > \nu, X_\nu \geq y) = \tilde{\mathbf{P}}_{y-1}(X_\nu \leq x - 1) \text{ if } x \geq 1 \text{ and}$$

$$\mathbf{P}_x(T_0 > \nu, X_\nu \geq y) = \tilde{\mathbf{P}}_{y-1}(X_{\nu-1} = 0) \text{ if } x = 0.$$

In particular, we have the following complement to Theorem 1.1 concerning the marginal distribution of T_0 under $\mathbf{P}$:

Corollary 4.2 *Let* $\mathbf{P}$ *and* $\tilde{\mathbf{P}}$ *govern Wall dual chains. Then*
(a) $\forall n \geq 0\ \forall x \geq 1:\ \mathbf{P}_{2x}(T_0 > 2n) = \tilde{\mathbf{P}}_0(X_{2n+1} \leq 2x - 1)$;
(b) $\forall n \geq 1\ \forall x \geq 0:\ \mathbf{P}_{2x+1}(T_0 > 2n - 1) = \tilde{\mathbf{P}}_0(X_{2n} \leq 2x)$.

Part (a) follows from Theorem 4.1 via the calculation

$$\mathbf{P}_{2x}(T_0 > 2n) = \mathbf{P}_{2x}(T_0 > 2n + 1) = \mathbf{P}_{2x}(T_0 > 2n + 1, X_{2n+1} \geq 1) = \tilde{\mathbf{P}}_0(X_{2n+1} \leq 2x - 1),$$

and part (b) follows similarly. So we need only prove Theorem 4.1.

Before beginning the proof, we record some preliminary observations concerning Siegmund duality. Let $\mathbf{Q}$ (with absorbing barrier at 0) and $\hat{\mathbf{Q}}$ be any two Markov chains with state space $\{0, 1, \ldots\}$ that are in Siegmund duality, as in Section 4.1. Then, for any $n \geq 0$ and $x \geq 1$ and $y \geq 1$, we have

$$\begin{aligned} \mathbf{Q}_x(T_0 > n, X_n \geq y) &= \mathbf{Q}_x(X_n \geq y) = 1 - \mathbf{Q}_x(X_n \leq y - 1) \\ &= 1 - \hat{\mathbf{Q}}_{y-1}(X_n \geq x) = \hat{\mathbf{Q}}_{y-1}(X_n \leq x - 1) \end{aligned}$$

In particular, if $\mathbf{P}$ and $\mathbf{Q}$ are birth and death chains related in the way described at (21)–(22), then

$$\mathbf{P}_x(T_0 > n, X_n \geq y) = \hat{\mathbf{Q}}_{y-1}(X_n \leq x - 1). \tag{26}$$

A similar result holds for initial state $x = 0$.

Proof of Theorem 4.1. We shall make further use of the notation and observations in Section 4.2. For brevity, we consider only the case that $x \geq 2$ is even; the cases $x = 0$ and x odd can be handled in similar fashion. That is, we show, for $\nu \geq 0$ and $x \geq 1$ and $y \geq 1$ with y and ν of the same parity, that

$$\mathbf{P}_{2x}(T_0 > \nu, X_\nu \geq y) = \tilde{\mathbf{P}}_{y-1}(X_\nu \leq 2x - 1). \tag{27}$$

First suppose that $\nu = 2n$ and $y = 2\eta$ are even. Then

$$\begin{aligned}
\mathbf{P}_{2x}(T_0 > \nu, X_\nu \geq y) &= \mathbf{P}_{2x}(T_0 > 2n, X_{2n} \geq 2\eta) \\
&= \mathbf{R}_x(T_0 > n, X_n \geq \eta) \quad \text{[by (23)]} \\
&= \hat{\mathbf{R}}_{\eta-1}(X_n \leq x-1) \quad \text{[by (26)]} \\
&= \tilde{\mathbf{P}}_{2\eta-1}(X_{2n} \leq 2x-1) \quad \text{[by (25)]} \\
&= \tilde{\mathbf{P}}_{y-1}(X_\nu \leq 2x-1),
\end{aligned}$$

as desired.

