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ASYMPTOTIC LAWS OF PLANAR BROWNIAN MOTION[1]

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Recent results on the asymptotic distribution of winding and crossing numbers are presented as part of a larger framework of asymptotic laws for planar Brownian motion. The approach is via random time changes, martingale calculus, and excursion theory.

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1. Introduction. Brownian motion in the plane has special features which set it apart from Brownian motion in other dimensions. What distinguishes dimension two from higher dimensions is that the planar motion is *neighborhood recurrent*. This means that for any open domain D in the plane, the random occupation time of D by a planar Brownian motion Z before time t,

$$A(D, t) = \int_0^t 1_D(Z_s)\, ds,$$

has limit ∞ almost surely as $t \to \infty$. For large t, the occupation time $A(D, t)$ is roughly proportional to the area of D and to $\log t$. More precisely, Kallianpur and Robbins (1953) showed that for all Borel sets D with $0 < \text{area}(D) < \infty$

$$(1.\text{a}) \qquad \frac{2\pi A(D, t)}{\text{area}(D)\log t} \xrightarrow{d} H \quad \text{as } t \to \infty,$$

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where H is a random variable with the *standard exponential distribution*

$$P(H \in dh)/dh = e^{-h}, \qquad 0 < h < \infty,$$

and $\xrightarrow{d}$ indicates convergence in distribution.

What distinguishes dimension two from dimension one is that the planar motion does not hit points: for each point z in the plane,

$$\text{(1.b)} \qquad P(Z_t = z \text{ for some } t > 0) = 0.$$

For points z other than the initial point Z_0 of the Brownian motion, this allows the definition of a winding random variable $\Phi^z(t)$, the continuous total angle wound by the Brownian motion Z around the point z up to time t. For large t, the winding $\Phi^z(t)$ is also roughly proportional to $\log t$. Spitzer (1958) showed that

$$\text{(1.c)} \qquad 2\Phi^z(t)/\log t \xrightarrow{d} W \quad \text{as } t \to \infty,$$

where W has the *standard Cauchy distribution*

$$P(W \in dw)/dw = \frac{1}{\pi(1 + w^2)}, \qquad -\infty < w < \infty.$$

We refer to these basic asymptotic distribution theorems (1.a) and (1.c) as the *Kallianpur–Robbins law* and *Spitzer's law*, respectively. These are typical of a large class of *asymptotic laws* for planar Brownian motion. Such a law asserts convergence in distribution as $t \to \infty$ of $G(t)/g(t)$ for some functional $G(t)$ of the Brownian path up to time t, and some normalizing function $g(t)$, typically $(\log t)^\alpha$ for $\alpha = \frac{1}{2}$, 1 or 2.

The purpose of this paper is to provide a survey of these asymptotic laws, explaining how they arise, and how they are linked together. We say that two or more asymptotic laws are *linked* if the laws hold jointly as $t \to \infty$, meaning that there is convergence of finite dimensional distributions toward some joint distribution of limiting random variables. For example, the Kallianpur–Robbins law (1.a) may be regarded as a collection of asymptotic laws parameterized by the set D. These laws are linked as D varies in a very simple way: they hold jointly with the same asymptotic exponential variable H for every D. This is by virtue of the *ratio ergodic theorem*, due also to Kallianpur and Robbins (1953):

$$\text{(1.d)} \qquad \frac{A(D,t)}{A(C,t)} \to \frac{\text{area}(D)}{\text{area}(C)} \quad \text{a.s.} \quad \text{as } t \to \infty,$$

for all Borel subsets C and D of the plane with finite and strictly positive area. We call H *the asymptotic local time variable.* Roughly speaking, H measures how much time the Brownian motion has spent per unit area up to a large time t, relative to $\log t$.

Spitzer's law (1.c) may be regarded similarly as a collection of asymptotic laws parameterized by the point z about which the winding $\Phi^z(t)$ is computed. Following the work of Messulam and Yor (1982) and Lyons and McKean (1984), Pitman and Yor (1984) showed that these Spitzer laws are linked in an interesting way, both to each other, and to the Kallianpur–Robbins law. For windings about points $z_1, \dots, z_n$ the asymptotic winding variables $W^1, \dots, W^n$ may be written as

$$W^j = W_+ + W_-^j, \tag{1.e}$$

where W_+ is a component in common to all points, attributable to *big windings* made when the Brownian motion is far from all points, and the W_-^j are individual components for each point z_j, attributable to *small windings* about z_j. The link with the Kallianpur–Robbins law is that the big winding variable W_+ and the n asymptotic small winding variables W_-^j are mutually conditionally independent given the asymptotic local time variable H.

Sections 2 through 7 give an exposition of these results for windings and occupation times. We start in Section 2 with the stochastic integral representation of the windings. This is used in Section 3 to give an elementary proof of Spitzer's law. The argument follows the method suggested by Williams (1974) and developed further by Messulam and Yor (1982) and Durrett (1982, 1984). Section 4 treats the asymptotic joint distribution of occupation times and big and small windings about a single point. In Section 5 the conditional independence of the asymptotic big and small windings given the asymptotic local time is explained using ideas of Brownian excursion theory. This leads to the representation in Section 6 of the asymptotic winding variables in (1.e) as functions of a collection of n linked complex Brownian motions, which have identical excursions in the right half plane, but independent excursions in the left half plane.

Section 7 concerns details of the joint distribution of the n asymptotic windings $W^1, \dots, W^n$ in (1.e). This is a multivariate distribution with Cauchy marginals and an interesting dependence structure which does not seem to have been encountered before. Every linear combination of $W^1, \dots, W^n$ has a distribution belonging to a two-parameter family of scale mixtures of symmetric Cauchy distributions. In particular, every positive linear combination is Cauchy distributed.

The development of Sections 2 through 6 leads to a large class of limit laws, including the Spitzer and Kallianpur–Robbins laws, which we call *log scaling laws*. These laws, described in Section 8, form a linked collection of limit laws for functionals of the Brownian motion which admit asymptotically simple expressions in terms of polar coordinates. The collection of log scaling laws is curious in that while large collections of random variables admit joint limit distributions, there does not seem to be any functional limit theorem of the usual kind, like Donsker's theorem.

While the collection of log scaling laws encompasses a great many asymptotic distribution theorems for functionals of planar Brownian motion, there are still further laws, linked to the log scaling laws, but involving more and more independent Brownian motions for the description of the limits. These include results of Kasahara and Kotani (1979) and Messulam and Yor (1982) for additive functionals, mentioned in Section 4, and further results related to windings, to appear in a sequel to this paper. The effect of probing deeper and deeper into the asymptotic fine structure of complex BM is the production of more and more independent Brownian motions in terms of which limit distributions can be written. This is so mainly as a consequence of Knight's theorem, which asserts that n continuous mutally *orthogonal* local martingales may be written as time changes of n *independent* Brownian motions.

Appendix A is a brief summary of Itô's calculus of continuous semimartingales, which is used throughout the paper. Appendix B is concerned with time changes to Brownian motion, including an asymptotic form of Knight's theorem which proves useful in Section 6. Immediately following is a paper which applies the results for windings to obtain asymptotic distributions for the numbers of level crossings of a Cauchy process.

The results of this paper can be extended to apply to some other processes in the plane. Le Gall and Yor (1986) give extensions to certain Brownian motions with drift. Belisle (1986) gives an analog of Spitzer's theorem for random walks. Interestingly, for random walks the limit law for the windings is not Cauchy, but the distribution of W_+, the asymptotic big winding for Brownian motion. What happens is that small windings of Brownian motion occur so fast they go unnoticed by a discrete time skeleton. Davis (1975, 1979a, b) uses the winding of Brownian motion to study analytic functions. McKean (1969) and Lyons and McKean (1984) give applications to the classification of Riemann surfaces. See also Doyle (1984) and McKean and Sullivan (1984) for related work.

2. Windings as stochastic integrals. Throughout this section let

$$Z = (Z_t,\ t \geq 0) = (X_t + iY_t,\ t \geq 0)$$

be a complex Brownian motion starting at 1, unless otherwise mentioned. Since Z a.s. never hits zero, Z may be written in polar coordinates as

$$Z_t = R_t \exp(i\Phi_t), \qquad t \geq 0, \tag{2.a}$$

where $R = |Z|$ is the *radial part* of Z, and Φ_t is the *total winding of Z about* 0 *up to time t*. More precisely, $(\Phi_t,\ t \geq 0)$ is the unique continuous process such that $\Phi_0 = 0$ and (2.a) holds. The total winding up to time t may be regarded as in complex analysis as the integral along the path of Z between times 0 and t of the closed differential form on $\mathbb{C} \setminus \{0\}$

$$\operatorname{Im}\left(\frac{dz}{z}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Thus if we define

$$\log Z_t = \int_{Z[0,t]} \frac{dz}{z}, \tag{2.b1}$$

the integral along the path of Z between times 0 and t of dz/z [see Yor (1977a)], then $(\log Z_t,\ t \geq 0)$ is the unique continuous determination of a logarithm of Z_t starting at $\log Z_0 = 0$, and

$$\log Z_t = \log R_t + i\Phi_t. \tag{2.b2}$$

It is an important property of complex Brownian motion that integrals of complex differential forms along Brownian paths, such as the one appearing in (2.b1), may be rewritten as Itô type stochastic integrals.

THEOREM 2.1. *If f is holomorphic on a domain D from which Z a.s. never escapes, then*

$$\int_{Z[0,t]} f(z)\,dz = \int_0^t f(Z_s)\,dZ_s. \tag{2.c}$$

See for example Ikeda and Manabe (1978, 1979) and Yor (1977a), where more general results of this type are discussed for a complex-valued semimartingale instead of the Brownian motion Z. The process defined by the integrals in (2.c) is an example of a *conformal martingale* relative to the σ-fields (F_t) generated by Z. This is a process $M + iN$ with continuous paths such that M and N are local martingales relative to (F_t), which are orthogonal, and have identical increasing processes:

$$\langle M, N\rangle = 0, \qquad \langle M, M\rangle = \langle N, N\rangle.$$

Appendices A and B give a brief review of the theory of continuous local martingales which is used in this paper. Getoor and Sharpe (1972) give a detailed treatment of conformal martingales.

If there exists a primitive F for f in the domain D, so $F'(z) = f(z)$, $z \in D$, then formula (2.c) becomes

$$F(Z_t) - F(Z_0) = \int_0^t f(Z_s)\,dZ_s. \tag{2.d}$$

This is a special case of Itô's formula, with less than the usual number of terms because

$$\frac{\partial F}{\partial \bar{z}} = 0 \quad \text{and} \quad \Delta F = 0.$$

Returning to the example $f(z) = 1/z$, where the primitive must be defined along the path, formulae (2.b) and (2.c) yield

$$\log Z_t = \int_0^t \frac{dZ_s}{Z_s}, \tag{2.e}$$

whence the *stochastic integral representations*

$$\log R_t = \int_0^t \frac{X_s\,dX_s + Y_s\,dY_s}{R_s^2}, \qquad \Phi_t = \int_0^t \frac{X_s\,dY_s - Y_s\,dX_s}{R_s^2}. \tag{2.f}$$

Thus the *log radial process* $\log R$ and the winding process Φ are orthogonal continuous local martingales, with common increasing process U, the *log clock* defined by

$$U_t = \int_0^t \frac{ds}{R_s^2}. \tag{2.g}$$

It was assumed above that Z started at $z_0 = 1$. If Z starts at $z_0 \neq 0$ then formulae (2.e), (2.f), and (2.g) still hold, provided $\log Z_t$ is replaced by $\log(Z_t/z_0)$, the unique continuous determination of the logarithm starting at zero, and $\log R_t$ is replaced by $\log(R_t/r_0)$ where $r_0 = |z_0|$. We shall continue to assume for simplicity that $z_0 = 1$. With the changes just indicated everything works also for a more general starting point $z_0 \neq 0$. But it is impossible to define the winding process starting at $z_0 = 0$. As explained in Section 7.16 of Itô and McKean (1965), if Z starts at zero, at time $t = 0$ the sample path of the naively defined angular process on the circle "comes in spinning like a circular Brownian motion defined for $-\infty < t \leq 0$ as t comes in from $-\infty$." This happens because $U_t = \infty$ for all $t > 0$ if $Z_0 = 0$, whereas $U_t < \infty$ for all $t > 0$ if $Z_0 \neq 0$.

The skew product representation. A conformal martingale may be represented by a time change of complex Brownian motion. This result, due to Getoor and Sharpe (1972), is a generalization of the theorem of Lévy (1948) that the composition of a nonconstant holomorphic function F with complex Brownian motion Z, as in (2.d) above, yields a time changed BM($\mathbb{C}$). The result for a conformal martingale may be regarded as an immediate corollary of the theorem of Knight (1971) stated in Appendix B. In the case at hand, the conformal martingale

$$\log Z_t = \log R_t + i\Phi_t$$

may be represented as

$$\log Z_t = \zeta(U_t), \tag{2.h}$$

where U is the log clock (2.g), and

$$\zeta = \beta + i\theta \tag{2.i}$$

is a complex Brownian motion starting at zero, which may be written in terms of $\log Z$ by inverting the time change. According to (2.h)

$$\log R_t = \beta(U_t), \qquad \Phi_t = \theta(U_t), \qquad t \geq 0, \tag{2.j}$$

where β and θ are two independent BM's, which we call the *log radial* and *angular* BM's, respectively. Since the log clock U_t can be expressed in terms of β as

$$U_t = \inf\left\{u\colon \int_0^u \exp(2\beta_v)\, dv > t\right\}, \tag{2.k}$$

the σ-fields generated by R and β are identical. As a consequence the angular BM is independent of the whole radial process, and in particular independent of the log clock U. This representation (2.j) of the winding process Φ as a BM run with an independent clock U determined by the radial motion is called the *skew product representation* of Φ. Similar representations for Brownian motions on spheres and Brownian motion with drift may be found in Section 7.15 of Itô and McKean (1965) and in Pitman and Yor (1981).

3. Spitzer's law. Spitzer proved (1.c) by explicit computation of the characteristic function of Φ_t. A variation of Spitzer's argument appears in Itô and

McKean (1965), page 270. The skew product is used to make the step

$$E\exp(i\alpha\Phi_t) = E\exp\left(-\frac{\alpha^2}{2}U_t\right). \tag{3.a}$$

After taking a Laplace transform, explicitly solving a differential equation, and inverting the transform, it is found that the expectation in (3.a) may be obtained by integrating the function of r

$$\frac{I_{|\alpha|}\left(\frac{r_0 r}{t}\right)}{I_0\left(\frac{r_0 r}{t}\right)} \tag{3.b}$$

with respect to the distribution of R_t. Here $I_\nu(x)$ is the usual modified Bessel function, and Z is supposed to start with $|Z_0| = r_0$. As this computation suggests, the function in (3.b) is identical to

$$E\left[\exp(i\alpha\Phi_t)|R_t = r\right] = E\left[\exp\left(-\frac{\alpha^2}{2}U_t\right)\middle|R_t = r\right]. \tag{3.c}$$

See Edwards (1967), Yor (1980), and Pitman and Yor (1981, 1982) for derivations of this identity and related results. In particular, Yor (1980) shows how to make the identification between (3.b) and (3.c) using Girsanov's theorem and formulae for the transition densities of Bessel processes.

