Stochastic Processes and their Applications 100 (2002) 223–231

stochastic processes and their applications

www.elsevier.com/locate/spa

Self-similar processes with independent increments associated with Lévy and Bessel processes☆

M. Jeanblanc[a], J. Pitman[b,*], M. Yor[c]

[a] Equipe d'analyse et probabilités, Université d'Évry Val d'Essonne, Boulevard F. Mitterrand, 91025 Evry Cedex, France

[b] Department of Statistics, University of California, 367 Evans Hall # 3860, Berkeley, CA 94720-3860, USA

[c] Laboratoire de probabilités et modèles aléatoires, Université Pierre et Marie Curie, 175, rue du Chevaleret, 75013 Paris, France

Received 13 September 2001; received in revised form 15 January 2002; accepted 16 January 2002

Abstract

Wolfe (Stochastic Process. Appl. 12(3) (1982) 301) and Sato (Probab. Theory Related Fields 89(3) (1991) 285) gave two different representations of a random variable X_1 with a self-decomposable distribution in terms of processes with independent increments. This paper shows how either of these representations follows easily from the other, and makes these representations more explicit when X_1 is either a first or last passage time for a Bessel process. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Self-decomposable distribution; Self-similar additive process; Independent increments; Generalized Ornstein–Uhlenbeck-process; First and last passage times; Bessel process; Background driving Lévy process

1. Introduction

The probability distribution of a random variable X_1 is said to be *self-decomposable*, or of *class L*, if for each u with $0 < u < 1$ there is the equality in distribution

$$X_1 \stackrel{\mathrm{d}}{=} uX_1 + \hat{X}_u \tag{1}$$

☆ Research supported in part by N.S.F. Grant DMS-0071448.

* Corresponding author.

E-mail addresses: jeanbl@maths.univ-evry.fr (M. Jeanblanc), pitman@stat.berkeley.edu (J. Pitman).

PII: S0304-4149(02)00098-4

for some random variable $\hat{X}_u$ independent of X_1. See Sato (1991, 1999, Chapter 3), for background and references to the work of Lévy and others on self-decomposable distributions. Here, we are primarily interested in real-valued random variables, but this definition, and the following general discussion and Theorem 1, are also valid for random variables with values in $\mathbb{R}^d$ or a real separable Banach space. In this paper, we discuss the relation between two different representations of self-decomposable distributions in terms of processes with independent increments. Following (Sato, 1999), we call a process $X=(X_t)_{t\geqslant 0}$ an *additive process* if X is stochastically continuous with càdlàg paths, with independent increments and $X_0=0$. An additive process X such that $X_{t+h}-X_t \stackrel{\mathrm{d}}{=} X_h$ for every $t,h\geqslant 0$ is a *Lévy process*.

Wolfe (1982) and Jurek and Vervaat (1983) showed that the distribution of a random variable X_1 is self-decomposable if and only if

$$X_1 \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}Y_s \tag{2}$$

for some Lévy process $Y=(Y_s, s\geqslant 0)$ with $E[\log(1\vee|Y_s|)]<\infty$ for all s. The process Y is called the *background driving Lévy process* (*BDLP*) of X_1. Here, the stochastic integral is understood as a suitable limit as $t\to\infty$ of an integral $\int_0^t$ defined by integration by parts, as in Jurek and Vervaat (1983). Recall that a Lévy process is a semi-martingale, which allows the integral in (2) to be defined as a stochastic integral. Later, Sato (1991, 1999) showed that a distribution is self-decomposable if and only if for any fixed $H>0$ it is the distribution of X_1 for some additive process $(X_r)_{r\geqslant 0}$ which is H-*self-similar*, meaning that for each $c>0$

$$(X_{cr})_{r\geqslant 0} \stackrel{\mathrm{d}}{=} (c^H X_r)_{r\geqslant 0}, \tag{3}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution of processes. In Sato's book (Sato 1999, Sections 16 and 17) these two representations of a self-decomposable distribution are derived by separate analytic arguments. The following result, proved in Section 2 of this paper, allows either representation to be derived immediately from the other:

