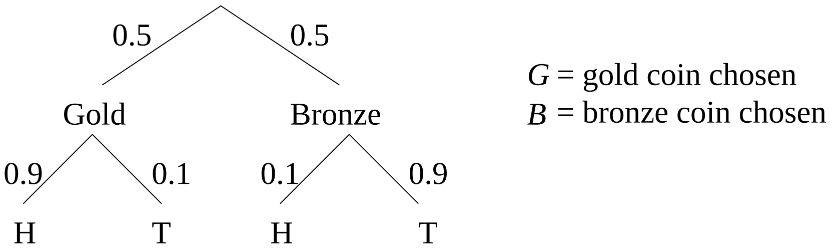
Lecture 14: Likelihoods, Likelihood Ratios, Hypotheses Testing and the Neyman-Pearson Lemma

Example: You have two coins, one gold, one bronze:

The gold coin comes up heads 90% of the time
The bronze coin comes up heads 10% of the time

One of these two coins is chosen by the toss of a fair coin. The coin chosen is tossed and a head comes up. What is the probability that the gold coin was chosen?



Solution:

$$\Pr(G|H) = \frac{\Pr(G\&H)}{\Pr(H)} = \frac{\Pr(G)\Pr(H|G)}{\Pr(H)}$$

Now,

$$\begin{aligned}\Pr(H) &= \Pr(G\&H) + \Pr(B\&H) \qquad \text{why?}\\ &= \Pr(G)\Pr(H|G) + \Pr(B)\Pr(H|B)\\ &= 0.5 \times 0.9 + 0.5 \times 0.1 = 0.5\end{aligned}$$

Therefore,

$$\begin{aligned}\Pr(G|H) &= \frac{0.5 \times 0.9}{0.5} = 0.9 = \Pr(H|G)\\ \Pr(B|H) &= \frac{0.5 \times 0.1}{0.5} = 0.1 = \Pr(H|B)\end{aligned}$$

Now suppose $\Pr(G) = 0.9$ and $\Pr(B) = 0.1$. Then,

$$\begin{aligned}\Pr(G|H) &= \frac{0.9 \times 0.9}{0.9 \times 0.9 + 0.1 \times 0.1} = 0.99\\ \Pr(G|H) &= \frac{0.1 \times 0.1}{0.9 \times 0.9 + 0.1 \times 0.1} = 0.01\end{aligned}$$

Again, suppose $\Pr(G) = 0.1$ and $\Pr(B) = 0.9$. Then,

$$\begin{aligned}\Pr(G|H) &= \frac{0.9 \times 0.1}{0.9 \times 0.1 + 0.1 \times 0.9} = \frac{1}{2} \\ \Pr(G|H) &= \frac{1}{2} \qquad \text{similarly}\end{aligned}$$

Conclusion:

$\Pr(G|H)$ depends critically on $\Pr(G)$. If $\Pr(G) = \Pr(B) = \frac{1}{2}$ (equally probable a priori) then:

$$\begin{aligned}\Pr(G|H) &= \Pr(H|G) \\ \Pr(B|H) &= \Pr(H|B),\end{aligned}$$

relations that are usually false.

Definition

The function $Pr(H|\cdot)$, defined by $G \longmapsto \Pr(H|G)$ and $B \longmapsto \Pr(H|B)$, is called the **likelihood function** generated by the observation of a head. Likelihood is probability of a fixed event (here H) viewed as a function of hypotheses (here G, B).

Suppose we have a coin which has either probability $p_0 = 0.1$ or $p_1 = 0.9$ of coming up heads. It is tossed 10 times and we obtain 4 heads and 6 tails. Is $p = p_0$ or $p = p_1$?