Williams (1974) made the key observation that all computations involving Bessel functions could be avoided by simply comparing the winding $\Phi(t)$ at a fixed time t with the winding $\Phi(T_r)$ at a radial hitting time

$$T_r = \inf\{t\colon R_t > r\}.$$

At time T_r the log clock reads simply

$$U(T_r) = \sigma_{\log r},$$

where

$$\sigma_h = \inf\{u\colon \beta_u > h\} \tag{3.d1}$$

and we have assumed for simplicity that $r_0 = 1$. The skew product formula (2.j) now gives

$$\Phi(T(e^h)) = \theta(\sigma_h), \qquad h \geq 0. \tag{3.d2}$$

As remarked by Spitzer (1958), and further explained in Section I.9 of Durrett (1984), the right-hand process defined in terms of independent BM's β and θ is a Cauchy process. Thus so is the left-hand process embedded in the windings. In particular, for $h = \log r$

$$h^{-1}\Phi(T_r) = h^{-1}\theta(\sigma_h) \stackrel{d}{=} \theta(\sigma_1) \tag{3.e}$$

by Brownian scaling, as will be discussed in more detail below. The common distribution here is the standard Cauchy distribution appearing as the limit in

Spitzer's theorem. To complete a proof of Spitzer's law, it now suffices to take $r = \sqrt{t}$, so $h = \log r = \frac{1}{2}\log t$, and show that as $t \to \infty$

$$(3.f) \qquad h^{-1}\left[\Phi(t) - \Phi(T_{\sqrt{t}})\right] \xrightarrow{P} 0.$$

Williams proved this by a tightness argument using Brownian scaling, which is presented and further exploited in Messulam and Yor (1982). Other variations of this argument are given by Durrett (1982, 1984). We remark here that (3.f) holds because an application of Brownian scaling shows that

$$(3.g) \qquad \Phi(t) - \Phi(T_{\sqrt{t}}) \xrightarrow{d} \Phi^{-} \quad \text{as } t \to \infty,$$

where Φ^{-} is the total angle wound between times 1 and T_1 for a complex Brownian motion started at zero. Thus (3.f) holds for any $h = h(t)$ tending to ∞ as $t \to \infty$, in particular for $h(t) = \frac{1}{2}\log t$. This completes the proof of Spitzer's theorem.

There is one point in the above argument which deserves further consideration. This is the Brownian scaling operation which yields the Cauchy scaling property (3.e). To bring out the important rôle played by Brownian scaling we introduce the following notation.

NOTATION. For $E = \mathbb{R}$ or $\mathbb{C}$, let $\Omega(E)$ be the set of all continuous E valued paths $\alpha = \alpha(u)$, $u \geq 0$.

For $h > 0$, α a path in $\Omega(\mathbb{R})$ or $\Omega(\mathbb{C})$, let $\alpha^{(h)}$ be the path

$$(3.h) \qquad \alpha^{(h)}(u) = h^{-1}\alpha(h^2 u), \qquad u \geq 0.$$

Of course, if α is a BM, then so is $\alpha^{(h)}$. To illustrate the notation, define a measurable map W: $\Omega(\mathbb{C}) \to R$ by

$$(3.i) \qquad W(\zeta) = \theta(\sigma_1),$$

where $\zeta = \beta + i\theta$ and σ_1 are as in (2.j) and (3.d). Then

$$W(\zeta^{(h)}) = h^{-1}\theta(\sigma_h),$$

so line (3.e) may be rewritten as

$$(3.j) \qquad h^{-1}\Phi(T_r) = W(\zeta^{(h)}) \stackrel{d}{=} W(\zeta), \quad \text{where } h = \log r.$$

This makes the role of Brownian scaling quite obvious, and leads to formulation of the following lemma.

LEMMA 3.1. *Let Z be a* BM($\mathbb{C}$) *starting at $z_0 \neq 0$. As t and $h = \frac{1}{2}\log t$ tend to ∞,*

$$h^{-1}\Phi(t) - W(\zeta^{(h)}) \xrightarrow{P} 0,$$

where $\zeta = \beta + i\theta$ is the BM($\mathbb{C}$) *obtained by time changing* $\log(Z/z_0)$ *via its clock U, and $W(\zeta) = \theta(\sigma_1)$.*

PROOF. Put together (3.f) and (3.j). □

Spitzer's law is an immediate consequence of the lemma, the invariance of Brownian motion under Brownian scaling, and the Cauchy distribution of $\theta(\sigma_1)$. It is this lemma behind Spitzer's law rather than Spitzer's law itself, which turns out to be the right kind of building block for further asymptotic laws.

4. Big windings, small windings, and additive functionals. An important step toward understanding the asymptotic behavior of Brownian windings about more than one point is to decompose each winding process into a process of big windings and a process of small windings. As in the previous section, let Φ be the winding process about zero of a complex Brownian motion Z starting at $z_0 = 1$. Let D_+, the *big domain* and D_- the *small domain* be the open sets outside and inside the unit circle. The sign used as a subscript is the sign of $\log|z|$ for z in the domain. The symbol $\pm$ is an index which may be $+$ or $-$, indicating big or small. Define processes $\Phi_{\pm}$ by the stochastic integrals

$$\Phi_{\pm}(t) = \int_0^t d\Phi(s)\, 1(Z(s) \in D_{\pm}), \tag{4.a}$$

where $1(A)$ stands for the indicator of A.

The process Φ_+ is the process of *big windings* and Φ_- is the process of *small windings*. Because the Lebesgue measure of the time spent by Z on the unit circle is a.s. zero,

$$\Phi = \Phi_+ + \Phi_-.$$

The idea now is that this decomposition describes the winding process Φ as alternating between typically very long stretches of time during which Z is far away from the origin in D_+, when Φ changes very slowly, but nonetheless significantly, according to the increments of Φ_+, and typically very short stretches of time when Z is in D_- approaching zero, when Φ changes very rapidly according to the increments of Φ_-. To help describe the dependence between the big and small windings, we introduce the *local time L on the unit circle*, defined as the local time process of the semimartingale R at 1, as in Appendix A. The local time process L may also be identified as the continuous additive functional of BM($\mathbb{C}$) associated with length measure on the circle. According to the ergodic theorem for additive functionals of complex Brownian motion [see Itô and McKean (1965), page 277], if $A(t)$ is another continuous increasing additive functional, such as

$$A(t) = \int_0^t a(Z_s)\, ds, \tag{4.b}$$

for a nonnegative measurable function a, then

$$\frac{L(t)}{A(t)} \to \frac{2\pi}{\|A\|} \quad \text{a.s.}, \tag{4.c}$$

where $\|A\|$ is the total mass of the measure representing A, with the normalization convention that

$$\|A\| = \int\int a(x+iy)\,dx\,dy$$

in case (4.b). This is a generalization of the result (1.d) for occupation times. Thus so far as asymptotic distributions are concerned, as in the next theorem, any other additive functional A with $\|A\| = 2\pi$ may be substituted for L. But it will be seen that the local time process on the circle enjoys exact distributional properties which distinguish it from other additive functionals and make it particularly easy to work with in association with the windings. Roughly speaking, $L(t)$ measures the amount of crossing back and forth over the unit circle which has occurred by time t, or the amount of alternation between the two kinds of windings.

In view of these remarks about additive functionals, the following theorem shows that the Kallianpur–Robbins law (1.a) holds jointly with Spitzer's law (1.c), after making the identifications $H = \frac{1}{2}\Lambda$, $W = W_+ + W_-$.

THEOREM 4.1. *As $t \to \infty$*

$$[\Phi_+(t), \Phi_-(t), L(t)]/h(t) \xrightarrow{d} [W_+, W_-, \Lambda],$$

where $h(t) = \frac{1}{2}\log t$, and the limit random variables are defined as follows in terms of a complex Brownian motion $\zeta = \beta + i\theta$:

$$W_\pm(\zeta) = \int_0^{\sigma_1} 1(\beta_u \in \mathbb{R}_\pm)\,d\theta_u,$$

where $\sigma_1 = \inf\{u\colon \beta_u = 1\}$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_- = (-\infty, 0)$, and

$$\Lambda = L(\beta, 0, \sigma_1)$$

is the local time of β at 0 up to time σ_1.

NOTE. Further descriptions of the joint limit law are given in Theorem 4.2 below.

PROOF. It will be shown that Lemma 3.1 holds with $\Phi_\pm$ or L substituted for Φ, and $W_\pm$ or Λ correspondingly substituted for W. By a time change in the stochastic integrals (4.a) defining $\Phi_\pm$ (see Lemma B.1) for $r > 0$, $h = \log r$, we obtain parallels of formula (3.j):

$$\Phi_\pm(T_r) = \int_0^{\sigma_h} 1(\beta_u \in \mathbb{R}_\pm)\,d\theta_u = hW_\pm(\zeta^{(h)}), \tag{4.d}$$

where we are taking $z_0 = 1$ for simplicity.

Similarly, by the time-change recipe for local times (B.1) and (A.8)

$$L(T_r) = L(\beta, 0, \sigma_h) = h\Lambda(\zeta^{(h)}). \tag{4.e}$$

Moreover, (3.f) still holds with either Φ_+ or L substituted for Φ, because (3.g) holds with these substitutions, except that $\Phi^\sim$ should be replaced by zero in the cases of Φ and L. Thus Lemma 3.1 holds with the substitutions indicated above, and the conclusion of the theorem is immediate. □

A useful consequence of the Kallianpur–Robbins component of the above theorem is if j and k are real valued functions in $L^2(\mathbb{C}, dx\,dy)$, then the martingale additive functional

$$M_t = \int_0^t \{ j(Z_s)\, dX_s + k(Z_s)\, dY_s \}$$

is $o(\log t)$ in probability as $t \to \infty$. In fact for any $\gamma > \frac{1}{2}$, as $t \to \infty$

$$\sup_{0 \le s \le t} |M_s|/(\log t)^{\gamma} \xrightarrow{P} 0. \tag{4.f}$$

Indeed, the increasing process of M is the additive functional

$$A_t = \int_0^t (j^2 + k^2)(Z_s)\, ds.$$

The Kallianpur–Robbins result implies

$$A_t/(\log t)^{2\gamma} \xrightarrow{P} 0, \qquad \gamma > \tfrac{1}{2},$$

which yields (4.f) by Lemma A.1.

As a further refinement Messulam and Yor (1982) showed that $M_t/\sqrt{h(t)}$ is convergent in distribution to $\sqrt{\Lambda/2\pi}\,\{\eta(j) + \chi(k)\}$, where η and χ are two Gaussian measures on $\mathbb{R}^2$, with intensity $dx\,dy$, independent of each other and of Λ. As j and k vary, these limit laws are linked, both to each other, and to companion results for additive functionals of bounded variation due to Kasahara and Kotani (1979). We will show elsewhere that if η and χ are taken to be independent of ζ, this collection of limit laws is linked to Theorem 4.1 and the broader class of log scaling laws described in Section 8.

The above remarks imply that just as the definition of $L(t)$ can be considerably perturbed without making any difference to the asymptotic behavior described in Theorem 4.1, so can the definition of the big and small windings $\Phi_{\pm}(t)$. For example, as an immediate application of (4.f) above, the indicator functions

$$1(Z(s) \in D_{\pm}) = f_{\pm}(Z(s))$$

used as the integrand in the stochastic integrals defining $\Phi_{\pm}$ can be replaced by any other bounded functions $g_{\pm}$ which agree with $f_{\pm}$ in neighborhoods of 0 and ∞. For instance, the domains D_+ and D_- defining the indicator can be replaced by any disjoint pair of neighborhoods of ∞ and 0, respectively. Thus windings in an annulus amount to only $o(\log t)$ in probability as $t \to \infty$. It is really only the very big windings and very small windings which count so far as the asymptotic behavior of the total winding is concerned.

There is a more geometric way to compute big and small windings, suggested by the work of Lyons and McKean (1984), which also gives the same results asymptotically. Simply add $\pm\pi$ every time the path crosses between the positive and negative parts of the real axis, classifying a crossing as big or small depending on whether it reaches its destination at a point inside or outside the unit circle, and adding $+\pi$ or $-\pi$ according to the total winding over the crossing. See Pitman and Yor (1986) for a proof of the asymptotic equivalence of the two definitions up to $o(\log t)$ in probability. As will be indicated in Section 5C, this makes a connection between the asymptotic laws described in Section 4 of Lyons and McKean (1984), for windings of spherical Brownian motion, and results considered here in the plane.

Some important features of the distribution of the limit triple in Theorem 4.1 are summarized by the following theorem.

THEOREM 4.2. *Let (W_-, W_+, Λ) be as in Theorem 4.1.*

(i) *For $a \geq 0$, $b, c \in \mathbb{R}$,*

$$E\exp(-a\Lambda + ibW_- + icW_+) = f(2a + |b|, c),$$

where

$$f(u,v) = [\cosh v + (u/v)\sinh v]^{-1}, \qquad v \neq 0$$
$$= [1+u]^{-1}, \qquad v = 0.$$

(ii) *Λ has exponential distribution with mean 2.*

(iii) *$P(W_+ \in dw)/dw = [2\cosh(\pi w/2)]^{-1}$.*

(iv) *W_- and W_+ are conditionally independent given Λ.*

(v) *The conditional distribution of W_- given W_+ and $\Lambda = l$ is Cauchy with scale parameter $l/2$.*

NOTE. As remarked in Section 3, the distribution of $W = W_+ + W_-$ is the standard Cauchy distribution, defined below (1.c). The conditional distribution in (v) is the distribution of $(l/2)W$

PROOF. (i) Since β and θ are independent, the conditional distribution of $W_\pm$ given β is Gaussian with mean zero and variance $s_\pm(\sigma_1)$, where $s_\pm(u)$ is the time spent by β in $\mathbb{R}_\pm$ up to time u. Thus

$$E\exp(-a\Lambda + ibW_- + icW_+) = E\exp[-\alpha(\sigma_1)], \tag{4.g}$$

where $(\alpha(u),\ u \geq 0)$ is the additive functional of β defined by

$$\alpha(u) = aL(\beta,0,u) + \tfrac{1}{2}b^2 s_-(u) + \tfrac{1}{2}c^2 s_+(u).$$

Such Laplace transforms of stopped additive functionals of Brownian motion can be calculated by traditional methods. See for example Problem 5 of Section 2.8 of Itô and McKean (1965) and Knight (1978). But our preference is to use martingale calculus. To complete the calculation of the right side of (4.g) it will suffice to find a function F such that

$$F(\beta(u \wedge \sigma_1))\exp[-\alpha(u \wedge \sigma_1)], \qquad u \geq 0,$$

is a bounded martingale, since the right side of (4.g) will then be $F(0)/F(1)$. Using

Itô's formula, it is enough for F to satisfy

$$F'' = \left[2a\delta_0 + b^2 1(x < 0) + c^2 1(x > 0)\right] F, \tag{1}$$

where δ_0 is the measure with mass 1 at 0,

$$F(0) = 1, \tag{2}$$

$$F(x) \text{ bounded for } x \le 0. \tag{3}$$

This implies

$$\begin{aligned} F(x) &= \cosh(cx) + k \sinh(cx), \qquad & x > 0 \\ &= e^{|b|x}, & x < 0, \end{aligned}$$

where the constant k is determined by

$$F'(0+) - F'(0-) = 2a.$$

This gives $ck - |b| = 2a$. Finally,

$$F(1) = \cosh c + \left[\frac{|b| + 2a}{c}\right] \sinh c,$$

whence the formula (i). Properties (ii) through (v) can of course be obtained by examination of the transform. But greater insight into these properties is provided by decomposing the Brownian motion β into its positive and negative excursions. □

See Kennedy (1976), Williams (1976), Lehoczky (1977), and Azéma and Yor (1979) for applications of similar martingales to compute Laplace transforms of functionals of Brownian motion. As a general rule, Itô's formula is the key to finding the right martingale.