Theorem 1. *If $(X_r)_{r\geqslant 0}$ is an H-self-similar additive process then the formulas*

$$Y_t^{(-)} := \int_{\mathrm{e}^{-t}}^1 \frac{\mathrm{d}X_r}{r^H} \quad \text{and} \quad Y_t^{(+)} := \int_1^{\mathrm{e}^t} \frac{\mathrm{d}X_r}{r^H} \tag{4}$$

define two independent and identically distributed Lévy processes $(Y_t^{(-)})_{t\geqslant 0}$ and $(Y_t^{(+)})_{t\geqslant 0}$ from which $(X_r)_{r\geqslant 0}$ can be recovered by

$$X_r = \begin{cases} \displaystyle\int_{\log(1/r)}^\infty \mathrm{e}^{-tH}\,\mathrm{d}Y_t^{(-)} & \text{if } 0\leqslant r\leqslant 1, \\ \displaystyle X_1 + \int_0^{\log r} \mathrm{e}^{tH}\,\mathrm{d}Y_t^{(+)} & \text{if } r\geqslant 1. \end{cases} \tag{5}$$

In particular, the BDLP of X_1 is $(Y_{s/H}^{(-)})_{s\geqslant 0}$. Conversely, given a BDLP $(Y_s, s\geqslant 0)$ associated with a self-decomposable distribution of X_1 via (2), *a corresponding H-self-similar additive process can be constructed by* (5) *from two independent copies $(Y_t^{(-)})_{t\geqslant 0}$ and $(Y_t^{(+)})_{t\geqslant 0}$ of $(Y_{tH}, t\geqslant 0)$.*

We note that while a priori the integrals in (4) should be understood as integrals over $[e^{-t}, 1]$ and $[1, e^t]$ defined by integration by parts, formula (5) implies that for every $a > 0$ the process $(X_{au}, u \geqslant 1)$ is a semi-martingale relative to its own filtration. So the integrals in (4) can also be understood in the usual sense of stochastic integration with respect to a semi-martingale.

As observed by Lamperti (1962), the formulae

$$X_r = r^H Z_{\log r}; \quad Z_u = e^{-uH} X_{e^u} \tag{6}$$

set up a one-to-one correspondence between H-self-similar processes $(X_r)_{r>0}$ and stationary processes $(Z_u)_{u\in\mathbb{R}}$. Call $(Z_u)_{u\in\mathbb{R}}$ the *stationary Lamperti transform* of $(X_r)_{r>0}$. On the other hand, given a Lévy process $(Y_t)_{t\geqslant 0}$, a number of authors (Adler et al., 1990; Barndorff-Nielsen and Shephard, 2001a,b; Hadjiev, 1985; Jacod, 1985; Sato, 1999) have studied the associated *Ornstein–Uhlenbeck process driven by* $(Y_t)_{t\geqslant 0}$, *with initial state* U_0 *and parameter* $c \in \mathbb{R}$, that is the solution of

$$U_t = U_0 + Y_t - c \int_0^t U_s \, ds, \tag{7}$$

which is

$$U_t = e^{-ct} \left(U_0 + \int_0^t e^{cs} \, dY_s \right). \tag{8}$$

If we compare the representation (5) of an H-self-similar additive process in terms of the Lévy process $(Y_t^{(+)})_{t\geqslant 0}$, we see that for $r \geqslant 1$

$$r^H Z_{\log r} = Z_0 + \int_0^{\log r} e^{tH} \, dY_t^{(+)} \tag{9}$$

so that, with $r = e^u$ for $u \geqslant 0$

$$Z_u = e^{-uH} \left(Z_0 + \int_0^u e^{tH} \, dY_t^{(+)} \right). \tag{10}$$

Together with similar considerations for $(Z_{-u})_{u\geqslant 0}$, we deduce the following:

