Stat 155 Game theory, Yuval Peres	**Fall 2004**

Lectures 7,8

Definition 1 *The sum of two combinatorial games G_1 and G_2 is that game G where, for any move, a player may choose in which of the games G_1 and G_2 to play. The terminal position in G is (t_1, t_2), where t_I is terminal in G_i for $i \in \{1, 2\}$. We will write $G = G_1 + G_2$.*

We say that two pairs $(G_i, x_i) : i \in \{1, 2\}$ of a game and a starting position are equivalent if the game $\{G_1 + G_2, (x_1, x_2)$ is a P-position. We will see that this notion of 'equivalent' games defines an equivalence relation.
Optional exercise: find a direct proof of transitivity for equivalent games.
As an example, we see that the nim position $(1, 3, 6)$ is equivalent to the nim position (7), because the nim-sim of the sum game $(1, 3, 6, 7)$ is zero.

Definition 2 *A game is said to be progressively bounded if, for any starting position x, the game must finish within some finite number $B(x)$ of moves.*

Theorem 1 (Sprague-Grundy theorem) . *Every progressively bounded combinatorial game G in normal play is equivalent to a nim pile, possibly empty. For any position x in G, the size $g(x)$ of this pile can be defined as follows: for terminal positions t, $g(t) = 0$, while for other positions,*

$$g(x) = mex\{g(y) : x \to y \text{ is a legal move}\},$$

where $mex(S) = \min\{n \geq 0 : n \notin S$, for a finite set $S \subseteq \{0, 1, \ldots\}$. (This is short for 'minimal excluded value').

We illustrate the theorem with an example:
Example: a game where a position consists of a pile of chips, and a legal move is to remove $1, 2$ or 3 chips. The following table shows the first few values of the Sprague-Grundy function for this game:

x	0	1	2	3	4	5	6
g(x)	0	1	2	3	0	1	2 .

That is, $g(2) = mex\{0, 1\} = 2$, $g(3) = mex\{0, 1, 2\} = 3$, and $g(4) = mex\{1, 2, 3\} = 0$. In general for this example, $g(x) = x \bmod 4$. This means that the P-positions are the naturals that are divisible by four.
Example: a game consisting of a pile of chips. A legal move from a position with n chips is to remove any positive number of chips strictly smaller than $n/2 + 1$. Here, the first few values of the Sprague-Grundy function are:

x	0	1	2	3	4
g(x)	0	1	0	2	1 .

Definition 3 *The substraction game with substraction set $\{a_1, \ldots, a_m\}$ is that game in which a position consists of a pile of chips, in which a legal move is to remove from the pile a_i chips, for some $i \in \{1, \ldots, m\}$.*

Example: Lasker's game. A position is finite collection of piles of chips. A player may remove chips from a given pile, or he may not remove chips, but instead break one pile into two, in any way that he pleases. To see that this game is progressively bounded, note that, if we define

$$B(x_1, \ldots, x_k) = \sum_{i=1}^{k} (2x_i - 1),$$

then the sum equals the total number of chips and gaps between chips in a position $(x_1, \ldots, x_k)$. It drops if the player removes a chip, but also if he breaks a pile, because, in that case, the number of gaps between chips drops by one. Hence, $B(x_1, \ldots, x_k)$ is an upper bound on the number of steps that the game will take to finish from the starting position $(x_1, \ldots, x_k)$.
The proof of the Sprague-Grundy theorem will follow from

Theorem 2 (Sum theorem) *If (G_1, x_1) and (G_2, x_2) are a pair of games and initial starting positions within those games, then, for the sum game $G = G_1 + G_2$, we have that*

$$g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2),$$

where g, g_1, g_2 respectively denote the Sprague-Grundy functions for the games G, G_1 and G_2.