5. Excursions. The main object of this section is to describe the decomposition of a complex Brownian motion $\zeta = \beta + i\theta$ into its excursions in the two open half planes

$$\mathbb{C}_+ = \{z: \operatorname{Re}(z) > 0\}, \qquad \mathbb{C}_- = \{z: \operatorname{Re}(z) < 0\}.$$

When ζ is the time change of $\log Z$, as in the previous section, these excursions correspond to excursions of Z in the domains D_+ and D_-. This decomposition of ζ explains why in Theorem 4.1 the asymptotic big winding W_+ and asymptotic small winding W_- are conditionally independent given the asymptotic local time Λ. Further, the decomposition into excursions is an essential part of our description of the joint limit distribution of windings and other functionals defined relative to several origins.

The timing of the excursions of $\beta + i\theta$ into $\mathbb{C}_+$ and $\mathbb{C}_-$ is entirely determined by the timing of the excursions of β into $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$. So the analysis of excursions of the complex Brownian motion $\beta + i\theta$ into $\mathbb{C}_+$ and $\mathbb{C}_-$ reduces quickly to the theory of excursions of one-dimensional Brownian motion. Instead of considering a point process of excursion, as in the general Markovian excursion theory of Itô (1970) and Maisonneuve (1975), we exploit the spatial homogeneity of Brownian motion to construct much simpler processes incorporating the excursions into $\mathbb{C}_+$ and $\mathbb{C}_-$. Roughly speaking, we take the excursions of $\beta + i\theta$ from the imaginary axis into $\mathbb{C}_+$, and knit them together to form a process $\rho_+ + i\theta_+$, a Brownian motion in $\mathbb{C}_+$ with reflection at the boundary axis. The same operation is performed on the excursions into $\mathbb{C}_-$ to obtain $\rho_- + i\theta_-$, a reflecting Brownian motion in $\mathbb{C}_-$. These operations are made precise in Section 5B, after first considering the one-dimensional aspect of splitting β into the processes $\rho_\pm$ obtained from its positive and negative excursions.

A. One-dimensional excursions. Given a one-dimensional Brownian motion β starting at zero, define

$$s_\pm(u) = \int_0^u 1(\beta_t \in \mathbb{R}_\pm)\, dt, \qquad u \geq 0,$$

so the process $s_\pm$ is a clock measuring time spent by β in $\mathbb{R}_\pm$. Let $u_\pm$ be the right-continuous inverse of $s_\pm$:

$$u_\pm(s) = \inf\{u: s_\pm(u) > s\}, \qquad s \geq 0.$$

Time changing via $u_\pm$ has the effect of closing up the gaps of time spent by β in the opposite interval $-\mathbb{R}_\pm$. Thus the processes $\rho_\pm$ defined by

$$\rho_\pm(s) = \beta(u_\pm(s)), \qquad s \geq 0,$$

are derived from β by throwing out the excursions of β in $-\mathbb{R}_\pm$, and closing up the gaps. Define

$$\beta_\pm(s) = \int_0^{u_\pm(s)} 1(\beta_t \in \mathbb{R}_\pm)\, d\beta_t.$$

The processes $\beta_\pm$ play important rôles as the martingale parts of $\rho_\pm$.

THEOREM 5.1. *Let $\rho_\pm$ and $\beta_\pm$ be defined as above in terms of a Brownian motion β.*

(i) *The processes ρ_+ and $-\rho_-$ are independent reflecting Brownian motions on $[0, \infty)$, related to β_+ and β_- via the formulae*

$$\rho_\pm = \beta_\pm \pm \tfrac{1}{2} l_\pm, \tag{5.a}$$

where $l_\pm$ is the local time process of $\rho_\pm$ at zero, and

$$\tfrac{1}{2} l_\pm(s) = -\min_{0 \leq v \leq s} [\pm\beta_\pm(v)]. \tag{5.b}$$

(ii) *The Brownian motion β may be recovered from ρ_+ and ρ_- as*

$$\beta = f(\rho_+, \rho_-)$$

for a measurable function f: $[\Omega(\mathbb{R})]^2 \to \Omega(\mathbb{R})$, and the same is true with $\beta_\pm$ instead of $\rho_\pm$, for a different f.

PROOF. (i) According to Section 2.11 of Itô and McKean (1965), ρ_+ and $-\rho$ are both reflecting Brownian motions on $[0, \infty)$. We now sketch a proof of this fact, following the argument of Knight (1971), which shows further that ρ_+ and ρ_- are independent. See also McKean (1975). The argument involves the processes $\beta_\pm$, which are time changes of the martingales

$$M_\pm(u) = \int_0^u 1(\beta_t \in \mathbb{R}_\pm)\, d\beta_t \tag{5.c}$$

via the inverses $u_\pm$ of their increasing processes $s_\pm$. Since the martingales M_+ and M_- are orthogonal, β_+ and β_- are independent Brownian motions by Knight's theorem (B.3). The Brownian motions $\beta_\pm$ are the martingale parts of $\rho_\pm$ appearing in the Tanaka formulae (5.a). Here $l_\pm$ is the local time process at zero of $\rho_\pm$, as discussed in Appendix A. Using the notation of that section, and abbreviating the local time $L(X, 0, t)$ to $L(X, t)$

$$l_\pm(s) = L(\rho_\pm, s) = L(\beta, u_\pm(s)), \tag{5.d}$$

by a variation of the time change formulae (B.1) and (B.2). More explicitly,

$$l_\pm(s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} T_\pm(\varepsilon, s) \quad \text{a.s.},$$

where $T_\pm(\varepsilon, s)$ is the time spent by $\rho_\pm$ in $[0, \varepsilon]$ up to time s, which is the time spent by $\pm\beta$ in $[0, \varepsilon]$ up to time $u_\pm(s)$. But by a result of Lévy (1948) [see also Skorokhod (1961) and El Karoui and Chaleyat-Maurel (1978)] $l_\pm$ may be expressed in terms of $\beta_\pm$ as in (5.b). Substituting this expression for $\frac{1}{2}l_\pm$ in (5.a) shows that $\rho_\pm$ and $-\rho_-$ are reflecting Brownian motions on $[0, \infty)$, by Lévy's representation of reflecting Brownian motion [see Lévy (1948) and Chung and Williams (1983), Section 8.2].

(ii) It is part of the folklore of Brownian excursion theory that the Brownian motion β can be recovered from its positive and negative excursions, incorporated here in the processes ρ_+ and ρ_-. To recover β from ρ_+ and ρ_-, notice first that by the definition of $\rho_\pm$,

$$\beta(t) = \beta(t)^+ - \beta(t)^- = \rho_+(s_+(t)) + \rho_-(s_-(t)),$$

so it suffices to show that the clocks s_+ and s_- are measurable functions of ρ_+ and ρ_-. But from (5.d), the local times of these processes at zero are linked via the clocks s_+ and s_-, according to the identity

$$L(\rho_+, s_+(t)) = L(\rho_-, s_-(t)) = L(\beta, t), \qquad t \ge 0. \tag{5.e}$$

Since $s_-(t) = t - s_+(t)$, this suggests

$$s_+(t) = \inf\{s \colon L(\rho_+, s) > L(\rho_-, t - s)\}, \tag{5.f1}$$

a formula which is readily confirmed. Thus the clocks s_+ and s_-, and hence β, are measurable functions of ρ_+ and ρ_-. □

The above theorem summarizes all the one-dimensional excursion theory which is needed for treatment of planar Brownian excursions in the next section. The remainder of this section is a digression from the main theme, to point out how the ideas behind Theorem 5.1 can be used to obtain some important results for one-dimensional Brownian motion, including Lévy's arcsin law for the distribution of $s_+(1)$.

There is a useful companion to the formula (5.f1), expressing the inverse u_+ of the clock s_+ in terms of ρ_+ and ρ_-:

$$(5.f2) \qquad u_+(s) - s = \inf\{v\colon L(\rho_-, v) > L(\rho_+, s)\}, \qquad s \ge 0.$$

This can easily be derived from (5.f1), or verified directly.

Notice that $u_+(s) - s = s_-(u_+(s))$ is the time spent negative by β up to the instant when β has spent time s positive. Together with the identity (5.b) this allows (5.f2) to be rewritten as

$$(5.f3) \qquad s_-(u_+(s)) = \inf\{v\colon \beta_-(v) > \tfrac{1}{2}L(\rho_+, s)\}, \qquad s \ge 0,$$

where β is a Brownian motion independent of ρ_+.

For $h > 0$ let the distribution of $\sigma_h = \inf\{u\colon \beta(u) = h\}$, which is stable with index $\frac{1}{2}$ and scale parameter h, be denoted Stable($\frac{1}{2}$, h). Then (5.f3) immediately implies the following results, due to Williams (1969):

(5.g1) The conditional distribution of $s_-(u_+(s))$ given ρ_+ is Stable($\frac{1}{2}$, H), where

$$H = \tfrac{1}{2}L(\rho_+, s) = \tfrac{1}{2}L(\beta, u_+(s)).$$

(5.g2) If U is a random time such that $s_+(U)$ is a measurable function of ρ_+, then $s_+(U)$ and $s_-(U)$ are conditionally independent given $L(\beta, U)$, and the conditional distribution of $s_-(U)$ given $s_+(U)$ and $L(\beta, U) = l$ is Stable($\frac{1}{2}$, $\frac{1}{2}l$).

To illustrate (5.g2) take $U = \sigma_1 = \inf\{u\colon \beta(u) = 1\}$. Then (5.g2) applies because

$$s_+(\sigma_1) = \inf\{u\colon \rho_+(u) = 1\}.$$

As remarked by Williams (1969), Lévy's arcsin law

$$(5.h+) \qquad P(s_+(1) \in ds) = \frac{ds}{\pi\sqrt{s(1-s)}}, \qquad 0 < s < 1,$$

is a consequence of (5.g1). Indeed, by the identity of events

$$(s_+(1) < s) = (u_+(s) > 1) = (s_-(u_+(s)) > 1 - s),$$

(5.g1) yields

$$(5.i+) \quad P\big(s_+(1) < s \mid \tfrac{1}{2}L(\rho_+, s) = h\big) = P(\sigma_h > 1 - s), \qquad h \ge 0, \quad 0 < s < 1.$$

This formula should be compared with a companion formula for

$$s_0(1) = \sup\{s \le 1\colon \beta(s) = 0\},$$

which is however much more obvious, by an application of the Markov property of β at time s:

$$(5.i0) \quad P\big(s_0(1) < s \mid |\beta(s)| = h\big) = P(\sigma_h > 1 - s), \qquad h \ge 0, \quad 0 < s < 1.$$

Since by (5.b), (5.d) and the reflection principle

$$\tfrac{1}{2}L(\rho_+, s) = -\inf_{0 \le v \le s} \beta_+(v) \stackrel{d}{=} |\beta(s)| \quad \text{for fixed } s,$$

comparison of (5.i +) and (5.i0) reveals Lévy's result that

$$s_+(1) \stackrel{d}{=} s_0(1). \tag{5.j}$$

It is well known that the arcsin law of $s_0(1)$ can be obtained from (5.i0) by integration, after some slightly tedious calculus. But we cannot resist including the following argument, which highlights some related distributional curiosities. By Brownian scaling,

$$\sigma_l \stackrel{d}{=} l^2 \sigma_1.$$

Substituting $\hat{\sigma}_1$ for σ_1, where $\hat{\sigma}_1$ is $\sigma_1(\hat{\beta})$ for another BM $\hat{\beta}$ independent β, (5.i0) yields

$$P(s_0(1) < s) = P(\beta^2(s)\hat{\sigma}_1 > 1 - s) = P\big((1 + V)^{-1} < s\big),$$

where

$$V = s^{-1}\beta^2(s)\hat{\sigma}_1.$$

But by scaling again $s^{-1}\beta^2(s) \stackrel{d}{=} \beta^2(1)$, and by a change of variable

$$\hat{\sigma}_1 \stackrel{d}{=} 1/\hat{\beta}^2(1).$$

Thus

$$V \stackrel{d}{=} \beta^2(1)/\hat{\beta}^2(1) \stackrel{d}{=} W^2,$$

where W has a standard Cauchy distribution, since the ratio of two independent standard normal variables is standard Cauchy. A final change of variable shows that $(1 + W^2)^{-1}$ has the arcsin law if W is standard Cauchy, and the proof is complete.

See Karatzas and Shreve (1984) for further results in this vein. More on excursion theory can be found in Walsh (1978), Greenwood and Pitman (1980), Ikeda and Watanabe (1981), and Pitman (1981).

B. Excursions in half planes. Suppose now that $\zeta = \beta + i\theta$ is a BM($\mathbb{C}$). Let $\rho_\pm$ be the reflecting Brownian motions derived from β as in the last section, and $\beta_\pm$ their martingale parts. The clocks $s_\pm$ now measure the time spent by ζ in the half planes $\mathbb{C}_\pm$ to the right and left of the imaginary axis. Let

$$\theta_\pm(s) = \int_0^{u_\pm(s)} 1(\beta_t \in \mathbb{R}_\pm)\, d\theta_t,$$

where $u_\pm$ is the inverse of $s_\pm$. Informally, the process $\theta_\pm$ is derived from θ by deleting the increments of θ over intervals when β is in $-\mathbb{R}_\pm$, and closing up the gaps. A little care is required to make this precise, since the sum of increments of

θ over the gaps is almost surely not absolutely convergent. But since β and θ are independent, any method of summation for the deletions, determined by β alone, will give the same answer almost surely as the stochastic integral.

Thus $\rho_{\pm} + i\theta_{\pm}$ is derived from $\beta + i\theta$ by taking the excursions of $\beta + i\theta$ in $\mathbb{C}_{\pm}$ and knitting them together to form a process with continuous paths in $\mathbb{C}_{\pm}$ and reflection at the imaginary axis.

THEOREM 5.2. (i) *The processes* ρ_+, ρ_-, θ_+, *and* θ_- *are mutually independent, as are the processes* β_+, β_-, θ_+, *and* θ_-.

(ii) *The processes* $\pm\rho_{\pm}$ *are reflecting Brownian motions on* $[0, \infty)$, *while the processes* $\beta_{\pm}$ *and* $\theta_{\pm}$ *are Brownian motions on* $(-\infty, \infty)$.

(iii) *The original Brownian motion* $\zeta = \beta + i\theta$ *may be recovered as a measurable function of either* $(\beta_+, \beta_-, \theta_+, \theta_-)$ *or of* $(\rho_+, \rho_-, \theta_+, \theta_-)$.