Corollary 2. *The stationary Lamperti transform* $(Z_u)_{u\in\mathbb{R}}$ *of an H-self-similar additive process* $(X_r)_{r>0}$ *is such that for the two independent Lévy processes* $(Y_t^{(+)})_{t\geqslant 0}$ *and* $(Y_t^{(-)})_{t\geqslant 0}$ *introduced in Theorem* 1:

(i) $(Z_u)_{u\geqslant 0}$ *is the Ornstein–Uhlenbeck process driven by* $(Y_t^{(+)})_{t\geqslant 0}$ *with initial state* X_1 *and parameter* $c = H$; and
(ii) $(Z_{-u})_{u\geqslant 0}$ *is the Ornstein–Uhlenbeck process driven by* $(-Y_t^{(-)})_{t\geqslant 0}$ *with initial state* X_1 *and parameter* $c = -H$.

Provided the integrals involved are well defined, Theorem 1 and Corollary 2 could even be generalized to an H-self-similar process (X_r) without the assumption of independent increments, to construct Ornstein–Uhlenbeck processes (Z_u) and (Z_{-u}) associated with two processes with stationary increments $(Y_t^{(+)})$ and $(Y_t^{(-)})$ derived from (X_r) via (4).

It is well known that if $(X_r)_{r>0}$ is an H-self-similar Lévy process, then necessarily $H \geqslant \frac{1}{2}$. The process $(X_r)_{r\geqslant 0}$, with $X_0 := 0$, is then commonly known as a *strictly α-stable* Lévy process for $\alpha = 1/H \in (0,2]$. The processes $(Y_t^{(+)})_{t\geqslant 0}$ and $(Y_t^{(-)})_{t\geqslant 0}$ introduced in Theorem 1 are then just two independent copies of $(X_r)_{r\geqslant 0}$. Corollary 2 then reduces to Breiman's (1968) well-known construction via (6) of an Ornstein–Uhlenbeck process driven by a copy of $(X_r)_{r\geqslant 0}$, as indicated by Sato (1999, E 18.17) and Bertoin (1996, VIII.5 Exercise 4). For some applications to the windings of a stable Lévy process in two dimensions, see Bertoin and Werner (1996).

Our formulation of Theorem 1 was suggested by consideration of the self-similar additive processes derived from the first and last passage times of a Bessel process $(R_t, t \geqslant 0)$ with positive real dimension $\delta = 2(1+v) > 0$, started at $R_0 = 0$. See Borodin and Salminen (1996), Getoor (1979), Itô and McKean (1965), Kent (1978), Revuz and Yor (1999) for background. It is well known Lamperti (1972) that a Bessel process is $\frac{1}{2}$-self-similar and hence that the first and last passage times

$$T_r = \inf\{t : R_t = r\} \text{ and } \Lambda_r = \sup\{t : R_t = r\} \tag{11}$$

define processes $(T_r)_{r\geqslant 0}$ and $(\Lambda_r)_{r\geqslant 0}$ which are two-self-similar. Sato (1999, Example 16.4) discusses the last passage process (Λ_r) as an example of a two-self-similar additive process, for integer dimensions δ with $\delta \geqslant 3$. If $-1 < v \leqslant 0$, that is $0 < \delta \leqslant 2$, the Bessel process is recurrent, which implies $\Lambda_r = \infty$ a.s.. So we consider the last passage process only in the transient case $v > 0$; then $0 < \Lambda_r < \infty$ a.s. because $R_t \to \infty$ a.s. as $t \to \infty$. Due to the strong Markov property of (R_t) at time T_r, and the last exit decomposition of (R_t) at time Λ_r, each of the processes (T_r) and (Λ_r) has independent increments. In Section 3.2, we recall some known descriptions of the laws of T_r and Λ_r, and deduce corresponding descriptions of their BDLP's from (2).