Now suppose that $\nu = 2n-1$ and $y = 2\eta - 1$ are odd, where $n \geq 1$ and $\eta \geq 1$. If $\eta = 1$, i.e., $y = 1$, then (27) holds since

$$\begin{aligned}
\mathbf{P}_{2x}(T_0 > \nu, X_\nu \geq 1) &= \mathbf{P}_{2x}(T_0 > \nu) = \mathbf{P}_{2x}(T_0 > 2(n-1)) \\
&= \mathbf{P}_{2x}(T_0 > 2(n-1), X_{2(n-1)} \geq 2) \\
&= \tilde{\mathbf{P}}_1(X_{2(n-1)} \leq 2x-1) = \tilde{\mathbf{P}}_0(X_\nu \leq 2x-1),
\end{aligned}$$

where parity considerations have been taken into account, the penultimate equality follows from the preceding paragraph, and the final equality follows from the fact that $\tilde{\mathbf{P}}$ has reflecting barrier at 0. If $\eta \geq 2$, then (27) again holds:

$$\begin{aligned}
&\mathbf{P}_{2x}(T_0 > \nu, X_\nu \geq y) \\
&\quad = \mathbf{P}_{2x}(T_0 > 2n-1, X_{2n-1} \geq 2\eta - 1) \\
&\quad = p_{2(\eta-1)}\mathbf{P}_{2x}(T_0 > 2(n-1), X_{2(n-1)} \geq 2(\eta-1)) \\
&\qquad\quad + q_{2(\eta-1)}\mathbf{P}_{2x}(T_0 > 2(n-1), X_{2(n-1)} \geq 2\eta) \\
&\quad = p_{2(\eta-1)}\tilde{\mathbf{P}}_{2\eta-3}(X_{2n-2} \leq 2x-1) \\
&\qquad\quad + q_{2(\eta-1)}\tilde{\mathbf{P}}_{2\eta-1}(X_{2n-2} \leq 2x-1) \quad \text{[by the preceding paragraph]} \\
&\quad = \tilde{\mathbf{P}}_{2(\eta-1)}(X_{2n-1} \leq 2x-1) \\
&\quad = \tilde{\mathbf{P}}_{y-1}(X_\nu \leq 2x-1). \blacksquare
\end{aligned}$$

A natural question about Wall duality is whether it can be extended to more processes, for example to (1) birth and death chains with non-zero holding probabilities r_x, or to (2) continuous-time birth-death processes. The generating functions $P_0(z)$ and $F_0(z)$ for such chains admit continued fraction representations (see, e.g., Good [10], Flajolet [8]), as do corresponding Laplace transforms for birth and death processes ([19], [12], [13]). We will give a partial answer to (1) in the following section.

5 Wall duality and spectral theory

In this section we discuss an approach via spectral theory (see [14]) to the duality in order to investigate if and how Wall duality can be transferred to more general birth and death

chains on the nonnegative integers. Our results are limited to birth and death chains started at $X_0 = 0$ but indicate some of the difficulties in generalizing Wall duality. Throughout this section let the probability $\mathbf{Q}_0$ govern a birth and death chain (X_n) on the nonnegative integers started at $X_0 = 0$ with one-step transition probabilities p_j, q_j, and r_j for births, deaths, and holds, where $p_j > 0$, $q_{j+1} > 0$, and $p_j + q_j + r_j \leq 1$ for $j = 0, 1, 2, \ldots$. The case $p_j + q_j + r_j < 1$ can be interpreted as adjoining an absorbing state j^* which can be reached from state j with probability $1 - p_j - q_j - r_j > 0$. The probability $\tilde{\mathbf{Q}}_0$ with transition probabilities $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{r}_j$ is called *dual* to $\mathbf{Q}_0$ if

$$\mathbf{Q}_0(T_0 = n) = \tilde{\mathbf{Q}}_0(X_{n-2} = 0) - \tilde{\mathbf{Q}}_0(X_n = 0) \tag{28}$$

holds for all $n = 2, 3, \ldots$. According to Theorem 1.1, a dual of a probability $\mathbf{Q}_0$ with $r_j = 0$, $q_0 = 0$ is obtained by switching the transition probabilities p_j and q_j for $j \geq 1$. We now address the question of characterizing all dual probabilities $\tilde{\mathbf{Q}}_0$ [in the sense of (28)] corresponding to a given probability $\mathbf{Q}_0$.