PROOF. The processes $\beta_{\pm}$ and $\theta_{\pm}$ are the time changes of the orthogonal martingales $M_{\pm}$ and $N_{\pm}$ via their clocks $s_{\pm}$, where $M_{\pm}$ was defined in (5.c), and $N_{\pm}$ is defined similarly with $d\theta_t$ instead of $d\beta_t$. So by another application of Knight's theorem (B.3), the processes β_+, β_-, θ_+, and θ_- are four mutually independent Brownian motions. Parts (i) and (ii) now follow immediately from Theorem 5.1. Since by Theorem 5.1 the Brownian motion β and hence the clocks s_+ and s_- are measurable functions of β_+ and β_-, or of ρ_+ and ρ_-, the process θ can be recovered via the formula

$$\theta(u) = \theta_+(s_+(u)) + \theta_-(s_-(u)),$$

which proves (iii). □

C. Application to windings. Suppose now that $\zeta = \beta + i\theta$ is obtained by time changing $\log Z$, via its clock U, where Z is a complex Brownian motion starting at 1, as before. By time changing the stochastic integrals, and local times [see Lemma B.1, and formulae (B.1) and (B.2)]

$$\Phi_{\pm}(t) = \int_0^{U(t)} 1(\beta_u \in \mathbb{R}_{\pm})\, d\theta_u = \theta_{\pm}(s_{\pm}(U_t)), \tag{5.j1}$$

$$L(t) = L(\beta, 0, U_t) = L(\rho_{\pm}, 0, s_{\pm}(U_t)), \tag{5.j2}$$

where $L(t)$ is the local time of Z on the circle of radius 1 up to time t. These formulae are especially informative at random times t such that $s_{\pm}(U_t)$ admits a simple expression in terms of ρ_+ and ρ_-, by variations of Williams' one-dimensional results (5.g). For example, if $(\tau_l,\ l \geq 0)$ is the inverse of the local time process L, then $(U(\tau_l),\ l \geq 0)$ is the inverse of the local time process of β at zero, whence from (5.e), (5.d), and (5.a).

$$s_{\pm}(U(\tau_l)) = \inf\{s\colon L(\rho_{\pm}, 0, s) > l\} = \inf\{s\colon \beta_{\pm}(s) < -\tfrac{1}{2}l\}.$$

Since β_+, β_-, θ_+, and θ_- are mutually independent Brownian motions, it follows that

(5.k1) the processes $(2\Phi_{\pm}(\tau_l),\ l \geq 0)$ are independent Cauchy processes.

The average of these two independent Cauchy processes is the further Cauchy process

$$(5.\text{k2}) \qquad \Phi(\tau_l) = \theta(v_l), \qquad l \geq 0,$$

where $v_l = U(\tau_l)$, so $(v_l,\ l \geq 0)$ is the inverse of $(L(\beta, 0, u),\ u \geq 0)$.

This Cauchy process embedded in the windings should be compared with the similarly embedded Cauchy process

$$(5.\text{l}) \qquad \Phi(T(e^h)) = \theta(\sigma_h), \qquad h \geq 0,$$

which was encountered in the proof of Spitzer's theorem. In contrast to sampling at the times τ_l, when Z has returned to the circle on which it started, the big and small windings are neither independent, nor Cauchy distributed, nor symmetric, when sampled at times $T(e^h)$ when Z hits larger circles. On the contrary, the joint distribution of $h^{-1}\Phi_{\pm}(T(e^h))$ is identical to the joint distribution of $W_{\pm}$ in Theorems 4.1 and 4.2. Indeed, taking $h = 1$ and substituting $t = T_e = T(e)$ in (5.j) yields

$$(5.\text{m1}) \qquad \Phi_{\pm}(T_e) = W_{\pm} = \theta_{\pm}(s_{\pm}(\sigma_1)),$$

$$(5.\text{m2}) \qquad L(T_e) = \Lambda = L(\rho_{\pm}, 0, s_{\pm}(\sigma_1)).$$

This excursion representation of the asymptotic winding variables and the asymptotic local time given by the right-hand formulae in (5.m) explains the conditional independence of W_+ and W_- given Λ, by a straightfoward extension of Williams' result (5.g). Because $\Phi_-(t)$ is flat on the interval from time $t = T_e$ to time $t = \tau_\Lambda$ when Z next hits the unit circle, there is the alternative expression

$$(5.\text{m3}) \qquad W_- = \Phi_-(T_e) = \Phi_-(\tau_\Lambda) = \tfrac{1}{2}Y_-(\Lambda),$$

where $(Y\ \ (l),\ l \geq 0)$ is the Cauchy process obtained as twice the small winding at time τ_l, as in (5.k1), which is a function of β_- and θ_-. But from (5.e)

$$\Lambda = L(\beta, 0, \sigma_1) = L(\rho_+, 0, s_+(\sigma_1)),$$

and

$$(5.\text{m4}) \qquad s_+(\sigma_1) = \inf\{s\colon \rho_+(s) > 1\}.$$

Thus both Λ and $s_+(\sigma_1)$ are functions of ρ_+, hence of β_+, while the Cauchy process $Y\ \ $ is a function of the Brownian motions β_- and θ_-. The mutual independence of the four Brownian motions $\beta_{\pm}$ and $\theta_{\pm}$ now implies that $W_+ = \theta_+(s_+(\sigma_1))$ and $W_- = \frac{1}{2}Y_-(\Lambda)$ are conditionally independent given Λ, and that the conditional distribution of W_- given W_+ and $\Lambda = l$ is Cauchy$(\frac{1}{2}l)$, as asserted in parts (iv) and (v) of Theorem 4.2.

Parts (ii) and (iii) of Theorem 4.2 can be derived similarly without considerations of Laplace transforms. The exponential distribution of $\Lambda = L(\beta, 0, \sigma_1)$ is a well-known consequence of the strong Markov property of β at the stopping times defined by the inverse local time process of β at zero. And formulae (5.m1) and (5.m4), plus the fact that ρ_+ is a reflecting Brownian motion identical in distribution to $|\beta|$, imply that

$$[W_+, \Lambda] \stackrel{d}{=} [\theta(\hat{\sigma}_1), 2L(\beta, 0, \hat{\sigma}_1)] = [\Phi(\hat{T}_e), L(\hat{T}_e)], \quad \text{where}$$

(5.n) $$\hat{\sigma}_1 = \inf\{u\colon \beta_u = +1 \text{ or } -1\},$$

$$\hat{T}_e = \inf\{t\colon R_t = e \text{ or } e^{-1}\}, \quad \text{so } U(\hat{T}_e) = \hat{\sigma}_1,$$

and $L(\hat{T}_e)$ is the local time Z on the unit circle up to time $\hat{T}_e$. From the middle expression, the distribution of the first component of each of these three pairs is the imaginary part of the hitting distribution of two lines parallel to the imaginary axis thorough ± 1, for the complex Brownian motion $\beta + i\theta$ starting at zero. The density of this distribution is well known to be the density given for W_+ in part (iii) of Theorem 4.2. See for example Section 5.1 of Durrett (1984).

It is obvious that this distribution of W_+ is different to that of W_-, which is the distribution of HY, where H and Y are independent random variables with standard exponential and standard Cauchy distribution. The former distribution has finite moments of all orders, and the latter finite moments of order α for $\alpha < 1$ only. The symmetry between big and small windings at the sampling times τ_l breaks down at time $T(e^h)$, because the analog of (5.m3) for big windings is false. While the small winding is flat between times T_e and τ_Λ, the big winding process is moving, so the first identity in (5.m3) fails when $+$ is substituted for $-$.

Thus there is no symmetry between big windings and small windings in the asymptotic distribution as $t \to \infty$, despite the symmetry in distribution displayed in (5.k1) at the random times τ_l tending to infinity. This result for the inverse of L implies by straightforward approximations using the ergodic theorem (4.c), that for any nonnegative additive functional $H(t)$ with $\|H\| = 2\pi$, and τ_h now the inverse of H,

(5.o) $$h^{-1}[\Phi_+(\tau_h), \Phi_-(\tau_h)] \to [\tfrac{1}{2}Y_+, \tfrac{1}{2}Y_-],$$

where Y_+ and Y_- are independent standard Cauchy random variables. This result is due in a slightly different form to Lyons and McKean (1984), who used the geometric definition of big and small windings mentioned at the end of Section 4.

The difference between the asymptotic distribution of big and small windings at fixed times tending to infinity and at the inverse local times τ_l tending to infinity may be understood as follows. So far as the windings are concerned, the fixed time t cannot be approximated by the random times τ_l at which there is a symmetry in distribution between big and small windings. But t is very well approximated by the radial hitting time $T_{\sqrt{t}}$, which necessarily catches Z in the middle of some big windings, thereby breaking the symmetry.

In retrospect, it is a remarkable coincidence that the limit laws for the total windings $\Phi(t)/h(t)$ and $\Phi(\tau_h)/h$ are both Cauchy, since one would naively expect this coincidence to extend to the big and small windings, which it does not. What the above arguments show is that

(5.p) $$h(t)^{-1}[\Phi(t), \Phi(\tau_{h(t)})] \stackrel{d}{\to} [\theta(\sigma_1), \theta(v_1)],$$

where β and θ are independent BM's, and

$$\sigma_1 = \inf\{u\colon \beta(u) = 1\}, \qquad v_1 = \inf\{u\colon L(\beta, 0, u) = 1\}.$$

That is to say, there is a very clear sense in which the asymptotic Cauchy variables governing the total winding at fixed times and the total winding at the inverse of an additive functional are *different*.

The Fourier transform of this limit law with Cauchy marginals can be calculated via the Laplace transform of (σ_1, v_1), which has Stable$(\frac{1}{2}, 1)$ marginals, but the resulting formula seems rather complicated. Numerous companion results for sampling of the windings by other families of random times are discussed in Section 8.

6. Windings about several points. The main object of this section is to prove the existence of an asymptotic joint distribution as $t \to \infty$ of the windings $\Phi^1(t), \dots, \Phi^n(t)$ of a Brownian motion Z about n distinct points $z_1, \dots, z_n$, distinct also from the starting point z_0. This result is most easily understood if, in addition to the windings $\Phi^j(t)$, we consider as in Theorem 4.1 an increasing additive functional $L(t)$ with $\|L\| = 2\pi$, as well as big windings $\Phi^j_+(t)$ and small windings $\Phi^j_-(t)$ about each point z_j:

$$\Phi^j_\pm(t) = \int_0^t 1\left(|Z(s) - z_j| \in I^j_\pm\right) d\Phi^j(s), \tag{6.a}$$

where

$$I^j_- = (0, r_j), \qquad I^j_+ = (r_j, \infty),$$

for some $r_j > 0$. As remarked below Theorem 4.1, windings in an annulus do not count so far as asymptotics with normalization by $h(t) = \frac{1}{2}\log t$ are concerned. So the values of r_j are irrelevant. Less obviously, the positions of the points $z_1, \dots, z_n$ are also irrelevant.

THEOREM 6.1. *As $t \to \infty$,*

$$h(t)^{-1}\left[\Phi^j_+(t), \Phi^j_-(t); 1 \le j \le n; L(t)\right]$$

converges in distribution to

$$\left[W_+, W^j_-; 1 \le j \le n; \Lambda\right],$$

where for each j the triple (W_+, W^j_-, Λ) has the joint distribution of (W_+, W_-, Λ) described in Theorems 4.1 *and* 4.2, *and the $n + 1$ random variables W_+ and $(W^j, 1 \le j \le n)$ are mutually conditionally independent given Λ.*

This theorem is a corollary of Theorem 6.2 below, which is the central result of the paper. Adding the big and small components in Theorem 6.1 shows that as $t \to \infty$

$$h(t)^{-1}\left[\Phi^j(t); 1 \le j \le n\right] \xrightarrow{d} \left[W_+ + W^j_-; 1 \le j \le n\right]. \tag{6.b}$$

So the asymptotic distribution of the windings about several points may be understood in terms of a common asymptotic big winding variable W_+, attributable to the Brownian movement in a neighborhood of ∞, and n small windings W^j, attributable to the Brownian movement in small neighborhoods of the points. These $n + 1$ contributions are not independent. Rather, they are conditionally independent given the asymptotic local time variable Λ. By virtue of the ergodic theorem, Λ governs the amount of crossing back and forth between the $n + 1$ neighborhoods. The larger the value of Λ, the larger in absolute value all the winding components tend to be. So there is a positive dependence between the absolute values of the various winding components which is explained by Λ.

To formulate the central theorem, let

$$\zeta^j = \beta^j + i\theta^j$$

be the BM($\mathbb{C}$) obtained as in (2.i) by time changing the logarithm of the process $(Z - z_j)/(z_0 - z_j)$. That is to say, β^j and θ^j are the Brownian motions obtained by time changing the two orthogonal local martingales

$$\log\left(|Z(t) - z_j|/|z_0 - z_j|\right), \qquad t \geq 0, \tag{6.c1}$$

and

$$\Phi^j(t), \qquad t \geq 0, \tag{6.c2}$$

via their common increasing process

$$U^j(t) = \int_0^t ds/\left|Z(s) - z_j\right|^2. \tag{6.c3}$$

Next, for $h > 0$ let $\zeta^{j,h}$ be the BM($\mathbb{C}$) derived from ζ^j by the Brownian scaling operation

$$\zeta^{j,h}(u) = h^{-1}\zeta^j(h^2 u), \qquad u \geq 0.$$

THEOREM 6.2. *As $h \to \infty$*

$$\left(\zeta^{j,h},\ j = 1, \dots, n\right) \xrightarrow{d} \left(\zeta^{j,\infty},\ j = 1, \dots, n\right),$$

where the limit is a family of n complex Brownian motions whose excursion processes $\rho_\pm^{j,\infty} + i\theta_\pm^{j,\infty}$ in the half planes $\mathbb{C}_\pm$ have the following two properties:

(i) *As j varies, the excursion processes $\rho_+^{j,\infty} + i\theta_+^{j,\infty}$ are identical to a common process $\rho_+^\infty + i\theta_+^\infty$.*

(ii) *The common excursion process $\rho_+^\infty + i\theta_+^\infty$ and the n excursion processes $\rho_-^{j,\infty} + i\theta_-^{j,\infty}$, $1 \leq j \leq n$, are mutually independent.*

In the statement of the theorem, the Brownian motions $\zeta^{j,h}$ and $\zeta^{j,\infty}$ are regarded as random elements in the path space $\Omega(\mathbb{C})$. By Lemma B.3 the theorem implies that for every measurable transformation ψ from $\Omega(\mathbb{C})$ to a separable metric space,

$$\left(\psi(\zeta^{j,h}),\ j = 1, \dots, n\right) \xrightarrow{d} \left(\psi(\zeta^{j,\infty}),\ j = 1, \dots, n\right). \tag{6.d}$$

Taking $\psi(\zeta) = W(\zeta) = \theta(\sigma_1)$, Lemma (3.1) implies that as t and h tend to ∞

with $h = \frac{1}{2}\log t$

(6.e) $$h^{-1}\Phi^j(t) - W(\zeta^{j,h}) \xrightarrow{P} 0.$$

This gives (6.b) with $W(\zeta^{j,\infty})$ instead of $W_+ + W_-^j$. Letting ψ be first W_+, then W and then Λ instead of W gives the more detailed account of Theorem 6.1, by the parallels of Lemma 3.1 described in the proof of Theorem 4.1.