In Section 3.1, we derive an alternative representation of the BDLP's associated with the distributions of T_1 and Λ_1. This involves the increasing process $(L_t, t \geqslant 0)$ of local time of the Bessel process R at level 1, that is

$$L_t := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1(|R_s - 1| \leqslant \varepsilon)\, \mathrm{d}s, \tag{12}$$

where the limit exists and defines a continuous increasing process almost surely Revuz and Yor (1999, VI). Let $(\tau_\ell, \ell \geqslant 0)$ denote the inverse local time process

$$\tau_\ell := \inf\{t : L_t > \ell\}.$$

It is known Pitman and Yor (1981, (9.s1)) that

$$P(\tau_\ell < \infty) = \begin{cases} 1 & \text{if } -1 < v \leqslant 0 \text{ (i.e. } 0 < \delta \leqslant 2), \\ \mathrm{e}^{-v\ell} & \text{if } v > 0 \text{ (i.e. } \delta > 2). \end{cases} \tag{13}$$

Theorem 3. *Let T_1, Λ_1 and τ_ℓ be defined as above in terms of the Bessel process $(R_t)_{t\geqslant 0}$ of index $v > -1$. Let $(Y_s^T)_{s\geqslant 0}$ denote the BDLP of T_1, and for $v > 0$ let $(Y_s^\Lambda)_{s\geqslant 0}$ denote the BDLP of Λ_1, each of which can be constructed as in Theorem 1 from the path of $(T_r, 0 \leqslant r \leqslant 1)$ or of $(\Lambda_r, 0 \leqslant r \leqslant 1)$, as the case may be. Then for* *each $\ell > 0$ and $\nu > -1$ there is the equality in distribution of Lévy processes*

$$(Y_s^T)_{0\leqslant s\leqslant \ell} \stackrel{d}{=} \left(\left(\int_{T_1}^{\tau_s} 1(R_t \leqslant 1)\, dt \right)_{0\leqslant s\leqslant \ell} \,\middle|\, \tau_\ell < \infty \right), \tag{14}$$

while for each $\ell > 0$ and $\nu > 0$

$$(Y_s^\Lambda)_{0\leqslant s\leqslant \ell} \stackrel{d}{=} \left(\left(\int_{0}^{\tau_s} 1(R_t > 1)\, dt \right)_{0\leqslant s\leqslant \ell} \,\middle|\, \tau_\ell < \infty \right). \tag{15}$$

According to an instance of Williams' time reversal theorem Williams (1974), Sharpe (1980), Pitman and Yor (1981) for $\nu > 0$ the process $(R_{\Lambda_1 - t}, 0 \leqslant t \leqslant \Lambda_1)$ is a Bessel process of index $-\nu$ started at 1 and stopped when it first hits 0. This allows Theorems 1 and 3 to be combined as follows:

Corollary 4. *For a recurrent Bessel process R of index $\nu \in (-1, 0)$ there are the following two equalities in distribution of Lévy processes*:

$$\left(\int_{T_1}^{\tau_\ell} 1(R_t \leqslant 1)\, dt, \ell \geqslant 0 \right) \stackrel{d}{=} \left(\int_{e^{-\ell}}^{1} \frac{dT_u}{u^2}, \ell \geqslant 0 \right), \tag{16}$$

$$\left(\int_{0}^{\tau_\ell} 1(R_t > 1)\, dt, \ell \geqslant 0 \right) \stackrel{d}{=} \left(\int_{e^{-\ell}}^{1} \frac{d\hat{\Lambda}_u}{u^2}, \ell \geqslant 0 \right), \tag{17}$$

where $\hat{\Lambda}_u$ is the last passage time at u for the transient Bessel process $\hat{R}$ of index $-\nu \in (0, 1)$. Consequently, there is the identity in distribution of additive processes

$$\left(\int_{T_1}^{\tau_\ell} e^{-L_s}\, ds, \ell \geqslant 0 \right) \stackrel{d}{=} \left(T_1 - T_{e^{-\ell/2}} + \hat{\Lambda}_1 - \hat{\Lambda}_{e^{-\ell/2}}, \ell \geqslant 0 \right), \tag{18}$$

where on the right-hand side it is assumed that the processes (T_r) and $(\hat{\Lambda}_r)$ are independent.

