A Fixed Point Theorem

Fixed point theorems play an important role in many parts of analysis and topology. The one that we shall now prove is due to Kakutani; it will be used to prove the existence of a Haar measure on any compact group. The proof of Kakutani's theorem involves only the most basic properties of locally convex spaces.

5.11 Theorem *Suppose*

(*a*) *K is a nonempty compact convex set in a locally convex space X,*
(*b*) *G is an equicontinuous group of linear mappings of X onto X, and*
(*c*) *$\Lambda(K) \subset K$ for every $\Lambda \in G$.*

Then G has a common fixed point in K; that is, there exists $p \in K$ such that $\Lambda p = p$ for every $\Lambda \in G$.

Part (*b*) of the hypothesis should perhaps be made more explicit. Equicontinuity is defined in Section 2.3. To say that G is a group means that every $\Lambda \in G$ is a one-to-one mapping of X onto X whose inverse Λ^{-1} also belongs to G and that $\Lambda_1\Lambda_2 \in G$ whenever $\Lambda_i \in G$ $(i = 1, 2)$. Here $(\Lambda_1\Lambda_2)x = \Lambda_1(\Lambda_2 x)$, of course. Hypothesis (*b*) is satisfied, for instance, when G is a group of linear isometries on a normed space X.

PROOF Let Ω be the collection of all nonempty compact convex sets $H \subset K$ such that $\Lambda(H) \subset H$ for every $\Lambda \in G$. Partially order Ω by set inclusion. Note that $\Omega \neq \varnothing$, since $K \in \Omega$. By Hausdorff's maximality theorem, Ω contains a maximal totally ordered subcollection Ω_0. The intersection H_0 of all members of Ω_0 is a minimal member of Ω. The theorem will be proved by showing that H_0 contains only one point. To do this, we shall consider a set $H \in \Omega$ which contains at least two points, and we shall prove that some $H_1 \in \Omega$ is a proper subset of H.

Before doing this, we prove that X has a local base consisting of balanced convex sets U that satisfy $\Lambda(U) \subset U$ for every $\Lambda \in G$.

Let V be a convex neighborhood of 0 in X. Since G is equicontinuous, there is a balanced neighborhood V_1 of 0 such that $\Lambda(V_1) \subset V$ for every $\Lambda \in G$. Let U be the convex hull of the union of all sets $\Lambda(V_1)$, as Λ ranges over G. Then U is convex and balanced, and $U \subset V$, since V is convex. Every $u \in U$ has the form

$$u = c_1\Lambda_1 v_1 + \cdots + c_n\Lambda_n v_n,$$

where $c_i \geq 0$, $\sum c_i = 1$, $\Lambda_i \in G$, $v_i \in V_1$. If $\Lambda \in G$, then

$$\Lambda u = c_1\Lambda\Lambda_1 v_1 + \cdots + c_n\Lambda\Lambda_n v_n$$

lies also in U, because $\Lambda\Lambda_i \in G$. Hence $\Lambda(U) \subset U$.

Now suppose $H \in \Omega$, and H contains at least two points. Then $H - H \neq \{0\}$, and some set U as above fails to cover $H - H$. Since $H - H$ is compact, $H - H \subset sU$ for some $s > 0$. Let t be the greatest lower bound of these numbers s. Then $t \geq 1$. Put $W = tU$. Then W is a convex balanced open set such that

$$\Lambda(W) \subset W \qquad \text{for every } \Lambda \in G, \tag{1}$$

$$H - H \subset (1 + r)W \qquad \text{if } r > 0, \tag{2}$$

$$(1 - r)\overline{W} \text{ does not cover } H - H \qquad \text{if } 0 < r < 1. \tag{3}$$

Properties (1) and (2) are obvious. Since W is convex,

$$(1 - r)\overline{W} \subset (1 - r)W + \tfrac{1}{2}rW = \left(1 - \frac{r}{2}\right)W;$$

this last set does not cover $H - H$; hence (3) holds.

Since H is compact, H contains points $x_1, \ldots, x_n$ such that

$$H \subset \bigcup_{i=1}^{n} (x_i + \tfrac{1}{2}W). \tag{4}$$

Put $r = 1/(4n)$, and define

$$H_1 = H \cap \bigcap_{y \in H} (y + (1 - r)\overline{W}). \tag{5}$$

It is clear that H_1 is compact and convex.

Suppose $x \in H_1$ and $y \in H$. Since $\Lambda^{-1}(H) \subset H$, $y = \Lambda y_1$ for some $y_1 \in H$. By (5), $x \in y_1 + (1 - r)\overline{W}$. Hence (1) implies that

$$\Lambda x \in \Lambda y_1 + (1 - r)\Lambda(\overline{W}) \subset y + (1 - r)\overline{W}.$$

It follows that $\Lambda(H_1) \subset H_1$ for every $\Lambda \in G$.

By (3), there are points $x \in H$, $y \in H$, such that $x - y$ does not lie in $(1 - r)\overline{W}$. Any such x is not in H_1. Thus $H_1 \neq H$.

To complete the proof, we have to show that $H_1 \neq \varnothing$. We do this by showing that H_1 contains the point

$$x_0 = \frac{1}{n}(x_1 + \cdots + x_n). \tag{6}$$

Since H is convex, $x_0 \in H$. Fix $y \in H$. By (4), there exists j such that

$$y \in x_j + \tfrac{1}{2}W. \tag{7}$$

If $i \neq j$, $1 \leq i \leq n$, property (2) implies that

$$y \in x_i + (1 + r)W. \tag{8}$$

Add the relations (7) and (8), divide by n, and use the convexity of W to obtain

$$y - x_0 \in \frac{1}{n}\left[\frac{1}{2} + (n-1)(1+r)\right] W \subset (1-r)W,$$

since $r = 1/(4n)$. Thus $x_0 \in y + (1-r)W$, for every $y \in H$. Hence $x_0 \in H_1$, and the proof is complete. ////