By the same token, to prove Theorem 6.2 it suffices to establish (6.d) for a transformation ψ which can be almost surely inverted. This is the case for the transformation ψ which splits a complex Brownian motion ζ into the four Brownian motions β_+, θ_+, β_-, and θ_-, from which ζ can be recovered according to Theorem 5.1. Thus Theorem 6.2 is an immediate consequence of the following lemma, in which each process denoted by a β with indices is the Brownian motion obtained as the martingale part of the reflecting Brownian motion ρ with the same indices, the processes $\rho_+^\infty + i\theta_+^\infty$ and $\rho_-^{j,\infty} + i\theta_-^{j,\infty}$ are as in the statement of Theorem 6.2, and $\rho_\pm^{j,h} + i\theta_\pm^{j,h}$ are the excursion processes of $\zeta^{j,h}$ in $\mathbb{C}_\pm$.

LEMMA 6.3. *As $h \to \infty$*

$$\left[\left(\beta_+^{j,h}, \theta_+^{j,h}, \beta_-^{j,h}, \theta_-^{j,h}\right),\ j = 1,\dots,n\right]$$

converges in distribution to

$$\left[\left(\beta_+^\infty, \theta_+^\infty, \beta_-^{j,\infty}, \theta_-^{j,\infty}\right),\ j = 1,\dots,n\right],$$

where $\beta_+^\infty, \beta_-^\infty$ and $\beta_-^{j,\infty}, \theta_-^{j,\infty}$, $j = 1,\dots,n$, are $2 + 2n$ mutually independent Brownian motions.

PROOF. Using the scaling notation (3.h), it should be noted from the definitions that for $\alpha = \beta$ or θ

(6.f) $$\alpha_\pm^{j,h} = \left[\alpha_\pm^j\right]^{(h)}.$$

Moreover, by a time change in the stochastic integrals, $\beta_\pm^j$ and $\theta_\pm^j$ are the Brownian motions associated with the local martingales $G_\pm^j$ and $\Phi_\pm^j$ which are the components of the conformal martingale

(6.g) $$G_\pm^j(t) + i\Phi_\pm^j(t) = \int_0^t \frac{1\left(|Z_s - z_j| \in I_\pm^j\right)}{\left(Z_s - z_j\right)}\, dZ_s,$$

where

$$I_-^j = \left(0, |z_0 - z_j|\right), \qquad I_+^j = \left(|z_0 - z_j|, \infty\right).$$

The processes $\Phi_\pm^j$ are processes of big and small windings about z_j, and $G_\pm^j$ is the martingale part of $[\log(|Z - z_j|/|z_0 - z_j|)]^\pm$. Consider first the case $n = 2$. It must be seen that as $h \to \infty$

(6.h1) $$\beta_+^{1,h} - \beta_+^{2,h} \xrightarrow{P} 0,$$

(6.h2) $$\theta_+^{1,h} - \theta_+^{2,h} \xrightarrow{P} 0,$$

and

$$(\beta_+^{1,h}, \theta_-^{1,h}, \beta_-^{1,h}, \theta_-^{1,h}, \beta_-^{2,h}, \theta_-^{2,h}) \tag{6.h3}$$

converges in distribution to a vector of six independent Brownian motions

$$(\beta_+^{\infty}, \theta_+^{\infty}, \beta_-^{1,\infty}, \theta_-^{1,\infty}, \beta_-^{2,\infty}, \theta_-^{2,\infty}).$$

These conclusions follow from general criteria for asymptotic identity and asymptotic independence of Brownian motions derived as time changes of continuous local martingales.

If M_1 and M_2 are two continuous local martingales relative to the same filtration, with $\langle M_1\rangle_\infty = \langle M_2\rangle_\infty = \infty$, and

$$\frac{\langle M_1 - M_2\rangle_t}{\langle M_i\rangle_t} \to 0 \quad \text{a.s.} \quad \textit{as } t \to \infty \textit{ for } i = 1,2, \tag{6.i}$$

then the Brownian motions β_i obtained by time changing M_i are such that

$$\beta_1^{(h)} - \beta_2^{(h)} \xrightarrow{P} 0 \textit{ as } h \to \infty.$$

This is a special case of Theorem B.4. For $M_1 = G_+^1$, $M_2 = G_+^2$, it is easily checked that

$$\langle M_1 - M_2\rangle_t = \int_0^t f(Z_s)\,ds,$$

where

$$f(z) = \frac{|z_1 - z_2|^2}{|z - z_1|^2|z - z_2|^2} 1(|z - z_1| > |z_1 - z_0|) 1(|z - z_2| > |z_2 - z_0|), \tag{6.j}$$

a function which is integrable over the plane, whereas $\langle M_i\rangle_t$ has the same form with functions f_i which are not integrable. Thus (6.i) in this case follows from the ergodic theorem (4.c), and (6.h1) follows. Formula (6.j) and the above argument apply just as well to the imaginary parts Φ_+^j of the conformal martingale (6.g), yielding (6.h2). Turning to the proof of (6.h3), we make use of the following general criterion, which is a case of Theorem B.2:

If M_i, $1 \le i \le k$, are k continuous local martingales relative to the same filtration with $\langle M_i\rangle_\infty = \infty$, $i = 1,\dots,k$, and for every $i < j$

$$\frac{1}{\langle M_i\rangle_t} \int_0^t |d\langle M_i, M_j\rangle_s| \to 0 \quad \text{a.s.} \quad \textit{as } t \to \infty, \tag{6.k1}$$

then the Brownian motions β_i obtained by time changing the M_i are such that

$$(\beta_i^{(h)}, i = 1,\dots,k) \xrightarrow{d} (\beta_i^{\infty}, i = 1,\dots,k) \quad \textit{as } h \to \infty, \tag{6.k2}$$

where the limit is a k-tuple of independent Brownian motions.

Applying this criterion to the six local martingales of the form G^j_+ and Φ^j_+ whose rescaled Brownian motions appear in (6.h3), it must be checked that (6.k1) holds for every pair. For a pair chosen from the first four components, or for the last pair, this is immediate since the processes are simply orthogonal. For the eight remaining pairs it is found that

$$\langle M_i, M_j\rangle_t = \int_0^t g_{ij}(Z_s)\,ds$$

for a function g_{ij} which is integrable in every case. For example, if $M_i = G^1$, $M_j = G^2$, the function $g_{ij}(z)$ is similar to the function $f(z)$ in (6.j), but with $|z_1 - z_2|^2$ replaced by the inner product of $z - z_1$ and $z - z_2$, and the inequalities reversed inside the indicators. Again, the denominator $\langle M_i\rangle_t$ in (6.k1) is of the same form for a function f_i which is not integrable, so (6.k1) holds by the ergodic theorem.

This proves the lemma in the case $n = 2$. The case $n > 2$ is quite similar. □

7. The joint limit law of the windings. We record in this section some properties of the joint limiting distribution of $(2\Phi_j(t)/\log t,\ 1 \le j \le n)$. According to Theorems 4.2 and 6.1, this is the joint law of $(W_1, \dots, W_n)$ defined by

(7.a1) $$W_j = W_-^j + W_+,$$

where

(7.a2) $W_-^j = HY_j,\ j = 1, \dots, n,$

(7.a3) $Y_1, \dots, Y_n$ are independent standard Cauchy random variables, independent also of the pair (H, W_+), which has Laplace–Fourier transform

(7.a4) $Ee^{-aH + ivW_+} = f(a, v) = [\cosh v + (a/v)\sinh v]^{-1},\ a \ge 0,\ v \in \mathbb{R}.$

By the scaling property of the Cauchy process, HY_j in (7.a2) can be replaced by $Y_j(H)$, where $Y_1(\cdot), \dots, Y_n(\cdot)$ are independent Cauchy *processes*. This is the representation provided by the proof of parts of Theorem 4.2 in Section 5, with $H = \frac{1}{2}\Lambda$, so H is exponentially distributed with mean 1. But the representation (7.a) has the merit that it presents the n random variables $W_1, \dots, W_n$ as a function of just $n + 2$ random variables $Y_1, \dots, Y_n$, H, and W_+.

Obvious properties of the joint law are that it is exchangeable with standard Cauchy marginals. And it is easy to see that it is infinitely divisible, but not stable. Denote by Cauchy(w, h) the distribution of $w + hY$, when Y has standard Cauchy distribution. From (7.a), conditional on $W_+ = w$ and $H = h$, $W_1, \dots, W_n$ are independent with identical Cauchy(w, h) distribution. This gives a formula for the joint density of $(W_1, \dots, W_n)$ as an integral with respect to the distribution of W_+ and H of products of Cauchy densities. But although the marginal distributions of W_+ and H are quite simple (see Theorem 4.2) the joint distribution of W_+ and H seems rather intractable. We are equally unable to give any useful formula for the Lévy measure of the law of $(W_1, \dots, W_n)$.

By an amazing coincidence, which we can explain only by invoking the Ray–Knight theorem on Brownian local times, Lévy (1951) encountered the joint distribution of H and W_+ as the joint law of $|Z_1|^2/2$ and $2A_1$, where $Z = X + iY$

is a complex Brownian motion starting at zero, and (A_t) is the stochastic area process traced out by Z:

$$A_t = \frac{1}{2}\int_0^t (X_s\,dY_s - Y_s\,dX_s).$$

This may be compared to the winding process

$$\Phi_t = \int_0^t (X_s\,dY_s - Y_s\,dX_s)/|Z_s|^2,$$

though it must be assumed for the winding that Z_0 is not zero. It would be most interesting to find a more direct connection between these two stochastic integrals. Lévy's results for $|Z_1|^2/2$ and $2A_1$ show that

$$E\big(e^{ivW_+}|H = h\big) = (v/\sinh v)e^{h(1 - v\coth v)}, \tag{7.b}$$

which implies that H and W_+ have a joint density. But we are unable to simplify the Fourier inversion formula for the conditional density of W_+ given $H = h$.

We note that the factor $v/\sinh v$ in (7.b) can be interpreted by a last exit decomposition in the representation of $H = \frac{1}{2}\Lambda$ and W_+ in Theorem 4.1, as the Fourier transform of $\theta(\sigma_1 - \sigma_0)$, where σ_0 is the last time β is at zero before time σ_1 when β first hits 1. According to Lévy, the corresponding distribution has density

$$P(\theta(\sigma_1 - \sigma_0) \in dx) = \pi/\big[4\cosh^2(\pi x/2)\big]. \tag{7.c}$$

In terms of windings, this is the limit distribution as $t \to \infty$ of

$$2\big[\Phi(t) - \Phi(t_0)\big]/\log t,$$

where t_0 is the last time before t that Z visits the unit circle (see Theorem 8.4). It is the second factor in (7.b),

$$e^{h(1 - v\coth v)} = E\big(e^{iv\theta(\sigma_0)}|H = h\big),$$

which seems difficult to invert.

Another way to describe the joint law of $(W_j,\ 1 \le j \le n)$ is to specify for each set of real coefficients $c_1, \dots, c_n$ the law of

$$\sum c_j W_j,$$

which is the limit in distribution as $t \to \infty$ of

$$(2/\log t)\sum c_j \Phi_j(t).$$

To describe this law, we introduce for $a, b \ge 0$ a distribution $\mathrm{MC}(a, b)$ on the line, the *mixed Cauchy distribution with weights a and b*, defined by

(7.d) $\mathrm{MC}(a, b)$ is the distribution of $aW_- + bW_+$,

where $W_\pm$ are asymptotic small and big winding random variables as in Theorems 4.1 and 4.2. The terminology "mixed Cauchy" will be explained shortly. The rôles of the coefficients a and b are indicated by the mnemonic a for sm*a*ll windings and b for *b*ig windings.

Elementary convolution and scaling properties of the Cauchy distribution imply that for all real coefficients $c_1, \ldots, c_n$ and d,

(7.e) the distribution of $\sum c_j W_-^j + dW_+$ is $\mathrm{MC}\big(\sum |c_j|, |d|\big)$.

In particular, for $W_j = W_-^j + W_+$,

(7.f) the distribution of $\sum c_j W_j$ is $\mathrm{MC}\big(\sum |c_j|, \big|\sum c_j\big|\big)$.

Thus, *every linear combination of windings about a finite number of points has asymptotic distribution belonging to the two parameter family of mixed Cauchy distributions.*

We record now a number of further consequences of (7.f), then give a more explicit description of the mixed Cauchy distributions. Since from (7.d) the $\mathrm{MC}(b, b)$ distribution is the symmetric Cauchy distribution with scale parameter b, it is a consequence of (7.f) that

(7.g) every positive linear combination of $W_1, \ldots, W_n$ has a symmetric Cauchy distribution.

Moreover, (7.a) and (7.d) yield,

(7.h) every linear combination $\Sigma c_j W_j$ with $\Sigma c_j = 0$ has the law of $(\Sigma_j |c_j|) H Y_1$, where H and Y_1 are independent random variables which are standard exponential and standard Cauchy, respectively.

From (7.d), (7.f), and Theorem 4.1 we obtain the Fourier transform of $W_1, \ldots, W_n$:

(7.i) $$E \exp\Big(\sum c_j W_j\Big) = f\Big(\sum |c_j|, \Big|\sum c_j\Big|\Big),$$

for $f(a, v)$ as in (7.a4), but this seems difficult to invert.

We turn now to a more detailed study of the two parameter family of distributions $\{\mathrm{MC}(a, b),\ a, b \geq 0\}$. An obvious feature of this family is the scaling property, that if the law of X is $\mathrm{MC}(a, b)$, then for any constant c the law of cX is $\mathrm{MC}(a|c|, b|c|)$. In particular, the $\mathrm{MC}(a, b)$ distribution is symmetric about zero. It is also easily seen that $\mathrm{MC}(a, b)$ is infinitely divisible.

From Theorem 4.2 the characteristic function of $\mathrm{MC}(a, b)$ is the function of $v \in \mathbb{R}$

(7.j1) $$f(a|v|, b|v|) = \left[\cosh a|v| + (a/b)\sinh b|v|\right]^{-1}.$$

An elementary calculation reveals that this characteristic function can be represented as

(7.j2) $$\int_0^\infty \mu_{a,b}(ds) e^{-s|v|},$$

where $\mu_{a,b}$ is the signed measure with total mass 1 on $(0, \infty)$ defined as follows:

For $b \neq 0$, $\mu_{a,b}$ is the discrete measure which puts mass

(7.j3) $$(2b/(a+b))((a-b)/(a+b))^n$$

at each of the points $(2n+1)b$, $n = 0, 1, 2, \ldots$, and $\mu_{a,0}$ is the exponential distribution with mean a: $\mu_{a,0}(ds) = a^{-1} e^{-s/a}\, ds$, $s \geq 0$.

Since $v \to e^{-s|v|}$ is the characteristic function of the Cauchy distribution, symmetric about 0 with scalar parameter s, and $\mu_{a,b}$ is a probability measure for $a \geq b$, we learn that

(7.k) for $a \geq b$, MC(a, b) is a scale mixture of symmetric Cauchy distributions, in the usual probabilistic sense. More precisely, for $a \geq b$, MC(a, b) is the distribution of $Y(T_{a,b})$ and $T_{a,b}Y(1)$, where Y is a Cauchy process, and $T_{a,b}$ is an independent random variable with distribution $\mu_{a,b}$.