2. Proof of Theorem 1

It is obvious that the processes $(Y_t^{(-)})_{t\geqslant 0}$ and $(Y_t^{(+)})_{t\geqslant 0}$ are independent, and that each of these processes has independent increments. So to show that $(Y_t^{(-)})_{t\geqslant 0}$ is a Lévy process, it just remains to check that $Y_{t+h}^{(-)} - Y_t^{(-)} \stackrel{d}{=} Y_h^{(-)}$ for $t, h \geqslant 0$. But

$$Y_{t+h}^{(-)} - Y_t^{(-)} = \int_{e^{-(t+h)}}^{e^{-t}} \frac{dX_u}{u^H} = \int_{e^{-h}}^{1} \frac{d_v(X_{e^{-t}v})}{(e^{-t}v)^H} \stackrel{d}{=} \int_{e^{-h}}^{1} \frac{dX_v}{v^H} = Y_h^{(-)},$$

where the equality in distribution appeals to the self-similarity (3) of X. The corresponding result for $(Y_t^{(+)})$ can be obtained by repetition of the same calculation, or by writing

$$Y_t^{(+)} = \int_{e^{-t}}^{1} \frac{d(-X_{1/v})}{v^{-H}}$$

and appealing to the previous case with X_v replaced by $-X_{1/v}$. Since both $(Y_t^{(+)})$ and $(Y_t^{(-)})$ have independent increments, to show they are identically distributed it suffices to show that they have the same one-dimensional distributions. But for each fixed t

$$\int_{e^{-t}}^{1} \frac{dX_u}{u^H} = \int_1^{e^t} \frac{d_v X_{e^{-t}v}}{(e^{-t}v)^H} \overset{d}{=} \int_1^{e^t} \frac{dX_v}{v^H}$$

by another application of the self-similarity of X. To obtain (5), write e.g.

$$Y_t^{(-)} = -\int_0^t \frac{d_v X_{e^{-v}}}{e^{-vH}}$$

so that

$$\int_0^\infty e^{-vH}\, dY_v^{(-)} = -\int_0^\infty d_v X_{e^{-v}} = X_1.$$

This is (5) for $r = 1$ and the general case of (5) is obtained by a similar calculation. Finally, the converse assertion is easily checked.

3. Application to Bessel processes

It is known Jurek (2001, Proposition 3) and easily verified that if $(Y_s, s \geqslant 0)$ is an increasing Lévy process (*subordinator*) with $E[\log(1 \vee Y_s)] < \infty$ for all s and

$$X_1 \overset{d}{=} \int_0^\infty e^{-s}\, dY_s$$

then the distribution of X_1 determines that of Y_s for each $s > 0$ by the formula

$$E[\exp(-\lambda Y_s)] = \exp\left(s\lambda \frac{d}{d\lambda} \ln E[\exp(-\lambda X_1)] \right). \tag{19}$$

3.1. Proof of Theorem 3

By the general theory of one-dimensional diffusions Itô and McKean (1965, 4.6), Borodin and Salminen (1996, II.10), Rogers and Williams (1987, V.50), for $r > 0$ the distribution of the first passage time T_r of the Bessel process $(R_t)_{t \geqslant 0}$ started at $R_0 = 0$ is determined by the Laplace transform

$$E(e^{-\lambda T_r}) = \frac{1}{\Phi_{\lambda\uparrow}(r)}, \tag{20}$$

where $\Phi_{\lambda\uparrow}$ is the unique increasing solution Φ of the differential equation $\mathscr{G}\Phi = \lambda\Phi$, with $\mathscr{G}$ the infinitesimal generator of the Bessel diffusion, and Φ subject to appropriate boundary conditions. Ciesielski and Taylor (1962) and Kent (1978) found the expression of $\Phi_{\lambda\uparrow}$ in terms of Bessel functions which can be read from (20) and the table in the next section. But this formula is not needed for the present argument. All that is required here is the immediate consequence of the two-self-similarity of $(T_r)_{r\geqslant 0}$ and (20) that

$$\Phi_{\lambda\uparrow}(r) = \phi(\sqrt{2\lambda}r) \tag{21}$$

for some differentiable function ϕ. For $(Y_s^T)_{s\geqslant 0}$ the BDLP of T_1, we obtain from (19) the formula

$$E[\exp(-\lambda Y_s^T)] = \exp\left[-s\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\phi(\sqrt{2\lambda})\right]. \tag{22}$$