Example: Wightoff sum. In this game, we have two piles. Legal moves moves are those of nim, but with the exception that it is also allowed to remove equal numbers of chips from each of the piles in a single move. This stops the positions $\{(n, n) : n \in \mathbb{N}\}$ from being P-positions. We will see that this game has an interesting structure.
Proof of sum theorem. We define $B(x_1, x_2)$ to be the maximum number of moves in which the game $(G, (x_1, x_2))$ will end. Note that this quantity is not merely an upper bound on the number of moves, it is the maximum. We will prove the statement by an induction on $B(x_1, x_2) = B(x_1) + B(x_2)$. Specifically, the inductive hypothesis at $n \in \mathbb{N}$ asserts that, for positions (x_1, x_2) in G for which $B(x_1, x_2) = n$,

$$g(x_1, x_2) = g_1(x_1) \oplus g_2(x_2). \tag{1}$$

If at least one of x_1 and x_2 is terminal, then (1) is clear: indeed, suppose that x_1 is terminal and that x_2 is not. In this case, the game G may only be played in the second co-ordinate, in its G_2 guise. Suppose then that neither of the positions x_1 and x_2 are terminal ones. Suppose also the inductive hypothesis for values strictly less than $B(x_1, x_2)$: that is, for each pair (y_1, y_2) of positions in G_1 and G_2, $g(y_1, y_2) = g_1(y_1) + g_2(y_2)$. We write in binary form:

$$g_1(x_1) = n_1 = n_1^{(m)} n_1^{(m-1)} \cdots n_1^{(0)}$$
$$g_2(x_2) = n_2 = n_2^{(m)} n_2^{(m-1)} \cdots n_2^{(0)},$$

so that, for example, $n_1 = \sum_{j=0}^{m} n_1^{(j)} 2^j$. We know that

$$\begin{aligned} g(x_1, x_2) &= mex\{g(y_1, y_2) : (x_1, x_2) \to (y_1, y_2) \text{ a legal move in } G\} \\ &= mex(A) \end{aligned}$$

where $A := \{g_1(y_1) \oplus g_2(y_2) : (x_1, x_2) \to (y_1, y_2) \text{ a legal move in } G\}$. The second inequality here follows from the inductive hypothesis, because we know that $B(y_1, y_2) < B(x_1, x_2)$ (the maximum number of moves left in the game G must fall with each move). Writing $s = n_1 \oplus n_2$, we must show that

$$\begin{aligned} &\text{(a):} \quad s \notin A \\ &\text{(b):} \quad t \in \mathbb{N}, 0 \le t < s \text{ implies that } t \in A. \end{aligned}$$

These two statements will imply that $mex(A) = s$, which yields (1).
Deriving (a): If $(x_1, x_2) \to (y_1, y_2)$ is a legal move in G, then either $y_1 < x_1$ and $x_2 \to y_2$ is a legal move in G_2, or $y_2 = x_2$ and x_1 to y_1 is a legal move in G_1. In the first case, we have that

$$g_1(y_1) \oplus g_2(y_2) = g_1(x_1) \oplus g_2(y_2) \neq g_1(x_1) \oplus g_2(x_2) :$$

for otherwise, $g_2(y_2) = g_1(x_1) \oplus g_1(x_1) \oplus g_2(y_2) = g_1(x_1) \oplus g_1(x_1) \oplus g_2(x_2) = g_2(x_2)$. This however is impossible, by the definition of the Sprague-Grundy function g_2.
Deriving (b): we take $t < s$, and observe that if $t^{(l)}$ is the leftmost digit of t that differs from the corresponding one of s, then $t^{(l)} = 0$ and $s^{(l)} = 1$. Since $s^{(l)} = n_1^{(l)} + n_2^{(l)}$, we may suppose that $n_1^{(l)} = 1$. We want to move in G_1 from x_1, for which $g_1(x_1) = n_1$, to a position y_1 for which

$$g_1(y_1) = n_1 \oplus s \oplus t. \tag{2}$$

Then we will have that

$$g_1(x_1) \oplus g_2(x_2) = n_1 \oplus s \oplus t \oplus n_2 = n_1 \oplus n_2 \oplus s \oplus t = s \oplus s \oplus t = t,$$

so that $g_1(x_1) \oplus g_2(x_2) \in A$ implies that $t \in A$, as we sought. But why is (2) possible? Well, note that

$$n_1 \oplus s \oplus t < n_1 : \tag{3}$$

indeed, the leftmost digit at which $n_1 \oplus s \oplus t$ differs from n_1 is l, at which at has a 1. Since number whose binary expansion contains a 1 in place l exceeds any number whose expansion has no ones in places l or higher, we see that (3) is valid. The definition of $g_1(x_1)$ now implies that there exists a legal move from x_1 to $n_1 \oplus s \oplus t$. This finishes case (b) and the proof of the theorem. $\square$