This case $a \geq b$ is the one relevant to the basic formula (7.f) for the asymptotic windings. Let us consider three subcases.

(i) If $a = b$ then

$$T_{b,b} = b$$

is constant. That is, as noted already,

$$\mathrm{MC}(b, b) = \mathrm{Cauchy}(0, b),$$

corresponding via (7.f) to the fact that the distribution of each W_j is Cauchy(0, 1).

(ii) If $b = 0$, then $T_{a,0}$ is exponentially distributed with mean a. Then by (7.d), (7.k) amounts to the description (5.a2) of W_-^j with the identification

$$T_{a,0} = aH.$$

So, in the context of the windings, $T_{a,0}$ is interpretable as the asymptotic local time variable. The MC$(a, 0)$ distribution has characteristic function

$$f(a|v|, 0) = (1 + a|v|)^{-1}, \qquad v \in \mathbb{R},$$

and probability density function

$$f_{a,0}(x) = \int_0^\infty \frac{e^{-s/a} s\, ds}{a\pi(s^2 + x^2)}, \qquad x \in \mathbb{R},$$

which is the mixture of Cauchy densities corresponding to the mixture of characteristic functions (7.j2).

(iii) If $a > b > 0$, then

$$T_{a,b} = (2N_{a,b} + 1)b,$$

where $N_{a,b}$ is geometrically distributed on $\{0, 1, \dots\}$ with parameter $2b/(a + b)$. Instead of an integral formula there is a corresponding series formula for the density $f_{a,b}$ of MC(a, b):

$$f_{a,b}(x) = \sum_{n=0}^{\infty} \frac{2}{(a+b)} \left[\frac{a-b}{a+b}\right]^n \frac{(2n+1)b}{\pi\left[(2n+1)^2 b^2 + x^2\right]}.$$

But we are quite unable to provide an explanation in terms of windings for the appearance in this case of the geometric mixing distribution.

In addition to the three subcases considered above of (7.k) with $a \geq b$, which suffices for the consideration of (7.f), in (7.d) and (7.e) there is the further case to consider when

(iv) $0 \le a < b$. In this case the mixing measure $\mu(a, b)$ on $(0, \infty)$ defined by (7.j3) is a discrete signed measure with masses on the points $(2n + 1)b$, $n = 0, 1, \dots,$ which *alternate in sign*. The probabilistic interpretation of the mixing operation breaks down in this case. Still, it follows analytically from (7.j) that the formula for the density of $\mathrm{MC}(a, b)$ in case (iii) is still valid.

In the subcase when $a = 0$, the series simplifies to

$$f_{0,b}(x) = \tfrac{1}{2}b/\cosh(\pi x/2b),$$

as noted by Lévy (1951). This yields the formula of Theorem 4.2 for the density of the asymptotic big winding W_+. This $\mathrm{MC}(0, b)$ distribution has all moments finite, in contrast to the $\mathrm{MC}(a, b)$ distributions for $a > 0$, which do not have a first moment.

We do not know of any reference in the literature either to this mixed Cauchy family of distributions on the line, or to the multivariate distribution of the asymptotic windings.

8. Log scaling laws. The preceding sections have largely been devoted to the asymptotic behavior of the winding numbers generated by a Brownian path, around a finite number of points in the plane. We now explore the extent to which the same methods can be applied to other functionals of the Brownian path. It turns out that there is a large and quite varied collection of Brownian functionals $G(t)$, for which there are asymptotic laws linked with the results for windings. This section concerns a particular family of such limit laws, which we call *log scaling laws*.

The most basic kind of log scaling law states that

$$2G(t)/\log(t) \xrightarrow{d} \gamma(\sigma_1) \quad \text{as } t \to \infty, \tag{8.a}$$

where G is a functional of a complex Brownian motion Z started at $z_0 \neq 0$,

$$\gamma = (\gamma(u),\ u \ge 0)$$

is a process associated with G, defined in terms of a single $\mathrm{BM}(\mathbb{C})$ starting at 0

$$\zeta = \beta + i\theta,$$

and

$$\sigma_v = \inf\{u \colon \beta_u = v\}.$$

Roughly speaking, the process γ is obtained from G as a limit by Brownian scaling, after time changing via the clock

$$U_t = \int_0^t \frac{ds}{R_s^2} \tag{8.b}$$

which transforms $\log Z$ into ζ. We shall say that G is *logarithmically attracted to* γ. A formal definition of logarithmic attraction is given in Definition 8.3 below. But it seems best to introduce the notion with some examples and operations for

TABLE 1
Logarithmic attractions

process $G(t)$	attracting process $\gamma(u)$
(1) $\int_0^t d\Phi_s 1(R_s \in I_\Delta)$	$\int_0^u d\theta_t 1(\beta_t \in \mathbb{R}_\Delta)$
(2) $\int_0^t d[\log R_s] 1(R_s \in I_\Delta)$	$\int_0^u d\beta_t 1(\beta_t \in \mathbb{R}_\Delta)$
(3) $\left[\int_0^t \frac{ds}{R_s^2} 1(R_s \in I_\Delta)\right]^{1/2}$	$\left[\int_0^u dv 1(\beta_t \in \mathbb{R}_\Delta)\right]^{1/2}$
(4) increasing additive functional $A(t)$	$(2\pi)^{-1}\|A\|L(\beta, 0, u)$
(5) $a_1G_1(t) + a_2G_2(t)$	$a_1\gamma_1(u) + a_2\gamma_2(u)$
(6) $G_1(t) \vee G_2(t)$	$\gamma_1(u) \vee \gamma_2(u)$
(7) $\sup_{0 \le s \le t} G(s)$	$\sup_{0 \le v \le u} \gamma(v)$

generating further examples. In Table 1 I_Δ stands for one of the intervals

$$I_+ = (r, \infty), \qquad I_- = (0, r), \qquad I = (0, \infty),$$

where $r > 0$ is arbitrary, and $\mathbb{R}_\Delta$ stands for a corresponding interval

$$\mathbb{R}_+ = (0, \infty), \qquad \mathbb{R}_- = (-\infty, 0), \qquad \mathbb{R} = (-\infty, \infty).$$

Thus Δ may be $+$, $-$, or absent altogether.

The processes of line (1) of Table 1 are the processes of big windings, small windings, and total winding about zero up to time t. It was proved in Theorem 4.1 that the limit law (8.a) holds for the processes G in lines (1) and (4). The same argument extends with no difficulty to the processes G in lines (2) and (3). The processes G in (2) are just the real parts of the three conformal martingales whose imaginary parts are the winding processes in line (1). And the processes in (3) are the square roots to the increasing processes $U_\Delta(t)$ of these three conformal martingales. After squaring both sides the limit law (8.a) in this case becomes

$$\frac{4U_\Delta(t)}{\log^2(t)} \xrightarrow{d} s_\Delta(\sigma_1), \tag{8.c}$$

where $s_+(\sigma_1)$ is the time spent positive by the Brownian motion β before time σ_1, $s_-(\sigma_1)$ is the corresponding time spent negative, and $s_\Delta(\sigma_1) = \sigma_1$ if $U_\Delta = U$ as in (8.b). It is shown in Pitman and Yor (1986) that $U_\Delta(t) - N_\Delta(t)$ is $o(\log^2(t))$ in probability as $t \to \infty$, where $N(t)$ is the number of crossings of Z between the positive and negative parts of the real axis up to time t, $N_+(t)$ is the number of big crossings, and $N_-(t)$ the number of small crossings, for the classification of crossings mentioned above Theorem 4.2. So the geometrically defined counting processes $N_\Delta(t)$ can be substituted for the Brownian clocks $U_\Delta(t)$ in (8.c).

In lines (5) through (7) of Table 1, it is assumed that G_i and G are processes which are logarithmically attracted to γ_i and γ, respectively. These lines indicate operations by which the table may be extended. Proof that logarithmic attraction respects these operations is immediate from Definition 8.3. Combining these operations shows that logarithmic attraction respects similarly the transformations from a process $G(t)$ to

$$|G(t)|, \inf_{0\le s\le t} G(s), \sup_{0\le s\le t} |G(s)|.$$

As illustrations, lines (1) and (2) may be extended via line (7) to obtain in (8.a)

$$2\sup_{0\le s\le t} \Phi(s)/\log(t) \xrightarrow{d} \sup_{0\le v\le \sigma_1} \theta(v) \tag{8.d1}$$

$$2\inf_{0\le s\le t} \log R(s)/\log(t) \xrightarrow{d} \inf_{0\le v\le \sigma_1} \beta(v). \tag{8.d2}$$

However, logarithmic attraction does not respect products: the square of a process G satisfying (8.a) has a limit with normalization by $\log^2(t)$ rather than $\log(t)$.

Looking at the proof of Spitzer's law and Theorem 4.1, we have argued over and over again that a limit theorem of the form (8.a) holds with limit $\psi(\zeta)$, for some measurable functional ψ of ζ, if

$$h^{-1}G(e^{2h}) - \psi(\zeta^{(h)}) \xrightarrow{P} 0, \tag{8.h}$$

where $\zeta^{(h)}$ is obtained from ζ by the Brownian scaling operation

$$\zeta^{(h)}(u) = h^{-1}\zeta(h^2u), \qquad u \ge 0. \tag{8.i}$$

Writing $\Psi(h)$ or $\Psi(h,Z)$ instead of $h^{-1}G(e^{2h})$ suggests the following definition.

DEFINITION 8.1. A process

$$(\Psi(h),\, h>0) = (\Psi(h,\omega),\, h>0,\, \omega\in\Omega(\mathbb{C}))$$

has *log scaling limit* ψ, where ψ is a measurable function on $\Omega(\mathbb{C})$, if

$$\Psi(h,Z) - \psi(\zeta^{(h)}) \xrightarrow{P} 0 \quad \text{as } h\to\infty \tag{8.j}$$

whenever Z is a BM($\mathbb{C}$) starting at a point $z_0 \ne 0$, $\zeta = \zeta(Z)$ is the BM($\mathbb{C}$) starting at 0 defined by time changing the conformal martingale $\int_0^t(dZ_s/Z_s)$ by its clock $U_t = \int_0^t(ds/R_s^2)$, and $\zeta^{(h)}$ is derived from ζ by the Brownian scaling operation (8.i).

Log scaling limits have most of the usual properties of a notion of convergence of random variables. For example, log scaling limits are unique a.s. with respect to the distribution of ζ, and limits are preserved under the usual algebraic operations and transformation by continuous functions. Because the distribution of $\zeta^{(h)}$ is the same for all h, if $\Psi(h)$ has log scaling limit ψ then

$$\Psi(h) \xrightarrow{d} \psi(\zeta). \tag{8.k}$$

We call such an asymptotic law a *log scaling law*. A remarkable fact, which is obvious from the definition, is that all log scaling laws are linked: *Every finite collection of log scaling laws holds jointly.*

According to Lemma 3.1, the winding process

$$\Psi(h) = \Phi(e^{2h})/h, \qquad h > 0, \tag{8.l}$$

has log scaling limit $\theta(\sigma_1)$. Moreover, the proof of Theorem 4.1 shows that *for every choice of G in lines* (1) *to* (4) *of Table* 1, *the process*

$$\Psi(h) = G(e^{2h})/h, \qquad h > 0, \tag{8.m}$$

has log scaling limit $\gamma(\sigma_1)$, where γ is the logarithmic attractor of G as defined by Table 1.

Furthermore, log scaling laws hold jointly relative to several origins. Thus Theorem 6.1 for windings is a particular case of the following general result:

THEOREM 8.2. *If $z_0, z_1, \dots, z_n$ are n distinct points in the plane, and $\Psi_j(Z, h)$ is a process with log scaling limit ψ_j, $j = 1, \dots, n$, then, as $h \to \infty$*

$$\left[\Psi_j(Z - z_j, h),\ j = 1, \dots, n\right] \xrightarrow{d} \left[\Psi_j(\zeta^{j,\infty}),\ j = 1, \dots, n\right],$$

where the $\zeta^{j,\infty}$ are the linked Brownian motions described in Theorem 6.2.

PROOF. Immediate from Theorem 6.2 and Lemma B.3. □

There is one important distinction between this notion of a log scaling limit and more usual notions of convergence of random variables: if $\Psi(h)$ has log scaling limit ψ, it will typically not be true that $\Psi(a(h))$ has log scaling limit ψ for functions $a(h)$ increasing to ∞ as $h \to \infty$. For example, it is easy to check that

if Ψ has log scaling limit ψ, and Ψ_a is defined by

$$\Psi_a(h) = \Psi(ah), \qquad a > 0, \tag{8.n}$$

then Ψ_a has log scaling limit ψ_a, where

$$\psi_a(\zeta) = \psi(\zeta^{(a)}).$$

In this case the limit exists and has the same distribution as ψ, but may be different almost surely. For other increasing functions $a(h)$ the log scaling limit may not exist at all.

As a particular case of (8.n), we may take Ψ as in (8.m). Without further calculation, the linkage between all log scaling laws implies that as $h \to \infty$

$$\left(G(e^{2ah})/h,\ a \geq 0\right) \xrightarrow{\mathrm{Fd}} \left(\gamma(\sigma_a),\ a \geq 0\right) \tag{8.o}$$

where $\xrightarrow{\mathrm{Fd}}$ indicates convergence of finite dimensional distributions. When G is the winding process Φ, γ is the Brownian motion θ, which is independent of the Brownian motion β whose hitting time of a is σ_a. Then, as remarked in Section 3, the process on the right is a Cauchy process. This extension of Spitzer's law may be found in Section 5.5 of Durrett (1984). When $G = A$ is an increasing additive functional,

$$\gamma(\sigma_a) = (2\pi)^{-1}\|A\|L(\beta, 0, \sigma_a), \qquad a \geq 0,$$

and we obtain a companion result due to Kasahara and Kotani (1979), extending the Kallianpur–Robbins theorem. According to the above discussion, these results (8.o) hold jointly with each other, as G varies, and jointly for several origins.