On the other hand, we also know from the theory of one-dimensional diffusions Itô and McKean (1965, 6.2), Pitman and Yor (1981, (9.8)), Pitman and Yor (2001), that the process on the right-hand side of (14) is a Lévy process with, for $0\leqslant s\leqslant \ell$,

$$E\left[\exp\left(-\lambda\int_{T_1}^{\tau_s}1(R_t\leqslant 1)\,\mathrm{d}t\right)\Bigg|\,\tau_\ell<\infty\right] = \exp\left(-\frac{s}{2}\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1}\frac{\phi(\sqrt{2\lambda}r)}{\phi(\sqrt{2\lambda})}\right). \tag{23}$$

But since

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\log\phi(\sqrt{2\lambda}r) = \frac{1}{2}\sqrt{2\lambda}r\frac{\phi'(\sqrt{2\lambda}r)}{\phi(\sqrt{2\lambda}r)} = \frac{r}{2\phi(\sqrt{2\lambda}r)}\frac{\mathrm{d}}{\mathrm{d}r}\phi(\sqrt{2\lambda}r)$$

the right-hand sides of (22) and (23) are identical, and conclusion (14) follows. The proof of (15) for $v>0$ is quite similar. The Laplace transform of Λ_r was found by Getoor (1979), as indicated in the table of the next section, while that of $\int_0^{\tau_s}1(R_t\leqslant 1)\,\mathrm{d}t$ given $\tau_\ell<\infty$ for $0\leqslant s\leqslant\ell$ can be read from Pitman and Yor (1981, (9.s7)) or Borodin and Salminen (1996, 6.4.4.1). See Pitman and Yor (2001) for further discussion.

3.2. Explicit formulae

Recall that the Lévy measure v_X of an infinitely divisible non-negative random variable X associated with a subordinator with no drift component is determined by the formula

$$E[\exp(-\lambda X)] = \exp\left[-\int_0^\infty(1-\mathrm{e}^{-\lambda x})v_X(dx)\right]$$

for all $\lambda>0$, or again by

$$-\frac{\mathrm{d}}{\mathrm{d}\lambda}\log E[\exp(-\lambda X)] = \int_0^\infty x\mathrm{e}^{-\lambda x}v_X(\mathrm{d}x).$$

Hence from (19), if $X_1\stackrel{\mathrm{d}}{=}\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}Y_s$ for $(Y_s, s\geqslant 0)$ a subordinator without drift, the Lévy measures of X_1 and Y_1 are related by

$$xv_{X_1}(dx) = v_{Y_1}[x,\infty)\,\mathrm{d}x. \tag{24}$$

For a detailed case study, see Knight (2001, p. 593). In particular, for the random variables $X_1 = T_1$ and $X_1=\Lambda_1$ defined by the first and last passage times of a Bessel process, we find from the sources cited in the previous proof that the distributions and Lévy measures of X_1 and the associated BDLP's are as presented in the following table. Here we employ the usual Bessel functions I_ν, K_ν, J_ν and Y_ν, as in Ismail (1977), Kent (1978), Pitman and Yor (1981), and the auxiliary functions

$$k_{\nu-1}(x) := \frac{1}{\pi^2}\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-tx}(J_\nu^2 + Y_\nu^2)^{-1}(\sqrt{2t}),$$

$$\Sigma_\nu(x) := \sum_{n=1}^{\infty} \exp(-j_{\nu,n}^2 x),$$

where $(j_{\nu,n}, n=1,2,\ldots)$ is the increasing sequence of the positive zeros of the Bessel function of the first kind J_ν. The formulae involving $k_{\nu-1}$ and Σ_ν can be read from Ismail (1977). See also Donati-Martin and Yor (1997, p. 1055).

