Stat 155 Game theory, Yuval Peres **Fall 2004**

Lectures 16,17

The minimax theorem: Suppose given a payoff matrix $A_{m\times n} = (a_{ij})_{1\le i\le m, 1\le j\le n}$, with a_{ij} equal to the payment of II to I is I picks i and II picks j. Player I can assure an expected payoff of $\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T Ay$. Player II can be assured of not paying more than $\min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T Ay$. The notation is:

$$\Delta_m = \{x \in \mathbb{R}^m : x_i \ge 0, \sum_{i=1}^{m} x_i = 1\}.$$

Having prepared some required tools, we will now prove:

Theorem 1 (Von Neumann minimax)

$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T Ay = \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T Ay.$$

Proof: That

$$\max_{x\in\Delta_m}\min_{y\in\Delta_n} x^T Ay \le \min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T Ay.$$

is easy, and has the same proof as that given for pure strategies. For the other inequality, we firstly assume that

$$\min_{y\in\Delta_n}\max_{x\in\Delta_m} x^T Ay > 0. \tag{1}$$

Let K denote the set of payoff vectors that player II can achieve, or that are better than some such vector. That is,

$$K = \Big\{ y_1 A^{(1)} + y_2 A^{(2)} + \ldots + y_n A^{(n)} + v : y = (y_1, \ldots, y_n)^T \in \Delta_n, v = (v_1, \ldots, v_m)^T, v_i \ge 0 \Big\},$$

where $A^{(i)}$ denotes the i-th column of A. That is, y is a strategy for player II, with the i-th component of a member of K at least as favourable to player I as the payoff to player I of playing move i against the mixed strategy y for player II. It is easy to show that K is convex and closed: this uses the fact that Δ_n is closed and bounded. Note also that

$$0 \notin K. \tag{2}$$

To see this, note that (1) means that, for every y, there exists x such that player I has a uniformly positive expected payoff $x^T Ay(> \delta > 0)$. If $0 \in K$, this means that, for some y, we have that

$$Ay = y_1 A^{(1)} + y_2 A^{(2)} + \ldots + y_n A^{(n)} \leq 0,$$

where by ≤ 0, we mean in each of the coordinates. However, this contradicts (1), and proves (2).

The separation theorem allows us to find $z \in \mathbb{R}^m$ and $c > 0$ such that $0 < c < z^T.w$ for all $w \in K$. This implies that $z^T.(Ay + v) > c > 0$ for all $y \in \Delta_n$, $v \geq 0$, $v \in \mathbb{R}^m$. We deduce that $z_i \geq 0$ by considering large and positive choices for v_i, and using

$$z^T.v = z_1 v_1 + \ldots + z_m v_m.$$

Not all of the z_i can be zero. This means that we may set $s = \sum_{i=1}^m z_i > 0$, so that $x = (1/s)(z_1, \ldots, z_m)^T = (1/s)z \in \Delta_m$ with $x^T Ay > 0$ for all $y \in \Delta_n$.

We now remove the assumption (1), and suppose that

$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T Ay < \lambda < \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T Ay.$$

Consider a new game with payoff matrix $\hat{A}$ given by $\hat{A}_{m \times n} = (\hat{a}_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, with $\hat{a}_{ij} = a_{ij} - \lambda$. We find that

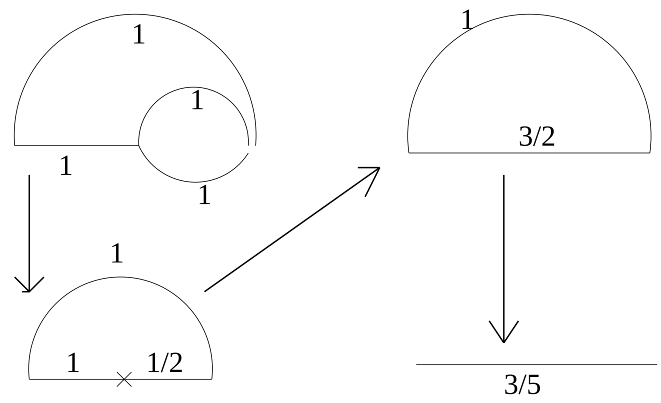
$$\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T Ay < 0 < \min_{y \in \Delta_n} \max_{x \in \Delta_m} x^T Ay,$$

contradicting $\max_{x \in \Delta_m} \min_{y \in \Delta_n} x^T Ay > 0$, which is implied by (1). □