With prior probabilities of the hypotheses p_0, p_1 we can proceed as in the previous example:

$$\Pr(p_0|4\text{Hs}, 6\text{Ts}) = \frac{\Pr(4\text{Hs}, 6\text{Ts}|p_0)\Pr(p_0)}{\ldots \quad + \quad \ldots}$$

Without prior probabilities, the conventional solution is to choose one of p_0, p_1 to be the null hypothesis, and test that. For example, if H_0 is $p = p_0$, we test against the alternative H_1: $p = p_1$. Intuitively, we should reject the null if we have "too many" heads. Is 4 too many? Fix the critical region $C = \{\# \text{ heads} \geq c\}$, and choose c so that $\Pr(C|p_0) = 0.05$ (say). If we do this, $c = 3$ corresponds to 0.07 , $c = 4$ to 0.015.

k	$\Pr(k$ heads$)$	$\Pr(\leq k$ heads$)$
0	0.348	0.348
1	0.387	0.735
2	0.193	0.928
3	0.057	0.985
4	0.011	0.996
5	0.001	0.997

The loaded die

A die is either fair (p_0) or biased in the manner indicated by p_1

	1	2	3	4	5	6
p_0:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
p_1:	$\frac{1}{5}$	$\frac{7}{40}$	$\frac{7}{40}$	$\frac{7}{40}$	$\frac{7}{40}$	$\frac{1}{10}$

It is rolled 10 times and the number n_1 of aces and n_6 of 6s is noted. The joint distribution of n_1 and n_6 based on p_1 is

$$\Pr(n_1 = x, n_6 = y) = \frac{10!}{x!y!(10-x-y)!}\left(\frac{1}{5}\right)^x\left(\frac{7}{10}\right)^{10-x-y}\left(\frac{1}{10}\right)^y$$

Suppose we want to test H$_0$: $p = p_0$ against H$_1$: $p = p_1$. What test statistic should be used?

Options: (a) Reject H$_0$ if “too many 1s”, i.e. large n_1
(b) Reject H$_0$ if “too few 6s”, i.e. small n_6
(c) Some other rejection rule, perhaps based on n_1 and n_6 jointly.

Which, if any, is “best” and how do we decide in general?

Look at (a): If we reject H$_0$ when $n_1 \geq c$ we have Type 1 error of:

$$\sum_{x \geq c}\binom{10}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{10-x}$$

k	$\Pr(x_1 = k)$	$\Pr(x_1 \leq k)$
0	0.1615	0.1615
1	0.3230	0.4845
2	0.2907	0.7752
3	0.1550	0.9303
4	0.0543	0.9845
5	0.0130	0.9976
6	0.0022	0.9997
7	0.0002	1.0000
8	0.0000	1.0000
9	0.0000	1.0000
10	0.0000	1.0000

Look at (c): the **likelihood ratio**

$$\frac{\frac{10!}{x!y!(10-x-y)!}\left(\frac{1}{5}\right)^x\left(\frac{7}{10}\right)^{10-x-y}\left(\frac{1}{10}\right)^y}{\frac{10!}{x!y!(10-x-y)!}\left(\frac{1}{6}\right)^x\left(\frac{2}{3}\right)^{10-x-y}\left(\frac{1}{6}\right)^y} = \frac{L_1}{L_0}, \text{ say}$$

$$
\begin{aligned}
\log\frac{L_1}{L_0} &= x\log\left(\frac{\frac{1}{5}}{\frac{1}{6}}\times\frac{\frac{2}{3}}{\frac{7}{10}}\right) + y\log\left(\frac{\frac{1}{10}}{\frac{1}{6}}\times\frac{\frac{2}{3}}{\frac{7}{10}}\right) \\
&= x\log\frac{120}{105} + y\log\frac{12}{21} \\
10\log\frac{L_1}{L_0} &= 1.3x - 5.6y = l
\end{aligned}
$$

We could calculate a cutoff for l, e.g. $l \geq c$, where c is such that $\Pr(l \geq c|p_0) \approx 0.05$. Then $\Pr(l \geq c|p_1)$ is the power (to reject, correctly).

This development is continued in Homework #4.

Neyman-Pearson Lemma

Suppose that the likelihood ratio test that rejects H_0 when $l \geq c$, has significance level α. Then any other test which has significance level $\alpha^* \leq \alpha$ has power less than or equal to that of the likelihood ratio test.