It is shown in [14] that the n-step transition probabilities of the chain (X_n) with governing probability $\mathbf{Q}_0$ can be represented as

$$\mathbf{Q}_0(X_n = j) = \frac{\int_{-1}^{1} x^n Q_j(x) d\psi(x)}{\int_{-1}^{1} Q_j^2(x) d\psi(x)} \tag{29}$$

where ψ is the spectral measure of the probability $\mathbf{Q}_0$ (with support contained in the interval $[-1, 1]$) and $Q_i(x)$ is a polynomial of degree i, recursively defined by

$$(x - r_i)Q_i(x) = q_i Q_{i-1}(x) + p_i Q_{i+1}(x) \tag{30}$$

whenever $p_i > 0$, and $Q_{-1}(x) = 0$, $Q_0(x) = 1$. If $n < j$, then (29) and $\mathbf{Q}_0(X_n = j) = 0$ show that the polynomials $Q_i(x)$ are orthogonal with respect to the distribution $d\psi(x)$ on the interval $[-1, 1]$. From (29) and [10] we obtain a continued fraction expansion for the Stieltjes transform of the spectral measure ψ:

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{\infty} \mathbf{Q}_0(X_n = 0) z^n = \int_{-1}^{1} \frac{d\psi(x)}{1 - zx} \\ &= \frac{1 \quad |}{|1 - r_0 z} - \frac{p_0 q_1 z^2 \ |}{|1 - r_1 z} - \frac{p_1 q_2 z^2 \ |}{|\ 1 - r_2 z} - \cdots . \end{aligned} \tag{31}$$

The spectral measure ψ is determined by its moments $c_1, c_2, \ldots$. In what follows it turns out to be useful to switch from the ordinary moments to the so-called *canonical moments* of ψ. A complete treatment of the theory of canonical moments can be found in Dette and Studden [5]; some basic facts are mentioned here for the sake of completeness. It is shown in [5], Section 8.2, that the Stieltjes transform of ψ can also be written as

$$P_0(z) = \frac{1 \quad |}{|1 + (1 - 2\zeta_1)z} - \frac{4\zeta_1\zeta_2 z^2 \quad |}{|1 + (1 - 2\zeta_2 - 2\zeta_3)z} - \frac{4\zeta_3\zeta_4 z^2 \quad |}{|1 + (1 - 2\zeta_4 - 2\zeta_5)z} - \cdots, \tag{32}$$

where $\zeta_j = (1 - m_{j-1})m_j$ $(j \geq 1)$, $m_0 = 0$, $\zeta_0 = 0$. The quantities m_j vary in the interval $[0,1]$ and are in one-to-one correspondence with the ordinary moments $c_1, c_2, \ldots$ (see Skibinsky [24]). We call m_j the jth *canonical moment* of the distribution ψ. If i_0 is the first index such that $m_{i_0} \in \{0,1\}$, then ψ is supported at a finite number of points and the continued fraction expansion of $P_0(z)$ terminates. If $m_i \in (0,1)$ for all $i \geq 1$, then ψ has infinite support and we put formally $i_0 = \infty$.

Observing (31) we see that the canonical moments of the spectral measure ψ can be determined from the probabilities of $\mathbf{Q}_0$ by solving recursively the equations

$$\begin{aligned} -1 + 2\zeta_{2j} + 2\zeta_{2j+1} &= r_j \qquad (0 \leq j < i_0), \\ 4\zeta_{2j-1}\zeta_{2j} &= p_{j-1}q_j \qquad (1 \leq j < i_0). \end{aligned} \tag{33}$$

For example, the first two canonical moments of ψ can be expressed in terms of the transition probabilities as

$$m_1 = \frac{1 + r_0}{2}, \quad m_2 = \frac{p_0 q_1}{1 - r_0^2}.$$

Let $\tilde{\psi}$ denote the measure corresponding to the sequence of canonical moments $\tilde{m}_j = 1 - m_j$ $(j \geq 1)$; then $\tilde{\psi}$ is called the *dual* of the spectral measure ψ of the probability $\mathbf{Q}_0$.

Note that $\tilde{\psi}$ is not necessarily the spectral measure of a probability $\tilde{\mathbf{Q}}_0$ governing the chain (X_n). By the previous discussion it follows that this property is only satisfied if one can find transition probabilities $\tilde{p}_j$, $\tilde{r}_j$, and $\tilde{q}_j$ corresponding to $\tilde{\mathbf{Q}}_0$ that solve the equations

$$\begin{aligned} -1 + 2\tilde{\zeta}_{2j} + 2\tilde{\zeta}_{2j+1} &= \tilde{r}_j \qquad (0 \leq j < i_0), \\ 4\tilde{\zeta}_{2j-1}\tilde{\zeta}_{2j} &= \tilde{p}_{j-1}\tilde{q}_j \qquad (0 \leq j < i_0) \end{aligned} \tag{34}$$

for all $j \geq 0$, where $\tilde{\zeta}_j = (1 - \tilde{m}_{j-1})\tilde{m}_j$ $(j \geq 1)$, $\tilde{\zeta}_0 = 0$.