As remarked by Kasahara (1982) and Durrett (1984), it is impossible to strengthen the convergence of finite dimensional distributions (8.o) to weak convergence in the function space $D[0, \infty)$ equipped with any of the usual topologies. This is because the processes converging have continuous paths, whereas the limit has jumps. As an example, for the winding process $G = \Phi$, the convergence in (8.o) cannot take place in any topology on $D[0, \infty)$ which makes the maximum on $[0, 1]$ a continuous functional. To see this, we recall Spitzer's remark that $(\Phi(t),\ t \geq 0)$ satisfies the same reflection identity as Brownian motion:

$$P(\overline{\Phi}(t) > x) = 2P(\Phi(t) > x),$$

where for a process G we write

$$\overline{G}(t) = \sup_{0 \leq s \leq t} G(s).$$

This follows directly from the time change to Brownian motion. Thus Spitzer's theorem implies that as $h \to \infty$

$$\overline{\Phi}(e^{2h})/h \xrightarrow{d} |W(1)|, \tag{8.p}$$

where the standard Cauchy random variable $W(1)$ may be obtained from the Cauchy process

$$(W(a) = \theta(\sigma_a),\ a \geq 0)$$

appearing in (8.o) in this case. On the other hand, were there convergence in (8.o) in a topology on $D[0, \infty)$ which made the maximum functional continuous, the limit in (8.p) would be identified as the distribution of

$$\overline{W}(1) = \sup_{0 \leq a \leq 1} \theta(\sigma_a). \tag{8.q}$$

But the jumps of the Cauchy process forbid a reflection principle, and the distributions of $|W(1)|$ and $\overline{W}(1)$ are *different.* The situation is clarified by finding the right representation in terms of $\zeta = \beta + i\theta$ for the limit random variable in (8.p). According to line (7) of Table 1, we should substitute

$$\overline{\theta}(\sigma_1) = \sup_{0 \leq v \leq \sigma_1} \theta(v)$$

instead of $|W(1)|$ in (8.p). This substitution turns (8.p) into yet another log scaling law, holding jointly with the convergence of finite dimensional distributions (8.o). Of course, $\overline{\theta}(\sigma_1)$ is with probability one strictly bigger than the maximum value

$\overline{W}(1)$ of the Cauchy process $W(a) = \theta(\sigma_a)$ up to time $a = 1$. The latter maximum looks only at those values of θ when β is hitting new maxima, and the maximum of θ will almost surely occur between such times. Thus the convergence of finite dimensional distributions in (8.o) does not in any sense determine the limit distribution of the maximum. Nonetheless this distribution can be obtained within the larger framework of log scaling laws.

This example brings us to the question of what it means for a process G defined in terms of the Brownian motion Z to be logarithmically attracted to a limit process γ defined in terms of the time changed Brownian motion ζ. Since Z can be expressed in terms of ζ via the formula

$$Z_t = z_0 \exp(\zeta(U_t)),$$

where U_t can be written in terms of ζ by (2.k), the Brownian functional $G(t) = G(t, Z)$ can always be rewritten as

$$G(t, Z) = \Gamma(U_t, \zeta) \tag{8.r}$$

for some process $\Gamma(u) = \Gamma(u, \zeta)$. Now let $\Gamma^{(h)}$ be obtained from Γ by the Brownian scaling operation

$$\Gamma^{(h)}(u) = \frac{1}{h}\Gamma(h^2 u), \qquad h > 0,$$

and regard $\Gamma^{(h)}$ as a process parameterized by $h > 0$, with values in the path space $\Omega(\mathbb{R})$, where we assume for simplicity that G has continuous paths.

DEFINITION 8.3. The Brownian functional G is logarithmically attracted to γ if the $\Omega(\mathbb{R})$ valued process $\Gamma^{(h)}$ has log scaling limit γ.

Spelled out, this condition states that

$$\Gamma^{(h)}(\zeta) - \gamma(\zeta^{(h)}) \xrightarrow{P} 0 \quad \text{as } h \to \infty, \tag{8.s}$$

where the convergence is uniform on compact sets. Equivalently, by Brownian scaling,

$$\Gamma^{(h)}(\zeta^{(1/h)}) \xrightarrow{P} \gamma(\zeta) \quad \text{as } h \to \infty, \tag{8.t}$$

in the same sense.

Straightforward calculations show that each of the processes G in the first four lines of Table 1 is logarithmically attracted by the corresponding process γ. Moreover the closure properties stated in lines (5) thorugh (7) of the table are easily verified. A large class of log scaling limit laws, including all those mentioned so far, can now be obtained by application of the following theorem:

THEOREM 8.4. *Suppose that a Brownian functional $G = (G(t),\ t \geq 0)$ is logarithmically attracted to $\gamma = (\gamma(u),\ u > 0)$. Suppose also that $(T_h,\ h \geq 0)$ is a family of random times such that*

$$\frac{1}{h^2} U(T_h) \text{ has log scaling limit } \tau, \tag{8.u}$$

where U is the logarithmic clock (8.*b*). *Then*

$$\frac{1}{h}G(T_h) \text{ has log scaling limit } \gamma(\tau). \tag{8.v}$$

PROOF. By the definition of Γ in (8.r),

$$\begin{aligned}\frac{1}{h}G(T_h) &= \frac{1}{h}\Gamma(U(T_h)) \\ &= \Gamma^{(h)}\left(\frac{1}{h^2}U(T_h)\right),\end{aligned}$$

so the result follows from the assumptions (8.u) and (8.v).

In particular, the hypothesis (8.u) holds for $T_h = e^{2h}$, with $\tau = \sigma_1$. The conclusion (8.v) then implies that the basic log scaling law (8.a) holds for every Brownian functional G which is logarithmically attracted to γ.

Table 2 indicates some families of random times (T_h, $h > 0$) and the corresponding asymptotic times τ, which are the log scaling limits of $(1/h^2)U(T_h)$. According to Theorem 8.4, for each selection of G and γ from Table 1, and each selection of (T_h) from Table 2, there is the log scaling law

$$\frac{1}{h}G(T_h) \xrightarrow{d} \gamma(\tau) \quad \text{as } h \to \infty. \tag{8.w}$$

PROOFS. From line (6) of Table 2, which is obvious from the definition of a log scaling limit, there is no restriction whatever on the collection of possible asymptotic times τ. Letting τ be σ_v yields line (2) from line (6). Lines (1) and (3) are deduced from line (2) by Brownian scaling, as in (3.g). Taking τ as in line (4), for $\|A\| = 2\pi$ the formula for T_h in line (6) gives the left side of line (4) for the particular additive functional

$$A_t = L(\log R, 0, t)$$

TABLE 2
Logarithmic time changes

family of times (T_h)	asymptotic time τ
(1) e^{2vh}, $v > 0$	σ_v
(2) $\inf\{t\colon R_t = e^{vh}\}$, all real v	σ_v
(3) $\inf\{t\colon A_t = e^{vh}\}$, $v > 0$, where A_t is the local time of X_t at 0	σ_v
(4) $\inf\{t\colon A_t = vh\}$ where A_t is an increasing additive functional with $\|A\| < \infty$	$\inf\{u\colon L(\beta, 0, u) = v/\|A\|\}$
(5) $\sup\{t \le e^{2vh};\ R_t \le r\}$	$\sup\{u \le \sigma_v;\ \beta_u = 0\}$
(6) $U^{-1}[h^2\tau(\zeta^{(h)})]$	τ

by the transformation of martingale local times under time and space changes according to (A.8), (B.1), and (B.2) . The result for a general additive functional A can then be derived by a comparison with this special case, using the ergodic theorem for additive functionals. □

Combining Theorems 8.2 and 8.4 for the times in lines (1) and (3) of Table 2 shows how the results (5.o) and (5.p) extend to hold jointly relative to several origins. This yields the joint asymptotic laws for windings described by Lyons and McKean (1984) and Pitman and Yor (1984). The times in line (3) yield asymptotic laws which have interesting implications for the Cauchy process obtained by watching Z only when it touches the real axis. See Pitman and Yor (1986).

Which additive functionals of the planar Brownian motion admit log scaling limit laws? For increasing additive functionals $A(t)$ the answer is simple if we assume that it is $A(t)$ itself and not some power $A(t)^{\alpha}$ that satisifes the log scaling limit law. By the ratio ergodic theorem, $A(t)$ must have $\|A\| < \infty$, and the result is then the extension due to Itô and McKean (1965) of the Kallianpur–Robbins theorem, given by (8.a) for G in line (4) of Theorem 8.1.

For martingale additive functionals, represented as say

$$G(t) = \int_0^t f(Z_s)\, dZ_s + \int_0^t g(Z_s)\, d\bar{Z}_s,$$

where f and g are Borel functions, the question is more interesting. The following result is a refinement of Theorem 6 in Messulam and Yor (1982), where it is assumed that $zf(z)$ converges both as $z \to 0$, $z \neq 0$, and as $z \to \infty$. Using complex conjugates, it is a straightforward matter to formulate a corresponding result for $G(t)$ as above.

THEOREM 8.5. *Let*

$$G(t) = \int_0^t f(Z_s)\, dZ_s \quad \textit{with}\ |f(z)| \leq \frac{c}{|z|}.$$

Then the following assumptions are equivalent:

(i) *G is logarithmically attracted to some process γ.*

(ii) $\exp[h(x+iy)]\, f[\exp(h(x+iy))]$ *converges as $h \to \infty$ in $L^1_{\mathrm{loc}}(dx\, dy)$.*

(iii) *There exist constants p_+ and p_- such that as $R \to \infty$,*

$$\frac{1}{\log R} \int_{D(R,\pm)} \frac{dx\, dy}{|z|^2} |zf(z) - p_{\pm}| \to 0,$$

where

$$D(R,+) = \{z \colon 1 \leq |z| \leq R\},$$

$$D(R,-) = \{z \colon R^{-1} \leq |z| \leq 1\}.$$

If these conditions are satisfied, then the logarithmic attractor γ is

$$\gamma(u) = \int_0^u p(\beta_v)\, d\zeta_v,$$

where

$$p(x) = p_+ 1(x \geq 0) + p_- 1(x \leq 0).$$

This theorem, and Theorem 8.6 below will be established in a forthcoming paper.

If f is holomorphic in $\{z \neq 0;\ |z| < r\}$ for some $r \in (0,1)$, and bounded on $\{r' \leq |z| \leq 1\}$ for some $r' < r$, then the limit p_- exists and is the residue of the function at zero. This lies outside the scope of the above theorem, since the assumption $|f(z)| \leq c/|z|$ will not be satisfied in general. But a log scaling law can still be obtained in this case. Put together for several origins, using the notation of Theorem 6.1, the result is the following asymptotic residue theorem:

THEOREM 8.6. *Let $z_1, \dots, z_n$ be a finite set of points in $\mathbb{C}$. Suppose that f is a complex valued function such that*

(i) *f is holomorphic in $D_j \setminus \{z_j\}$ for a neighborhood D_j of each point z_j,*
(ii) *f is bounded on the complement of the union of these neighborhoods,*
(iii) *f has compact support.*

Then, as $t \to \infty$

$$\frac{2}{\log t} \int_0^t f(Z_s)\, dZ_s \xrightarrow{d} \sum_j \operatorname{Res}(f, z_j)\left(\frac{\Lambda}{2} + iW_-^j\right).$$

If instead of (iii), *f is holomorphic in a neighborhood of ∞ and*

$$\lim_{z \to \infty} f(z) = 0,$$

then the same conclusion holds with the addition in the limit of

$$\operatorname{Res}(f, \infty)\left(\frac{\Lambda}{2} - 1 + iW_+\right).$$

APPENDICES

A. Continuous semimartingales. Unless otherwise stated, all processes considered here are defined on a probability space $(\Omega, \mathcal{F}, P)$, and are adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ of sub σ-fields of $\mathcal{F}$. A real-valued process M is a continuous local martingale, to be abbreviated as CLM, if the trajectories of M are continuous, and there exists a sequence (T_n) of stopping times increasing to ∞ a.s., such that $(M(t \wedge T_n),\ t \geq 0)$ is a martingale for every n.

A very important process attached to a continuous local martingale M is its *increasing process* $\langle M \rangle$ also known as its *quadratic variation* or *variance* process, which is characterized by

$$M^2 - \langle M \rangle \text{ is a CLM.}$$

As explained in Yor (1982), following Lenglart and Sharpe, the process $\langle M \rangle$ both allows the construction of the stochastic integrals $\int H\, dM$ of predictable processes with respect to M, and can be obtained during their construction via the Itô formula

$$M_t^2 - M_0^2 = 2\int_0^t M_s\, dM_s + \langle M\rangle_t.$$

Among other things, the process $\langle M\rangle$ governs the speed at which $M_t^* \equiv \sup_{s\le t}|M_s|$ increases. This shows up clearly in the following distributional inequalities which can be used to establish the Burkholder–Gundy inequalities in L^p for $0 < p < \infty$. See for example Burkholder (1973). If M is a CLM, and $M_t^* = \sup_{s\le t}|M_s|$, then for any $\lambda > 0$, $\delta > 0$

$$\text{(A.1)} \qquad P\big(M_\infty^* \ge 2\lambda, \langle M\rangle_\infty^{1/2} < \delta\lambda\big) \le 4\delta^2 P(M_\infty^* \ge \lambda).$$

This immediately implies

$$\text{(A.2)} \qquad P(M_\infty^* \ge 2\lambda) \le 4\delta^2 + P\big(\langle M\rangle_\infty^{1/2} \ge \delta\lambda\big).$$

Moreover, the same inequalities hold with M_∞^* and $\langle M\rangle_\infty^{1/2}$ switched. The following lemma is an easy consequence, the proof of which is left to the reader.

LEMMA A.1. (i) *Let* $(M_t^n,\ t \ge 0)$ *be a sequence of* CLM's. *Then, as* $n \to \infty$,

$$(M^n)_\infty^* \xrightarrow{P} 0 \quad \textit{if and only if } \langle M^n\rangle_\infty \xrightarrow{P} 0.$$

(ii) *In particular, if* $(\tau(l),\ l \ge 0)$ *is a family of stopping times,* M *is a* CLM, *and* $g(l)$ *is a real valued function of* l, *then, as* $l \to \infty$,

$$M_{\tau(l)}^*/g(l) \xrightarrow{P} 0 \quad \textit{if and only if } \langle M\rangle_{\tau(l)}/g^2(l) \xrightarrow{P} 0.$$

A fundamental link between martingale theory and Brownian motion is Lévy's characterization of an $(\mathscr{F}_t)$BM, defined as a process (B_t) which is $(\mathscr{F}_t)$ adapted, has the distribution of BM, and such that, for any $t \ge 0$, the increments $(B_{t+s} - B_t,\ s \ge 0)$ are independent of $\mathscr{F}_t$. Lévy's characterization reads

$$\text{(A.3)} \qquad (B_t) \text{ is an } (\mathscr{F}_t)\text{BM iff } (B_t) \text{ is a CLM with } \langle B\rangle_t = t.$$

A *continuous semimartingale* is the sum of a CLM and a process with continuous paths of bounded variation on any compact of $\mathbb{R}_+$. Semimartingales are precisely the right objects for stochastic integration. See Dellacherie's survey (1980), for example. Further developments are found in Meyer (1976) and Dellacherie and Meyer (1978, 1980).

A cornerstone of stochastic calculus is Itô's formula

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\, ds + \tfrac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s,$$

for continuous semimartingales X and functions f with two continuous derivatives. From among the numerous extensions of Itô's formula we extract the following:

(A.4) If X is a continuous semimartingale which takes its values in an open interval $I \subset \mathbb{R}$, and $f\colon I \to \mathbb{R}$ is a difference of two convex functions, then $Y = f(X)$ is a continuous semimartingale.

In particular, Tanaka's formula

$$(\text{A.5}) \qquad (X_t - a)^+ = (X_0 - a)^+ + \int_0^t dX_s 1_{(X_s > a)} + \tfrac{1}{2}L(X, a, t)$$

and its companion

$$(\text{A.6}) \qquad (X_t - a)^- = (X_0 - a)^- - \int_0^t dX_s 1_{(X_s \le a)} + \tfrac{1}{2}L(X, a, t)$$

serve to define the local time $L(X, a, t)$ of the semimartingale X at a up to time t. For each a the process $(L(X, a, t),\ t \ge 0)$ is an increasing process, whose set of points of increase is contained in $\{t\colon X_t = a\}$. All the local times we consider are well known to admit versions which are jointly continuous in the time and space variables. (When dealing with $\mathbb{R}_+$ valued processes, we restrict the levels a to belong to $\mathbb{R}_+$.) Such a version of local times, with which we shall work henceforth, satisfies the *occupation density formula*:

$$(\text{A.7}) \qquad \int_0^t d\langle M\rangle_s\, h(X_s) = \int da\, h(a) L(X, a, t)$$

for any positive Borel function h, with M the martingale part of X. Proofs may be found in Yor (1978a, b).

