Statistics 150: Spring 2008

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1 Classification of states

Definition 1.1. State i is called *persistent* (or *recurrent*) if

$$\mathbb{P}\{X_n = i \text{ for some } n \geq 1 | X_0 = i\} = 1,$$

which is to say that the probability of eventual return to i, having started from i, is 1. If this probability is strictly less than 1, the state is called *transient*.

Let

$$f_{ij}(n) = \mathbb{P}\{X_1 \neq j, X_2 \neq j, \ldots, X_{n-1} \neq j, X_n = j | X_0 = i\}$$

be the probability that the first visit to state j, starting from i, takes place at the nth step. Define

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

to be the probability that the chain ever visits j, starting from i. Of course, j is persistent if and only if $f_{ij} = 1$.

We seek a criterion for persistence in terms of the n-step transition probabilities. We define the generating functions

$$P_{ij}(s) = \sum_{n=0}^{\infty} s^n p_{ij}(n), \quad F_{ij}(s) = \sum_{n=0}^{\infty} s^n f_{ij}(n)$$

with the convention that $p_{ij}(0) = \delta_{ij}$, and $f_{ij}(0) = 0$ for all i and j. Clearly $f_{ij} = F_{ij}(1)$.

Proposition 1.2. *1.* $P_{ii}(s) = 1 + F_{ii}(s)P_{ii}(s)$*;*

2. $P_{ij}(s) = F_{ij}(s)P_{jj}(s)$ *if* $i \neq j$*.*

Proof. We have

$$p_{ij}(m) = \sum_{r=1}^{m} f_{ij}(r)p_{jj}(m-r), \quad m = 1, 2, \ldots .$$

Multiply throughout by s^m, where $|s| < 1$, and sum over $m(\geq 1)$ to find that $P_{ij}(s) - \delta_{ij} = F_{ij}(s)P_{ij}(s)$ as required.

$\diamondsuit$

Corollary 1.3. *1. State j is persistent if $\sum_n p_{jj}(n) = \infty$, and if this holds then $\sum_n p_{ij}(n) = \infty$ for all i such that $f_{ij} > 0$.*

2. State j is transient if $\sum_n p_{jj}(n) < \infty$, and if this holds then $\sum_n p_{ij}(n) < \infty$ for all i.

Proof. First we show that j is persistent if and only if $\sum_n p_{jj}(n) = \infty$. From Proposition 1.2 part (1),

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)} \quad \text{if } |s| < 1.$$

Hence, as $s \uparrow 1$, $P_{jj}(s) \to \infty$ if and only if $f_{jj} = F_{jj}(1) = 1$. Now $\lim_{s \uparrow 1} P_{jj}(s) = \sum_n p_{jj}(n)$ and our claim is shown. Use Proposition 1.2 part (2) to complete the proof.

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Corollary 1.4. *If j is transient then $p_{ij}(n) \to 0$ as $n \to \infty$ for all i.*

Each state is either persistent or transient. The number $N(i)$ of times which the chain visits its starting point i satisfies

$$\mathbb{P}\{N(i) = \infty\} = \begin{cases} 1 & \text{if } i \text{ is persistent,} \\ 0 & \text{if } i \text{ is transient.} \end{cases}$$

Let $T_j = \min\{n \geq 1 : X_n = j\}$ be the time of the first visit to j, with the convention that $T_j = \infty$ if this visit never occurs.

$\mathbb{P}\{T_i = \infty | X_0 = i\} > 0$ if and only if i is transient, and in this case $\mathbb{E}[T_i | X_0 = i] = \infty$.

Definition 1.5. The *mean recurrence time* μ_i of a state i is defined as

$$\mu_i = \mathbb{E}\left[T_i | X_0 = i\right] = \begin{cases} \sum_n n f_{ii}(n) & \text{if } i \text{ is persistent,} \\ \infty & \text{if } i \text{ is transient} \end{cases}$$

Definition 1.6.

A persistent state i is called $\begin{cases} \textit{null} & \text{if } \mu_i = \infty, \\ \textit{non-null} \text{ (or } \textit{positive}\text{)} & \text{if } \mu_i < \infty. \end{cases}$

Proposition 1.7. *A persistent state is null if and only if* $p_{ii}(n) \to 0$*. If this hold then* $p_{ji}(n) \to 0$ *for all* j*.*

We will see why this true later.

Definition 1.8. The *period* $d(i)$ of a state i is defined by $d(i) = gcd\{n : p_{ii}(n) > 0\}$, the greatest common divisor of the epochs at which return is possible. We call i *periodic* if $d(i) > 1$ and *aperiodic* if $d(i) = 1$.

This is to say, $p_{ii}(n) = 0$ unless n is a multiple if $d(i)$, and $d(i)$ is maximal with this property.

Definition 1.9. A state is called *ergodic* if it is persistent, non-null, and aperiodic.

Example [Random walk] The states of the simple one-dimensional random walk that goes up with probability p and down with probability $1 - p$ are all periodic with period 2, and

1. transient, if $p \neq \frac{1}{2}$
2. null persistent, if $p = \frac{1}{2}$

2 Classification of chains

Definition 2.1. We say i *communicates with* j, written $i \to j$, if the chain may ever visit state j with positive probability, having started from i. That is, $i \to j$ if $p_{ij}(m) > 0$ for some $m \geq 0$. We say i and j *intercommunicate* if $i \to j$ and $j \to i$, in which case we write $i \leftrightarrow j$.

Note If $i \neq j$, then $i \to j$ if and only if $f_{ij} > 0$. Clearly $i \to i$ since $p_{ii}(0) = 1$, and it follows that $\leftrightarrow$ is an equivalence relation. The state space S can be partitioned into the equivalence classes of $\leftrightarrow$.

Proposition 2.2. *If $i \leftrightarrow j$ then:*

1. *i and j have the same period,*
2. *i is transient if and only if j is transient,*
3. *i is null persistent if and only if j is null persistent.*

Definition 2.3. A set C of states is called:

1. *closed* if $p_{ij} = 0$ for $i \in C, j \notin C$,
2. *irreducible* if $i \leftrightarrow j$ for all $i, j \in C$.

Proposition 2.4 (Decomposition theorem). *The state space S can be partitioned uniquely as*

$$S = T \cup C_1 \cup C_2 \cup \dots$$

where T is the set of transient states, and the C_i are irreducible closed sets of persistent states.

The decomposition theorem tells us that if $X_0 \in C_r$, the chain never leaves C_r and we might as well take C_r to be the whole state space. On the other hand, if $X_0 \in T$ then the chain either stays in T for ever or moves eventually to one of the C_k where it subsequently remains. Thus, either the chain always takes values in the set of transient state or it lies eventually in some irreducible closed set of persistent states.

Lemma 2.5. *If S is finite, then at least one state is persistent and all persistent states are non-null.*

Proof If all states are transient, then take the limit through the summation sign to obtain the contradiction

$$1 = \lim_{n \to \infty} \sum_j p_{ij}(n) = 0$$

by Corollary 1.4. The same contradiction arises by Proposition 1.7 for the closed set of all null persistent states, should this set be non-empty.

$\diamondsuit$

3 Exercises

1) Suppose that the number of states is n. Show that if state j is accessible from state i, then it is accessible in n or fewer steps.

2) Show that the symmetric random walk is recurrent in two dimensions and transient in three dimensions.