Theorem 5.1 *Let ψ denote the spectral measure of the probability $\mathbf{Q}_0$ and $\tilde{\psi}$ the dual of ψ. Then for all $n \geq 2$*

$$\mathbf{Q}_0(T_0 = n) = \int_{-1}^{1} (1 - x^2)x^{n-2} d\tilde{\psi}(x).$$

Proof. From Lau [17] it follows that the Stieltjes transform of the measure $(1 - x^2)d\tilde{\psi}(x)$ is given by

$$\begin{aligned} H(z) &= \int_{-1}^{1} \frac{(1 - x^2)d\tilde{\psi}(x)}{1 - zx} \\ &= \frac{4\tilde{\gamma}_1\tilde{\gamma}_2}{|1 + (1 - 2\tilde{\gamma}_2 - 2\tilde{\gamma}_3)z} - \frac{4\tilde{\gamma}_3\tilde{\gamma}_4 z^2}{|1 + (1 - 2\tilde{\gamma}_4 - 2\tilde{\gamma}_5)z} - \cdots, \end{aligned}$$

where $\tilde{\gamma}_j = \tilde{m}_{j-1}(1-\tilde{m}_j) = (1-m_{j-1})m_j = \zeta_j$ and $\tilde{\gamma}_1 = \zeta_1$. Now (32) and (33) give

$$P_0(z) = \left[1 - r_0 z - z^2 H(z)\right]^{-1},$$

and from the extension of (7) to the present setting we obtain

$$F_0(z) = r_0 z + z^2 H(z) = r_0 z + z^2 \sum_{n=0}^{\infty} \int_{-1}^{1} (1-x^2) x^n d\tilde{\psi}(x) z^n,$$

which implies the assertion of the theorem. ■

Example 5.2 Consider the situation of Section 1, where $\mathbf{P}_0$ governs (X_n) with $p_0 = 1$ and $r_j = 0$, $p_j + q_j = 1$ $(j = 0, 1, 2, \ldots)$. It is easy to see that in this case the solution to (33) is

$$m_{2j-1} = \frac{1}{2}, \quad m_{2j} = q_j \qquad (j = 1, 2, \ldots)$$

and the dual $\tilde{\psi}$ of the spectral measure has canonical moments $\tilde{m}_{2j-1} = \frac{1}{2}$, $\tilde{m}_{2j} = p_j$. Thus there is a one-to-one correspondence between the transition probabilities of $\mathbf{P}_0$ and $\tilde{\mathbf{P}}_0$ and the canonical moments of the distributions ψ and $\tilde{\psi}$. Observing (34) we obtain that $\tilde{\psi}$ is the spectral measure of a probability $\tilde{\mathbf{P}}_0$ where the corresponding transition probabilities satisfy $\tilde{r}_j = 0$ $(j \geq 0)$ and $\tilde{p}_{j-1}\tilde{q}_j = q_{j-1}p_j$ $(j \geq 1)$. By the special choice $\tilde{p}_0 = 1$ it follows that $\tilde{\psi}$ is the spectral measure of a birth and death chain governed by $\tilde{\mathbf{P}}_0$ which is obtained from $\mathbf{P}_0$ by interchanging p_j and q_j for $j \geq 1$. This is the Wall duality described and investigated in Sections 1 through 3.

Theorem 5.3 *Let $\mathbf{Q}_0$ govern (X_n) started at $X_0 = 0$ with one-step transition probabilities p_j, q_j, and r_j for births, deaths, and holds.*

a) There exists a probability $\tilde{\mathbf{Q}}_0$ governing (X_n) such that the duality relation (28) holds for $\mathbf{Q}_0$ and $\tilde{\mathbf{Q}}_0$ if and only if $r_j = 0$ for all $j = 0, 1, 2, \ldots$.

b) If $r_j = 0$ for all $j = 0, 1, 2, \ldots$, define

$$S_n = 1 + \sum_{k=1}^{n} p_0 \left(\prod_{j=1}^{k} \frac{p_j}{q_j} \right) Q_k(1) Q_{k+1}(1) \quad \text{for } n \geq 1, \quad S_{-1} = 0, \quad S_0 = 1, \tag{35}$$

and $S = \lim_{n\to\infty} S_n$ $(= \infty$ if the sum diverges). For every $t \in [0, \frac{1}{S}]$ the probability $\tilde{\mathbf{Q}}_0^t$ with transition probabilities $\tilde{r}_n(t) = 0$,

$$\tilde{q}_n(t) = 1 - \tilde{p}_n(t) = \frac{1 - tS_{n-2}}{1 - tS_{n-1}} \cdot \frac{Q_{n+1}(1)}{Q_n(1)} p_n \quad \text{for } n \geq 1, \quad \tilde{q}_0(t) = 1 - \tilde{p}_0(t) = t$$

is a dual of $\mathbf{Q}_0$ [in the sense of (28)], and these are the only such duals. In particular, the dual of the probability $\mathbf{Q}_0$ is unique if and only if the sum in (35) diverges.