Semimartingale local times respond well to deterministic space transformations. Using the same notation as in (A.4), and assuming further that f is strictly increasing, it is easily shown that

$$(\text{A.8}) \qquad L(Y, f(a), t) = f'(a) L(X, a, t), \qquad a \in I.$$

These local times also respond well to random time changes, as discussed in the next appendix.

For a pair of CLM's M and N, the *covariance process* of M and N, denoted $\langle M, N\rangle$, is defined by the requirement

$$(\text{A.9}) \qquad MN - \langle M, N\rangle \text{ is a CLM}.$$

According to Itô's formula for integration by parts,

$$(\text{A.10}) \qquad M_t N_t - M_0 N_0 = \int_0^t M_s\, dN_s + \int_0^t N_s\, dM_s + \langle M, N\rangle_t.$$

B. Time changes to Brownian motions. Let M be a CLM with increasing process $U = \langle M\rangle$, such that $U_\infty = \infty$. We introduce the right-continuous inverse $(T_u,\ u \ge 0)$ of the increasing process U:

$$T(u) = \inf\{t\colon U(t) > u\}.$$

Dubins and Schwartz (1965) and Dambis (1965) have shown that

$$(\beta_u \equiv M_{T_u},\ u \ge 0) \text{ is an } (\mathscr{F}_{T_u})\text{BM},$$

and moreover

$$M_t = \beta_{U(t)}, \qquad t \ge 0.$$

This result by itself explains a lot about the ubiquity of BM in stochastic processes. We refer to β as the BM associated with M. Stochastic integration with respect to M responds well to the time change $(T_u,\ u \geq 0)$ as the following lemma shows.

LEMMA B.1. (i) *If H is an $(\mathscr{F}_t)$-predictable process with*

$$\int_0^t H_s^2\, dU_s < \infty, \qquad t > 0,$$

then

$$\int_0^t H_s\, dM_s = \int_0^{U_t} H_{T_v}\, d\beta_v, \qquad t \geq 0.$$

(ii) *If K is an $(\mathscr{F}_{T_u})$-predictable process with*

$$\int_0^u K_v^2\, dv < \infty, \qquad u > 0,$$

then

$$\int_0^u K_v\, d\beta_v = \int_0^{T_u} K_{U_s}\, dM_s.$$

The local times of M are simply time changes of those of β. To be precise,

$$L(M, a, t) = L(\beta, a, U_t), \qquad t \geq 0, \tag{B.1}$$

$$L(\beta, a, u) = L(M, a, T_u), \qquad u \geq 0. \tag{B.2}$$

For details, and more on questions of measurability of M with respect to β, see Stroock and Yor (1980).

Continuous local martingales M and N are called *orthogonal* if their covariance process $\langle M, N\rangle$ is identically zero, that is if their product MN is a CLM. Knight (1971) showed that

(B.3) if $(M_j;\ 1 \leq j \leq k)$ are CLM's with respect to the same filtration, such that $\langle M_j\rangle_\infty = \infty$ a.s. for every j, and M_i and M_j are orthogonal for every $i < j$, then the associated Brownian motions β_j are mutually independent.

Meyer (1971) gave a very quick proof of this, which is now sketched. For the sake of clarity, we assume $k = 2$. It is sufficient to show that if F_j $(j = 1, 2)$ are $\sigma(\beta_j)$-measurable, bounded functionals, then

$$E(F_1 F_2) = E(F_1)E(F_2). \tag{B.4}$$

Since every square integrable Brownian functional may be represented as a stochastic integral, we have

$$F_j = E(F_j) + \int_0^\infty f_j(u, w)\, d\beta_j(u) \tag{B.5}$$

with $(f_j(u, w))$ a predictable process with respect to the natural filtration of β_j. By Lemma B.1, we may write

$$F_j = E(F_j) + \int_0^\infty f_j(\langle M_j\rangle_t)\, dM_j(t). \tag{B.6}$$

Now (B.4) follows from Itô's integration by parts formula (A.10), since the Itô correction term disappears by the orthogonality assumption. A slightly different proof of Knight's theorem, using exponential martingales is presented in Cocozza and Yor (1980).

In Section 6 we make use of the following extension of Knight's theorem. We expect this result should also prove to be useful in other studies of limit theorems in distribution, such as those conducted by Papanicolaou, Stroock and Varadhan (1977), Kasahara (1977), etc. The formulation was suggested in a particular case by Varopoulos.

THEOREM B.2. *Let $(M_j^n;\ 1 \le j \le k)$ be a sequence of k-tuples of continuous local martingales such that $\langle M_j^n\rangle_\infty = \infty$ for every j and n, and for every pair of distinct indices i and j there exists a sequence of positive random variables H_n such that*

(i) $$\int_0^{H_n} |d\langle M_i^n, M_j^n\rangle_s| \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

(ii) *at least one of the sequences $(\langle M_i^n\rangle_{H_n})$ and $(\langle M_j^n\rangle_{H_n})$ converges in probability to ∞ as $n \to \infty$.*

Then, as $n \to \infty$ the Brownian motions β_j^n associated with M_j^n are asymptotically independent, in the sense that

$$(\beta_j^n;\ 1 \le j \le k) \xrightarrow{d} (\beta_j^\infty;\ 1 \le j \le k),$$

where $(\beta_j^\infty;\ 1 \le j \le k)$ are k independent BM's.

REMARKS. A sufficient condition for (i) and (ii) is easily seen to be

(iii) $$\text{for all } u,\ \int_0^{\langle M_i^n\rangle_u^{-1}} |d\langle M_i^n, M_j^n\rangle| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

A particularly important special case arises when the M_j^n are defined in terms of k CLM's M_j, $1 \le j \le k$, by

$$M_j^n = M_j/\sqrt{n}.$$

Then the BM's β_j^n are obtained from the BM β_j associated with M_j by the Brownian scaling operation

$$\beta_j^n(u) = \frac{1}{\sqrt{n}}\beta_j(nu), \qquad u \ge 0.$$

Condition (iii) in this case reduces to

(iii′) $$\frac{1}{n}\int_0^{\langle M_i\rangle_n^{-1}} |d\langle M_i, M_j\rangle_s| \xrightarrow{P} 0 \quad \text{as } n \to \infty,$$

a condition which is obviously implied by

(iv) $$\frac{1}{\langle M_i\rangle_t}\int_0^t |d\langle M_i, M_j\rangle_s| \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Thus if the covariance processes between the M_j grow more slowly than their variance processes, the associated Brownian motions become asymptotically independent when viewed on a sufficiently large scale.

PROOF. For simplicity we again treat only the case $k = 2$. Write M^n instead of M_1^n, N^n instead of M_2^n. Following the same track as in the proof of Knight's theorem above, we consider two bounded Brownian functionals F and G, and let

$$F_n = F(\beta^n), \qquad G_n = G(\gamma^n), \tag{B.7}$$

where β^n and γ^n are the Brownian motions associated with M^n and N^n. We denote by $f_n(u, w)$ and $g_n(u, w)$ the predictable processes which appear in the representation (B.5) of F_n and G_n as stochastic integrals with respect to $d\beta^n$ and $d\gamma^n$. We shall show that, as $n \to \infty$

$$E(F_n G_n) \to E(F)E(G). \tag{B.8}$$

By Itô's integration by parts formula (A.10),

$$E(F_n G_n) - E(F)E(G) = E(X_n),$$

where

$$X_n = \int_0^\infty \tilde{f}_n(u)\tilde{g}_n(u)\, d\langle M^n, N^n\rangle_u,$$

and we use the abbreviations

$$\tilde{f}_n(t) = f_n(\langle M^n\rangle_t), \qquad \tilde{g}_n(t) = g_n(\langle N^n\rangle_t).$$

We prove (B.8) by showing that

$$X_n \text{ is bounded in } L^2, \tag{B.9}$$

and

$$X_n \xrightarrow{P} 0, \tag{B.10}$$

so that X_n converges in L^1 to 0.

To see (B.9), notice that by the Cauchy–Schwarz inequality and time changes

$$X_n^2 \le \left[\int_0^\infty f_n^2(u)\, du\right]\left[\int_0^\infty g_n^2(u)\, du\right],$$

implying

$$E(X_n^2) \le \sqrt{cd},$$

where

$$c = E\left[\int_0^\infty f_n^2(u)\, du\right]^2$$

is a number which does not depend on n by definition of f_n, and is finite by the inequalities of Burkholder–Davis–Gundy and Doob, and the same is true of d defined similarly in terms of g_n.

To prove (B.10), we argue that for $K > 0$

$$(1)\qquad |X_n| \le \int_0^\infty |d\langle M^n, N^n\rangle_u| |(\tilde{f}_n \tilde{g}_n)(u)| 1(|\tilde{f}_n(u)| \ge K)$$

$$(2)\qquad + \int_0^\infty |d\langle M^n, N^n\rangle_u| |(\tilde{f}_n \tilde{g}_n)(u)| 1(|\tilde{g}_n(u)| \ge K)$$

$$(3)\qquad + K^2 \int_0^{H_n} |d\langle M^n, N^n\rangle_t|$$

$$(4)\qquad + \left[\int_{\langle M^n\rangle_{H_n}}^\infty f_n^2(u)\, du\right]^{1/2} \left[\int_0^\infty g_n^2(u)\, du\right]^{1/2}.$$

Call these terms $X_n^{(i)}$, $i = 1,2,3,4$. We first show that for $i = 1,2$, $X_n^{(i)}$ converges to 0 in L^1, and hence in probability, uniformly in n as k tends to ∞. Consider that

$$E(|X_n^{(1)}|) \le E\left[\int_0^\infty du\, f_n^2(u) 1(|f_n(u)| \ge K)\right]^{1/2} E\left[\int_0^\infty du\, g_n^2(u)\right]^{1/2}.$$

Each expectation on the right does not depend upon n, by definition of f_n and g_n, so the right side tends to zero as $n \to \infty$ by dominated convergence. The same argument applies to $X_n^{(2)}$. Thus it suffices to show that for fixed K the terms $X_n^{(3)}$ and $X_n^{(4)}$ converge in probability to zero as $n \to \infty$. For $X_n^{(3)}$ this follows from the hypothesis (i). Finally, for $\varepsilon > 0$ and $v > 0$

$$P(X_n^{(4)} \ge \varepsilon) \le P(\langle M^n\rangle_{H_n} \le v) + P\left\{\left(\int_v^\infty du\, f_n^2(u)\, du\right)^{1/2} \left(\int_0^\infty g_n^2(u)\, du\right)^{1/2} \ge \varepsilon\right\}$$

$$\le P(\langle M^n\rangle_{H_n} \le v) + \frac{1}{\varepsilon}\left[E\int_v^\infty du\, f_n^2(u)\right]^{1/2} \left[E\int_0^\infty g_n^2(u)\, du\right]^{1/2}.$$

Fix $\varepsilon > 0$. As before, the last term does not depend on n, and converges to zero as $v \to \infty$, while for each v the first term tends to zero as $n \to \infty$ by the hypothesis (ii). This completes the argument. □

It appears at first sight that the above argument proves more than convergence in distribution of $(\beta_j^n;\ 1 \le j \le k)$ toward $(\beta_j^\infty;\ 1 \le j \le k)$. The argument shows that

$$\left(\varphi_j(\beta_j^n);\ 1 \le j \le k\right) \xrightarrow{d} \left(\varphi_j(\beta_j^\infty);\ 1 \le j \le k\right)$$

for any k-tuple of measurable Brownian functionals $(\varphi_j,\ 1 \le j \le k)$. However, the following lemma, which proves useful in Section 6, shows that this reinforcement has nothing to do with either stochastic integration or independence.

LEMMA B.3. *Suppose that for each $j = 1,\dots,k$ and $n = 1,2,\dots$, X_j^n is a random variable with values in a separable metric space S_j, such that for each j the distribution of X_j^n does not depend on n, and as $n \to \infty$*

$$(X_j^n;\ j = 1,\dots,k) \xrightarrow{d} (X_j;\ j = 1,\dots,k).$$

Then for Borel measurable functions φ_j from S_j to further separable metric spaces T_j,

$$\left(\varphi_j(X_j^n);\ j = 1, \dots, k\right) \xrightarrow{d} \left(\varphi_j(X_j);\ j = 1, \dots, k\right).$$

REMARK. The hypothesis that X_j^n has the same distribution for every n can be weakened to the hypothesis that the distribution of X_j^n converges in total variation.

PROOF. Consider first the special case when $k = 2$ and φ_1 is the identity function on S_1. The result in this case is a straightforward consequence of the fact that the Borel σ-field on S_2 is generated by $C_b(S_2)$, the space of bounded continuous functions on S_2, so $C_b(S_2)$ is dense in $L^1(\mu)$, with μ the common distribution of X_2^n. Repeated application of the special case yields the general case. □

Returning to the consideration of continuous local martingales M^n and N^n, the associated Brownian motions β^n and γ^n may become asymptotically identical rather than asymptotically independent as $n \to \infty$. A sufficient condition is provided by the following theorem:

THEOREM B.4. *If there exists a sequence of time changes $(R_n(t))$ such that*

(i) $$\langle M^n \rangle_{R_n(t)} \xrightarrow{P} t \quad \text{and} \quad \langle N^n \rangle_{R_n(t)} \xrightarrow{P} t,$$

(ii) $$\langle M^n - N^n \rangle_{R_n(t)} \xrightarrow{P} 0,$$

then, for every $k > 0$

$$\sup_{t \le k} |\beta^n(t) - \gamma^n(t)| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Moreover, the hypotheses (i) *and* (ii) *are satisfied if and only if*

$$\text{for every } t > 0, \quad \langle M^n - N^n \rangle_{\tau_n(t) \vee \sigma_n(t)} \to 0 \quad \text{as } n \to \infty,$$

with τ_n and σ_n the right-continuous inverses of $\langle M^n \rangle$ and $\langle N^n \rangle$.

PROOF. This is a simple application of the equivalence between the convergence to 0 in probability for the maximal processes and quadratic processes associated with CLM's, as presented in Lemma A.1. □

In the case when $M^n = M/\sqrt{n}$, $N^n = N/\sqrt{n}$, the conditions (i) and (ii) may be rewritten as below Theorem B.2. In this case, a sufficient condition for (i) and (ii) is provided by

(iii) $$\frac{1}{\langle X \rangle_t} \langle M - N \rangle_t \to 0 \quad \text{a.s.} \quad \text{as } t \to \infty$$

both for $X = M$ and $X = N$.

Again, the conclusion of the theorem may be reinforced by Lemma B.3.

Of course, the Brownian motions obtained by time changing a sequence of pairs of CLM's may be neither asymptotically independent nor asymptotically identical. Section 6 of this paper shows how the above two criteria may be combined to describe an example in which the positive excursions of the Brownian motions become asymptotically identical, while the negative excursions become asymptotically independent.

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