X_1	$E\left[\exp\left(-\frac{\alpha^2}{2}X_1\right)\right]$	$E\left[\exp\left(-\frac{\alpha^2}{2}Y_1\right)\right]$	$x\nu_{X_1}(\mathrm{d}x)/\mathrm{d}x$	$\nu_{Y_1}(\mathrm{d}y)/\mathrm{d}y$
T_1 $(\nu > -1)$	$\frac{\alpha^\nu}{2^\nu \Gamma(\nu+1) I_\nu(\alpha)}$	$\exp\left(-\frac{\alpha I_{\nu+1}(\alpha)}{2I_\nu(\alpha)}\right)$	$\Sigma_\nu(x/2)$	$-\frac{1}{2}\Sigma'_\nu(y/2)$
Λ_1 $(\nu > 0)$	$\frac{2}{\Gamma(\nu)}\left(\frac{\alpha}{2}\right)^\nu K_\nu(\alpha)$	$\exp\left(-\frac{\alpha K_{\nu-1}(\alpha)}{2K_\nu(\alpha)}\right)$	$k_{\nu-1}(x)$	$-k'_{\nu-1}(y)$

In the particular case $\nu=\frac{1}{2}$ (that is for a three-dimensional Bessel process), the results simplify as indicated in the next table. In this case the process $(\Lambda_r, r \geqslant 0)$ has stationary increments, and is a stable subordinator of index $\frac{1}{2}$, due to the close connection between the three-dimensional Bessel process and one-dimensional Brownian motion (Pitman, 1975). See also Biane et al. (2001) for further developments related to the distribution of T_r in this case.

X_1	$E\left[\exp\left(-\frac{\alpha^2}{2}X_1\right)\right]$	$E\left[\exp\left(-\frac{\alpha^2}{2}Y_1\right)\right]$	$x\nu_{X_1}(\mathrm{d}x)/\mathrm{d}x$	$\nu_{Y_1}(\mathrm{d}y)/\mathrm{d}y$
T_1 $(\nu = \frac{1}{2})$	$\frac{\alpha}{\sinh\alpha}$	$\exp\left(-\frac{1}{2}(\alpha\coth\alpha - 1)\right)$	$\sum_{n=1}^\infty \mathrm{e}^{-n^2\pi^2 x/2}$	$\sum_n \frac{n^2\pi^2}{2}\mathrm{e}^{-n^2\pi^2 y/2}$
Λ_1 $(\nu = \frac{1}{2})$	$\mathrm{e}^{-\alpha}$	$\mathrm{e}^{-\alpha/2}$	$\frac{1}{\sqrt{2\pi x}}$	$\frac{1}{2y\sqrt{2\pi y}}$

References

Adler, R.J., Cambanis, S., Samorodnitsky, G., 1990. On stable Markov processes. Stochastic Process. Appl. 34, 1–17.

Barndorff-Nielsen, O.E., Shephard, N., 2001a. Modelling by Lévy processes for financial econometrics. In: Barndorrf-Nielsen, O.E., Mikosch, T., Resnick, S.I. (Eds.), Lévy Processes. Birkhäuser, Boston, MA, pp. 283–318.

Barndorff-Nielsen, O.E., Shephard, N., 2001b. Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. J. R. Statist. Soc. Ser. B Stat. Methodol. 63 (2), 167–241.

Bertoin, J., 1996. Lévy Processes. Cambridge University Press, Cambridge. Cambridge Tracts in Math. 126.

Bertoin, J., Werner, W., 1996. Stable windings. Ann. Probab. 24 (3), 1269–1279.

Biane, P., Pitman, J., Yor, M., 2001. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. Bull. Amer. Math. Soc. 38, 435–465.