Proof. a) Let ψ denote the spectral measure of $\mathbf{Q}_0$ and $\tilde{\psi}$ the corresponding dual spectral measure. By the Karlin and McGregor representation (29) and the determinacy of the Hausdorff moment problem it follows from Theorem 5.1 that the assertion can be proved by showing that $\tilde{\psi}$ is the spectral measure of some probability $\tilde{\mathbf{Q}}_0$ if and only if $r_j = 0$ for all $j = 0, 1, 2, \ldots$.

Now let m_j and $\tilde{m}_j = 1 - m_j$ denote the canonical moments of ψ and $\tilde{\psi}$, respectively, and assume that $\tilde{\psi}$ is the spectral measure of $\tilde{\mathbf{Q}}_0$ with transition probabilities $\tilde{p}_j$, $\tilde{q}_j$, and $\tilde{r}_j$. Obviously a measure is a spectral measure of a birth and death chain if and only if $r_j \geq 0$ for all $j \geq 0$. If so, then from (33) and (34) we have

$$0 \leq r_0 = -1 + 2m_1 = 1 - 2\tilde{m}_1 = -\tilde{r}_0 \leq 0,$$

which implies $m_1 = \frac{1}{2}$, $r_0 = \tilde{r}_0 = 0$. Assume that $r_i = \tilde{r}_i = 0$ and $m_{2i+1} = \tilde{m}_{2i+1} = \frac{1}{2}$ for all $0 \leq i \leq j-1$; then

$$\begin{aligned} 0 \leq r_j &= -1 + 2\zeta_{2j} + 2\zeta_{2j+1} = -(1 - m_{2j})(1 - 2m_{2j+1}) \\ &= -(1 - m_{2j})(2\tilde{m}_{2j+1} - 1) = -\frac{1 - m_{2j}}{1 - \tilde{m}_{2j}}\tilde{r}_j \leq 0, \end{aligned}$$

which proves $r_j = \tilde{r}_j = 0$ and $m_{2j+1} = \tilde{m}_{2j+1} = \frac{1}{2}$. Thus, if $\tilde{\psi}$ is the spectral measure of a birth and death chain governed by $\tilde{\mathbf{Q}}_0$, then $r_j = \tilde{r}_j = 0$ for all $j = 0, 1, 2, \ldots$.

Conversely, assume that $\mathbf{Q}_0$ satisfies $r_j = 0$ for all $j = 0, 1, 2, \ldots$. Then it is easy to see that $m_{2j-1} = \tilde{m}_{2j-1} = \frac{1}{2}$ for all j, and the proof of the existence of a probability $\tilde{\mathbf{Q}}_0$ with spectral measure $\tilde{\psi}$ reduces to the problem of finding transition probabilities $\tilde{p}_j$, $\tilde{q}_j$ (with $\tilde{r}_j = 0$) such that

$$\tilde{p}_{j-1}\tilde{q}_j = (1 - \tilde{m}_{2j-2})\tilde{m}_{2j} \tag{36}$$

(with $\tilde{m}_0 = 0$) holds for all $j \geq 1$. Observing $m_{2j-1} = \frac{1}{2}$, it follows by an induction argument that the solution to (33) is given by

$$m_{2j} = 1 - \tilde{m}_{2j} = \frac{Q_{j-1}(1)}{Q_j(1)}q_j, \quad 1 - m_{2j} = \tilde{m}_{2j} = \frac{Q_{j+1}(1)}{Q_j(1)}p_j$$

where the polynomials $Q_j(x)$ are defined in (30). Now (36) can be rewritten as

$$\tilde{p}_{j-1}\tilde{q}_j = q_{j-1}p_j\frac{Q_{j-2}(1)Q_{j+1}(1)}{Q_{j-1}(1)Q_j(1)} \quad \text{for } j \geq 2, \quad \tilde{p}_0\tilde{q}_1 = \frac{Q_2(1)}{Q_1(1)}p_1, \tag{37}$$

and putting $\tilde{p}_0 = 1$ it follows that there is at least one solution of (37), namely,

$$\tilde{p}_j = \frac{Q_{j-1}(1)}{Q_j(1)}q_j, \quad \tilde{p}_0 = 1. \tag{38}$$

Thus $\tilde{\psi}$ is the spectral measure of $\tilde{\mathbf{Q}}_0$ with transition probabilities (38) ($\tilde{r}_j = 0$, $\tilde{q}_j = 1 - \tilde{p}_j$) and the assertion a) follows.

