Lecture 3

Suppose that in tossing two coins together, we recognize all **four** combinations of outcomes hh, ht, th, tt as equally likely. Here ht means head on the first coin and tail on the second coin. Then the outcomes on the coins are independent.

Check: pr(head on 1st coin|head on 2nd coin) = 1/2.
pr(head on 1st coin) = 2/4.

D'Alembert thought at one time that the 3 cases: 2 heads, one head & one tail, 2 tails were equally likely. Notice that this seems to force **dependence** between the outcomes of the two coins.

$$\text{pr(head on 1st coin)} = 1/2 = \text{pr(head on 2nd coin)}$$

$$\text{pr(head on 1st coin|head on 2nd coin)} = \frac{\text{pr(heads on both coins)}}{\text{pr(head on 2nd coin)}} = \frac{1/3}{1/2} = 2/3$$

Mendel's Pea Experiments

Mendel's pea experiments led him to believe the yellow/green seed color (for example) was caused by two different variants y and g, say (he used A, a) of what we now call a gene. Each cell apart from the sex cells (pollen, ovules) had a pair, and this pair determined seed color as follows:

$$\text{y/y, y/g, g/y} \rightarrow \text{yellow seed}$$
$$\text{g/g} \rightarrow \text{green seed}$$

In this case, yellow is dominant w.r.t. green. He further postulated that 50% of the sex cells in a y/g or g/y plant carried y and 50% carried g, while all sex cells in y/y (resp. g/g) plants carried y (resp. g).

Finally, he postulated that the gene pair of a new seed was determined by the combination of its pollen and ovule, and that these were passed on independently, i.e. the events that a seed gets y (or g) from its pollen parent and its seed parent are probabilistically independent.

y g	y g
pollen parent	seed parent

choices: at random & independent

yy	yg	gy	gg
1/4	1/4	1/4	1/4

Notice how this explains Mendel's observed 3:1 frequencies (assuming dominance of y) in the first generation after the hybrids (F_2).

Bernoulli trials

Bernoulli trials (named after James Bernoulli, a pioneer of probability theory) are a series of events with

- **binary outcomes**, conventionally S = success or F = failure
- **constant probability** of success, p say
- **mutually independent**, i.e. each independent of all combinations of the others

Examples

What is p in these cases?

1. Tossing a fair coin : S = head, say; F = tail $\rightarrow p = 1/2$
2. Rolling a fair die: $S = 6$, say; $F = 1$ or 2 or 3 or 4 or 5 $\rightarrow p = 1/6$
3. Rolling a pair of fair dice: S = double ace, say; F = the other possibilities $\rightarrow p = 1/36$
4. Sex of a new born child at Alta Bates: S = female, say $\rightarrow p \approx 0.51$

Main Fact 1 about Bernoulli trials:

If S_i denotes success on the ith trial, and F_i failure,

$$\text{pr}(S_1 F_2 F_3 S_4 \ldots) = p \times (1-p) \times (1-p) \times p \ldots$$

More generally,

$$\text{pr}(\text{any sequence of outcomes } S \text{ or } F) = p^{\#\text{Successes}}(1-p)^{\#\text{Failures}}$$

Proof:

$$\begin{aligned}
\text{pr}(S_1 F_2 F_3 S_4 \ldots) &= \text{pr}(S_1)\text{pr}(F_2|S_1)\text{pr}(F_3|S_1 F_2)\text{pr}(S_4|S_1 F_2 F_3)\ldots \\
&= p \times (1-p) \times (1-p) \times p \times \ldots
\end{aligned}$$

where we make use of the mutual independence assumption.

The more general result is proved similarly.

Main fact 2 about Bernoulli trials:

In 2 trials,

- pr(0 successes) $= (1-p)^2$
- pr(1 success) $= 2p(1-p)$
- pr(2 successes) $= p^2$

More generally, in n trials

$$\text{pr}(k \text{ successes}) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \qquad k = 0, \ldots, n$$

Proof: It's just counting, main fact 1 and the addition rule, since **distinct** sequences of trials with the **same** number of successes, e.g. S_1F_2 or F_1S_2 are mutually exclusive. (Let's now make the subscripts implicit)

$$\begin{aligned} \text{pr}(1 \text{ success in 2 trials}) &= \text{pr}(SF \text{ or } FS) \\ &= \text{pr}(SF) + \text{pr}(FS) \\ &= p(1-p) + (1-p)p \\ &= 2p(1-p). \end{aligned}$$

Similarly,

$$\text{pr}(1 \text{ success in } n \text{ trials}) = \text{pr}(SF\ldots F \text{ or } FS\ldots F \text{ or } \ldots \text{ or } FF\ldots S) = np(1-p)^{n-1}$$

The main result presupposes a familiarity with permutations and combinations $n!$ and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Random quantities, a.k.a. random variables

Events are binary (true or false) notions to which assign probabilities. We also need to assign probabilities to more general numerical quantities whose value is unknown at the time of assignment).

Examples

1. Toss a fair coin. Let $X = 1$ if head, 0 if a tail. Still binary.
2. Roll a fair die. Let $X =$ score face up.
3. Roll a pair of fair dice. Let $X =$ total score.
4. Consider a day at Alta Bates. Let $X = \#$ girls newly born.
5. Toss a fair coin until you get a head, and let $X = \#$ tosses needed.

6. Roll a fair die until you get a 6, and let $X = \#$ rolls needed.
7. Start at a position in the *E.coli* sequence and let
   - $e = \#$ Rs (R = A or G) in the next 100 bp
   - $W = \#$ occurrences of TAG in the next 5,000 bp
   - $X = \#$ bp until the occurrence of TATAAT.

The two important notions associated with a random variable are

- the **distribution** of X given H

$$\text{pr}(X = x|H), \qquad \text{as } x \text{ runs over the possible values of } X$$

- the **expectation**, $\text{E}(X|H)$, of x given H

$$\text{E}(X|H) = \sum_x x \times \text{pr}(X = x|H), \qquad \text{as } x \text{ runs over the possible values of } X$$

Examples

1. Toss a fair coin and let $X = 1$ if a head comes up, $X = 0$ if a tail:
   $\text{pr}(X = 1|H) = \text{pr}(\text{head}|H) = 1/2$.

$$\text{E}(X|H) = 1/2 \times 0 + 1/2 \times 1 = 1/2$$

2. Roll a fair die and let $X =$ face up value. $\text{pr}(X = 1|H) = ... = \text{pr}(X = 6|H) = 1/6$.

$$\text{E}(X|H) = 1/6 \times (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

3. Roll a pair of fair dice and let $X =$ total score.

   $\text{pr}(X = 2|H) = 1/36$, $\text{pr}(X = 3|H) = 2/36$, etc. $\text{E}(X|H) = 7$ (check!)