Borodin, A.N., Salminen, P., 1996. Handbook of Brownian motion—Facts and Formulae. Birkhäuser, Boston, MA.

Breiman, L., 1968. A delicate law of the iterated logarithm for non-decreasing stable processes. Ann. Math. Statist. 39, 1818–1824.

Ciesielski, Z., Taylor, S.J., 1962. First passage times and sojourn density for Brownian motion in space and the exact Hausdorff measure of the sample path. Trans. Amer. Math. Soc. 103, 434–450.

Donati-Martin, C., Yor, M., 1997. Some Brownian functionals and their laws. Ann. Probab. 25 (3), 1011–1058.

Getoor, R.K., 1979. The Brownian escape process. Ann. Probab. 7, 864–867.

Hadjiev, D.I., 1985. The first passage problem for generalized Ornstein–Uhlenbeck processes with nonpositive jumps. In: Azéma, J., Yor, M. (Eds.), Séminaire de probabilités, XIX, 1983/84, Springer, Berlin, pp. 80–90.

Ismail, M.E.H., 1977. Integral representations and complete monotonicity of various quotients of Bessel functions. Canad. J. Math. 29 (6), 1198–1207.

Itô, K., McKean, H.P., 1965. Diffusion Processes and their Sample Paths. Springer, Berlin.

Jacod, J., 1985. Grossissement de filtration et processus d'Ornstein–Uhlenbeck généralisé. In: Jeulin, Th., Yor, M. (Eds.), Grossissements de filtrations: exemples et applications. Séminaire de Calcul Stochastique, Paris 1982/83, Lecture Notes in Mathematics, Vol. 1118. Springer, Berlin, pp. 36–44.

Jurek, Z.J., 2001. Remarks on the selfdecomposability and new examples. Demonstratio Math. 34 (2), 241–250.

Jurek, Z.J., Vervaat, W., 1983. An integral representation for self-decomposable Banach space valued random variables. Z. Wahrsch. Verw. Gebiete. 62 (2), 247–262.

Kent, J., 1978. Some probabilistic properties of Bessel functions. Ann. Probab. 6, 760–770.

Knight, F.B., 2001. On the path of an inert object impinged on one side by a Brownian particle. Probab. Theory Related Fields 121, 577–598.

Lamperti, J., 1962. Semi-stable stochastic processes. Trans. Amer. Math. Soc. 104, 62–78.

Lamperti, J., 1972. Semi-stable Markov processes. I. Z. Wahrsch. Verw. Gebiete 22, 205–225.

Pitman, J., 1975. One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Probab. 7, 511–526.

Pitman, J., Yor, M., 1981. Bessel processes and infinitely divisible laws. In: Williams, D. (Ed.), Stochastic Integrals, Lecture Notes in Mathematics, Vol. 851, Springer, Berlin, pp. 285–370.

Pitman, J., Yor, M. 2001. Hitting, occupation, and inverse local times of one-dimensional diffusions: martingale and excursion approaches. Technical Report 607, Department of Statistics, U.C. Berkeley, 2001.

Revuz, D., Yor, M., 1999. Continuous Martingales and Brownian Motion, 3rd Edition. Springer, Berlin.

Rogers, L.C.G., Williams, D., 1987. Diffusions, Markov Processes and Martingales, Vol. II: Itô Calculus. Wiley, New York.

Sato, K., 1991. Self-similar processes with independent increments. Probab. Theory Related Fields 89 (3), 285–300.

Sato, K., 1999. Lévy processes and infinitely divisible distributions. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, revised by the author.

Sharpe, M.J., 1980. Some transformations of diffusions by time reversal. Ann. Probab. 8, 1157–1162.

Williams, D., 1974. Path decomposition and continuity of local time for one dimensional diffusions I. Proc. London Math. Soc. 28 (3), 738–768.

Wolfe, S.J., 1982. On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. Stochastic Process. Appl. 12 (3), 301–312.