Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 29,30,31

Proof of Brouwer's fixed point theorem (from the no retraction theorem): Recall that we are given a continuous map $T : K \to K$, with K a closed, bounded and convex set. Suppose that T has no fixed point. Define $F : K \to \partial K$ as follows. For each $x \in K$, we draw a line segment from $T(x)$ through x until it meets ∂K. We set $F(x)$ equal to this point of intersection. (note that, in the case that $T(x) \in \partial K$, we set $F(x)$ equal that point of intersection of the line segment with ∂K which is not equal to $T(x)$). In the case of the domain $K = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$, the map F may be written explicitly in terms of T. With some checking, it follows that $F : k \to \partial K$ is continuous. Thus, F is a retraction of K - but none exists, by the no retraction theorem. This contradiction establishes the theorem. □

Potential games: We now discuss a collection of games called *potential games*. These are k-player general sum games that have a special feature: let $F_i(s_1, s_2, \ldots, s_k)$ denotes the payoff to player i if the players respectively adopt the pure strategies $s_1, s_2, \ldots, s_k$. In a potential game, we suppose that there is a function $\Psi : S_1 \times \ldots S_k \to \mathbb{R}$, defined on the product of the players' strategy spaces, and such that

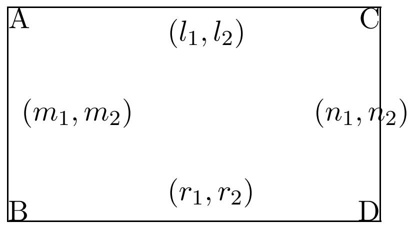
$$\begin{aligned} & F_i(s_1, \ldots, s_{i-1}, \tilde{s}_i, s_{i+1}, \ldots, s_k) - F_i(s_1, \ldots, s_k) \\ = \; & \psi(s_1, \ldots, s_{i-1}, \tilde{s}_i, s_{i+1}, \ldots, s_k) - \psi v(s_1, \ldots, s_k). \end{aligned} \quad (1)$$

We call the function $\psi : S_1 \times \ldots S_k \to \mathbb{R}$ the 'potential' function associated with the game. We have the following result:

Theorem 1 (Shmeidler-Shapley) *Every potential game has a Nash equilibrium in pure strategies.*

Proof: this appears after the following example.

Example of a potential game: a simultaneous congestion game: In this sort of game, the cost of using each road depends on the number of users of the road. For the road AC, it is l_i if there are i users, for $i \in \{1, 2\}$, in the case of the game depicted in the figure. Note that the cost paid by a given driver depends only on the number of users, not on which user she is.



More generally, we may define $\mathbb{R}$-valued map C on the product space of the road-index set and the set $\{1, \ldots, k\}$, so that $C(j, u_j)$ is equal to the cost incurred by any driver using road j in the case that the total number of drivers using this road is equal to u_j. Note that the vector $s = (s_1, s_2, \ldots, s_k)$ determines the usage of each road. That is, it determines $u_i(s)$ for each $i \in \{1, \ldots k\}$, where

$$u_i(s) = \Big|\Big\{ j \in \{1, \ldots, k\} : \text{ player } j \text{ uses road } i \text{ under strategy } s_j \Big\}\Big|,$$

for $i \in \{1, \ldots, R\}$ (with R being the number of roads.)

In the case of the game depicted in the figure, we suppose that two drivers, I and II, have to travel from A to D, or from B to C, respectively.

In general, we set

$$\psi(s_1, \ldots, s_k) = -\sum_{r=1}^{R} \sum_{l=1}^{u_r(s)} c(r, l).$$

We claim that ψ is a potential function for such a game. We show why this is so in the specific example. Suppose that driver 1, using roads 1 and 2, makes a decision to use roads 3 and 4 instead. What will be the effect on her cost? The answer is a change of

$$\Big(c(3, u_3(s) + 1) + c_3(4, u_4(s) + 1) \Big) - \Big(c(1, u_1(s)) + c(2, u_2(s)) \Big).$$

How did the potential function change as a result of her decision? We find that, in fact,

$$\psi(s) - \psi(\tilde{s}) = c(3, u_3(s) + 1) + c_3(4, u_4(s) + 1) - c(1, u_1(s)) - c(2, u_2(s))$$

where $\tilde{s}$ denotes the new joint strategy (after her decision), and s denotes the previous one. Noting that payoff is the negation of cost, we find that the change in payoff is equal to the change in the value of ψ. To show that ψ is indeed a potential function, it would be necessary to reprise this argument in the case of a general change in strategy by one of the players.

Proof of Shmeidler-Shapley theorem: Choose s that maximizes $\psi(s)$. Note that the expression in (1) is at most zero, for any $i \in \{1, \ldots, k\}$ and any choice of $\tilde{s}_i$. This implies that s is a Nash equilibrium. $\square$

New topics: We will discuss two further topics in this class. One is that of finding evolutionally stable strategies: which Nash equilibria arise naturally? We will discuss only some examples, of which the game of 'hawks and doves' on a recent homework will be one. The topic we now turn to is that of games involving coalitions.