b) By the discussion in the proof of the converse part in a) we have to find all solutions of (37). Now the right hand side of (37) is a chain sequence with minimal parameter sequence

$$\tilde{m}_{2j} = 1 - m_{2j} = \frac{Q_{j+1}(1)}{Q_j(1)} p_j \qquad (j = 0, 1, 2, \ldots)$$

(see Chihara [1]), and therefore the problem of solving (37) reduces to the problem of finding all parameter sequences of the given chain sequence $\{(1 - \tilde{m}_{2j-2})\tilde{m}_{2j}\}_{j=1}^{\infty}$. But this problem is solved in [1] and therefore all solutions of (37) are obtained from $\tilde{q}_0(t) = t$ and

$$\tilde{q}_n(t) = \frac{1 - tS_{n-2}}{1 - tS_{n-1}} \cdot \frac{Q_{n+1}(1)}{Q_n(1)} p_n \qquad (n = 1, 2, 3, \ldots)$$

where $0 \leq t \leq \frac{1}{S}$ (note that there is a typographical error in (3.8) of [1]). This proves assertion b). ■

We will conclude this section with a discussion of the situation in Section 1 where $\mathbf{P}_0$ governs the birth and death chain with transition probabilities $r_x = 0$, $p_0 = 1$, $p_x = 1 - q_x$. We assume that $p_x > 0$ and $q_x > 0$ for $x \geq 1$. Observing (30) it follows that $Q_k(1) = 1$ $(k \geq 0)$ and, by Theorem 5.3, a dual of $\mathbf{P}_0$ in the sense of (28) is unique if and only if the dual $\tilde{\mathbf{P}}_0$ obtained from $\mathbf{P}_0$ by switching the p_x and q_x for $x \geq 1$ is recurrent. Applying (2) to both $\mathbf{P}_0$ and $\tilde{\mathbf{P}}_0$, one can show that $\tilde{\mathbf{P}}_0$ is positive recurrent, null recurrent, or transient according as $\mathbf{P}_0$ is transient, null recurrent, or positive recurrent (note that this provides an alternative proof of a result in [9]). By these arguments we obtain:

Corollary 5.4 *Let $\mathbf{P}_0$ govern a reflecting birth and death chain (X_n) on the nonnegative integers $\{0, 1, 2, \ldots\}$ started at $X_0 = 0$ with birth probabilities p_i and death probabilities q_i, where $q_0 = 0$, and $p_i + q_i = 1$ for all $i \geq 0$. The dual of $\mathbf{P}_0$ in the sense of (28) is unique if and only if the chain governed by $\mathbf{P}_0$ is transient or null recurrent.*

Example 5.5 Let $\mathbf{P}_0$ govern the random walk with transition probabilities $p_0 = 1$, $p_x = 1 - q_x = p$ $(x \geq 1)$. Then it is easy to see that

$$S_n = \frac{1 - \rho^{n+1}}{1 - \rho} \qquad (n \geq 0), \quad S_{-1} = 0$$

where $\rho = p/q$. If $p \geq q$ we have $S = \infty$ (the random walk is transient or null recurrent) and there exists only one dual probability of $\mathbf{P}_0$ obtained by switching the p_x and q_x (as we did in Section 1). If $p < q$ (the random walk is positive recurrent) we have $S = (1 - \rho)^{-1}$ and it follows that for every $t \in [0, 1 - \rho]$ the probability $\tilde{\mathbf{Q}}_0^t$ with transition probabilities

$$\tilde{p}_n(t) = 1 - \frac{1 - \rho - t(1 - \rho^{n-1})}{1 - \rho - t(1 - \rho^n)} p \quad (n \geq 1), \quad \tilde{p}_0(t) = 1 - t, \quad \tilde{q}_n(t) \equiv 1 - \tilde{p}_n(t), \quad \tilde{r}_n(t) \equiv 0$$

is a dual of $\mathbf{P}_0$. Note that the special choice $t = 1 - \rho$ gives the probability $\tilde{\mathbf{Q}}_0$ with transition probabilities $\tilde{p}_0 = p/q$, $\tilde{p}_x = p$ $(x \geq 1)$ as a dual of $\mathbf{P}_0$, while the choice $t = 0$ gives the dual probability $\tilde{\mathbf{P}}_0$ considered in Sections 1 through 3.