Resistor networks and troll games: Suppose that two cities, A and B, are connected by a 'parallel-series' network of roads. Such a network is built by modifying an initial straight road that runs from A to B. The modification takes the form of a finite number of steps, these steps being of either *series* or *parallel* type. A series step is to find a road in the current network and to replace it by two roads that run one after the other along the current path of the road. The parallel step consists of replacing one current road by two, each of which runs from and to the start and finish of the road being replaced.

Imagine that each road has a cost attached to it. A troll and a traveller will each travel from city A to city B along some route of their choosing. The traveller will pay the troll the total cost of the roads that both of them choose.

To solve these games, we interpret the network as an electrical one, and the costs as resistances. Resistances add in series, whereas conductances, which are their reciprocals, add in parallel. We claim that the optimal strategies for both players are the same. Under the optimal strategy, a player who reaches a fork should move along any of the edges emanating



Adding in series and parallel for the troll−traveller game

from the fork with a probability proportional to the conductance of that edge.
Example: see the figure.

A way of seeing why these games are solved like this is to introduce the notion of two games being summed in parallel or in series. Suppose given two zero-sum games G_1 and G_2 with values v_1 and v_2. Their series addition just means: play G_1, and then G_2. The series sum game has value $v_1 + v_2$. In the parallel-sum game, each player chooses either G_1 or G_2 to play. If each picks the same game, then it is that game which is played. If they differ, then no game is played, and the payoff is zero. We may write a big payoff matrix as follows:

II I		
	G_1	0
	0	G_2

If the two players play G_1 and G_2 optimally, however, the payoff matrix is effectively:

II I		
	v_1	0
	0	v_2

The optimal strategy for each player comsists of player G_1 with probability $v_2/(v_1 + v_2)$, and G_2 with probability $v_1/(v_1 + v_2)$. Given that

$$v_2/(v_1 + v_2) = \frac{1}{1/v_1 + 1/v_2},$$

this explains the form of the optimal strategy in troll-traveller games on series-parallel graphs.

Hide and seek games: this is another class of two-person zero-sum games that we will analyse. For this, we need a tool:

Lemma 1 (Hall's marriage lemma) *Suppose given a set B of boys and a set G of girls, with $|B| = b$ and $|G| = g$ satisfying $g \geq b$. Let $f : B \to 2^G$ be such that $f(c)$ denotes the subset of G that are the girls known to boy c. A necessary and sufficient condition for the existence of a matching of each boy to a girl that he knows is that, for each $b' \subseteq B$, we have that $|f(B')| \geq |B'|$. (Here, the function f has been extended to subsets B' of B by setting $f(B') = \cup_{c \in B'} f(c)$.)*

Proof: It is clear from the definition of a matching that the existence of a matching implies the other condition. We will prove the opposite direction by an induction on b, the number of boys. The case when $b = 1$ is easy. For larger values of b, suppose that the statement we seek is known for $b' < b$. Two cases: firstly, suppose that there exists $B' \subseteq B$ satisfying $|f(B')| = |B'|$ with $|B'| < b$. We perform a matching of B' to $f(B')$ by using the inductive hypothesis. If $A \subseteq B \setminus B'$, then $|f(A) \setminus f(B')| \geq |A|$: this is because $|f(A \cup B')| = |f(B')| + |f(A) \setminus f(B')|$ and $|f(A \cup B')| \geq |A| + |B'|$ by assumption. Hence, we may apply the inductive hypothesis to the set $B \setminus B'$ to find a matching of this set to girls in $G \setminus f(B')$. We have found a matching of B into G as required.

In the second case, $|f(B')| > |B'|$ for each $' \subseteq B$. This case is easy: we just match a given boy to any girl he knows. Then the set of remaining boys still satisfy the second condition in the statement of the lemma. By the inductive hypothesis, we match them, and we have finished the proof. □