Coalitions and Shapley value: Suppose given a group of $k > 2$ players. Each seeks a part of a given prize, but may achieve that prize only by joining forces with some of the other players. The players have varying influence - but how much power does each have? This is a pretty general summary. Let us look at an example, mentioned at the beginning of the semester:

the glove market: A customer enters a shop seeking to buy a pair of gloves. In the store are the three players. Player 1 has a left glove and players 2 and 3 each have a right glove. The customer will make a payment of 100 dollars for a pair of gloves. In their negotiations prior to the purchase, how much can each player realistically demand of the payment made by the customer? To resolve this question, we introduce a 'characteristic' function v, defined on subsets of the player set. By an abuse of notation, we will write v_{12} in place of $v_{\{12\}}$, and so on. The function v will take the values zero or one, and will take the value one precisely when the subset of players in question are able between them to affect their aim. In this case, this means that the subset includes one player with a left glove, and one with a right one - so that, between them, they may offer the customer a pair of gloves. Thus, the values are

$$v_{123} = v_{23} = v_{13} = 1,$$

with v taking the value zero on every other subset of $\{1, 2, 3\}$. Note that a $\{0, 1\}$-valued monotone function always satisfies superadditivity.

Shapley was searching for a value function ψ_i, $i \in \{1, \ldots k\}$, such that $\psi_i(v)$ would be the 'arbitration value' (now called Shapley value) for player i in a game whose characteristic function is v. Shapley analysed this problem by introducing the following axioms:

$$\text{Symmetry} : \text{that } \psi_i(v) = \psi_j(v) \text{ if } v(S \cup \{i\}) = v(S \cup \{j\}) \text{ for all } S \text{ with } i, j \notin S$$

$$\text{No power / no value} : \text{if } v(S \cup \{i\}) = V(S) \text{ for all } S, \text{ then } \psi_i(v) = 0.$$

$$\text{Additivity} : \psi_i(v + u) = \psi(v) + \psi(u),$$

$$\text{Efficiency} : \sum_{i=1}^n \psi_i(v) = v(\{1, \ldots, n\}).$$

The third axiom is the most problematic: it assumes that there is no effect on players of earlier games on later ones.

Theorem 2 (Shapley) *There exists a unique solution for ψ.*

A simpler example first: To each subset $S \subseteq \{1, \ldots, n\}$, consider the S-veto game, in which the effective coalitions are those that contain each member of S. It has characteristics function w_S, given by $w_S(T) = 1$ if and only if $S \subseteq T$. In this case, the Shapley value is given by

$$\psi_i(w_S) = \frac{1}{|S|} \text{ if } i \in S,$$

with $\psi_i(w_S) = 0$ if $i \notin S$. Moreover, we have that $\psi_i(cW_s) = c\psi_i(W_s)$ for each $c \in [0, \infty)$.

Note that the glove market game has the same payoffs as $w_{23} + w_{13}$, except for the case of the set $\{1, 2, 3\}$. In fact, we have that

$$w_{23} + w_{13} = v + w_{123}.$$

In particular, we have that $\psi_i(w_{23}) + \psi_i(w_{12}) = \psi_i(v) + \psi_i(w_{123})$. If $i = 1$, then

$$0 + 1/2 = \psi_1(v) + 1/3,$$

while, if $i = 3$, then

$$1/2 + 1/2 + \psi_3(v) + 1/3.$$

Hence, $\psi_3(v) = 2/3$, and $\psi_1(r) = \psi_2(r) = 1/6$. This means that player I has two-thirds of the arbitration value, while player II and III have one-third between them.

Example: the four stockholders. Four people own stock in ACME. Player i holds i units of stock, for each $i \in \{1, 2, 3, 4\}$. Six shares are needed to pass a resolution at the board meeting. How much is the position of each player worth in the sense of Shapley value? Note that

$$1 = v_{1234} = v_{24} = v_{34},$$

with $v = 1$ on any 3-tuple, and $v = 0$ in each other case.

We will assume that the value v may be written in the form

$$v = \sum_{\emptyset \neq S} c_S w_S.$$

Later, we will see that there always exists such a way of writing v. For now, however, we assume this, and compute the coefficients c_S. To do so, note that

$$0 = v_1 = c_1$$

(we write c_1 for $c_{\{1\}}$), and so on. Similarly,

$$0 = c_2 = c_3 = c_4.$$

Also,

$$0 = v_{12} = c_1 + c_2 + c_{12},$$

implying that $c_{12} = 0$. Similarly,

$$c_{13} = c_{14} = 0.$$

Next,

$$1 = v_{24} = c_2 + c_4 + c_{24},$$

implying that $c_{24} = 0$. Similarly, $c_{34} = 1$. We have that

$$1 = v_{123} = c_{123},$$

while

$$1 = v_{124} = c_{24} + c_{124},$$

implying that $c_{124} = 0$. Similarly, $c_{134} = 0$, and

$$1 = v_{234} = c_{24}+c_{34}+c_{123}+c_{124}+c_{134}+c_{234}+c_{1234} = 1+1+1+0+0-1+c_{1234},$$

implying that $c_{1234} = -1$. Thus,

$$v = w_{24} + w_{34} + w_{123} - w_{234} - w_{1234},$$

whence

$$\psi_1(v) = 1/3 - 1/4 = 1/12,$$

and

$$\psi_2(v) = 1/2 + 1/3 - 1/3 - 1/4 = 1/4,$$

while $\psi_3(v) = 1/4$, by symmetry with player 2. Finally, $\psi_4(v) = 5/12$.

Probabilistic interpretation of Shapley value: Suppose that the players arrive at the board meeting in a uniform random order. With the arrival of precisely one stockholder, the coalition present in the board-room becomes effective. Then the Shapley value of a given player is the probability that that player is the one to make the existing coalition effective, where the players enter the board-room in this particular random fashion. We will write this statement in symbols and prove it later.