6 Miscellaneous remarks

1. As an example where probabilities can be calculated much more easily on one side of the identity (2) than on the other, fix $0 < p = 1 - q < 1$, and let $\mathbf{P}_0$ govern the birth and death chain (X_n) with transition probabilities

$$p_x = \frac{p^{x+1} + q^{x+1}}{p^x + q^x}; \qquad q_x = 1 - p_x \quad (x = 1, 2, \ldots).$$

By Theorem 5.3 there exists a unique dual $\tilde{\mathbf{P}}_0$ of $\mathbf{P}_0$ [in the sense of (28)] which is obtained by switching the transition probabilities p_x and q_x for $x \geq 1$. Then for $n = 0, 1, 2, \ldots$,

$$\tilde{\mathbf{P}}_0(T_0 > 2n) = \mathbf{P}_0(X_{2n} = 0) = (pq)^n \binom{2n}{n}, \tag{39}$$

$$\tilde{\mathbf{P}}_0(X_{2n} = 0) = \mathbf{P}_0(T_0 > 2n) = 1 - \sum_{m=1}^{n} \frac{(pq)^m}{2m-1} \binom{2m}{m}. \tag{40}$$

In each case the second equality follows from the fact that the $\mathbf{P}_0$ distribution of (X_n) is the distribution of $(|Y_1 + \cdots + Y_n|)$ for independent Y_i with $P(Y_i = 1) = p$, $P(Y_i = -1) = q$. This fact follows from the criterion of Rogers and Pitman [21] for a function of a Markov chain to be Markov; or see Ross [22], Example 4.1(d). One can get more general formulae from the arguments of Section 4.3. For example, one can explicitly obtain the entire $\tilde{\mathbf{P}}_0$ distribution of X_{2n}: for $n \geq 0$ and $y \geq 0$,

$$\tilde{\mathbf{P}}_0(X_{2n} \leq 2y) = \mathbf{P}_{2y+1}(T_0 > 2n - 1) = 1 - \frac{(2y+1)(pq)^y}{p^{2y+1} + q^{2y+1}} \sum_{m=y+1}^{n} \frac{(pq)^m}{2m-1} \binom{2m}{m-y} \left(1 - \frac{y}{m}\right);$$

we omit the details.

There does not seem to be any similar representation (as the absolute value of a random walk or some such simple process) for the $\tilde{\mathbf{P}}_0$ distribution of (X_n). So without exploiting the duality relation it is not evident why there should be such simple formulae for the $\tilde{\mathbf{P}}_0$ probabilities. Due to the representation in terms of (Y_n), there is a fairly simple formula for the n-step transition probabilities $\mathbf{P}_i(X_n = j)$.

2. While it does not seem obvious how to extend Wall duality to birth and death processes in continuous time, the expression of the duality in (5) allows passage to the limit to obtain a duality relation for pairs of reflecting diffusions on $[0, \infty)$ obtained as weak limits of birth and death chains. Compare remark (e) on p. 923 in Siegmund [23] . It appears that apart from boundary conditions, one obtains exactly the same pairs of dual processes in the limit from either Siegmund duality or Wall duality. The simplest special case of the diffusion result is the following fact about reflecting Brownian motion (BM) with drift. Let $\mathbf{P}_0^\delta$ govern a BM $(X_t, t \geq 0)$ on $[0, \infty)$ with drift δ and instantaneous reflection at 0. Let $L_t = \sup\{s \leq t : X_s = 0\}$. Then for all $t > 0$ the $\mathbf{P}_0^\delta$ distribution of L_t is identical to the $\mathbf{P}_0^{-\delta}$ distribution of $t - L_t$. This can also be seen easily from the well-known representation of $(X_t, t \geq 0)$ with

distribution $\mathbf{P}_0^\delta$ as $X_t = B_t - M_t$, where $(B_t, t \geq 0)$ is a BM with drift δ and $M_t := \inf_{0 \leq s \leq t} B_s$, by a simple time reversal.

3. Continuing the previous remark, it seems clear that for diffusions the analog of (2) will be a corresponding identity between the transition density $p(0,0,t)$ for one diffusion and the right tail $\tilde{\Lambda}(t,\infty)$ of the Lévy measure $\tilde{\Lambda}$ of the inverse local time process at zero for the dual diffusion. Recently, Jansons [11] has indicated how to obtain such a result using excursion theory. This duality is related to the question, studied by various authors, of what are the possible Lévy measures Λ for the inverse local time of a diffusion process. See, e.g., Knight [15], Küchler [16].

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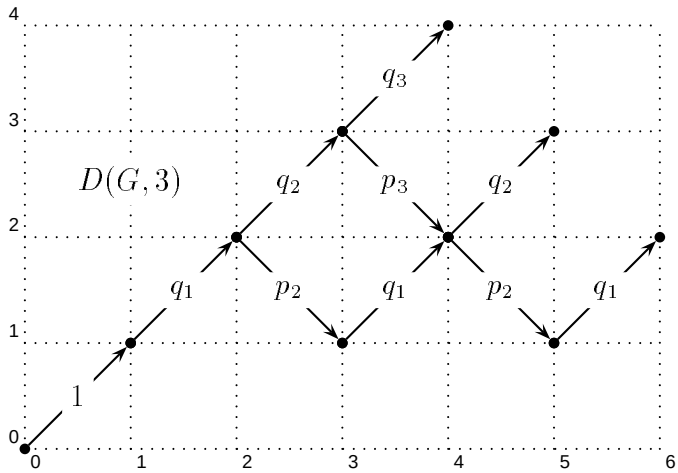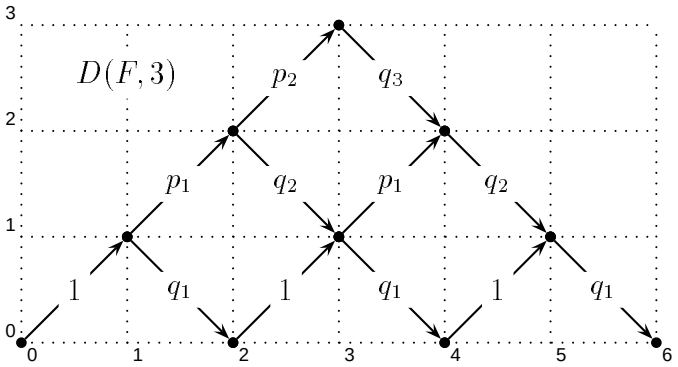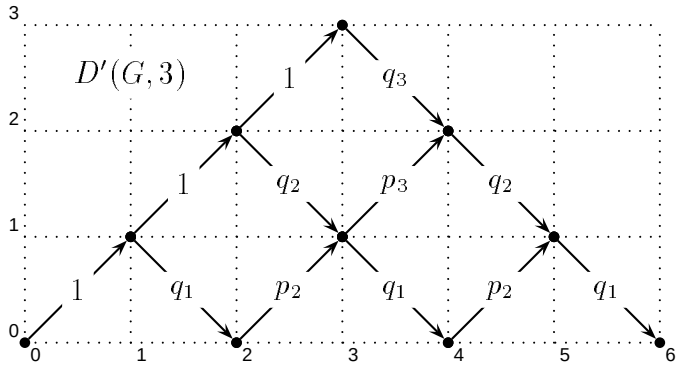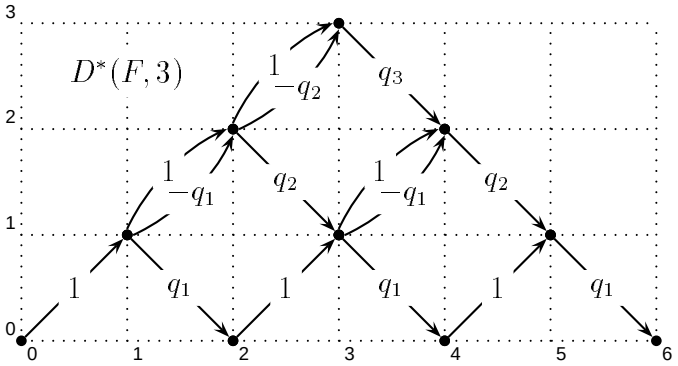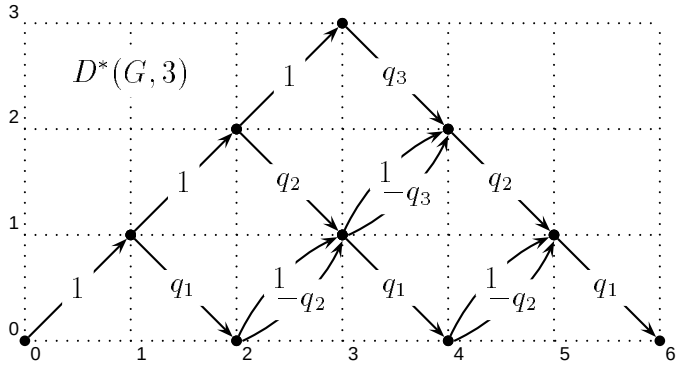
Figure 1: Diagrams for $n = 3$