UC Berkeley
Department of Statistics

STAT 210A: Introduction to Mathematical Statistics

Review problems
Fall 2007

Problem 12.1
Suppose that X_i, $i = 1, \ldots, n$ are i.i.d. samples from the uniform Uni$[0, \theta]$ distribution.

(a) Find a one-dimensional sufficient statistic for estimating θ.

(b) Compute the maximum likelihood estimate θ_{MLE} based on $(X_1, \ldots, X_n)$. Using an elementary argument, show that $\theta_{MLE} \xrightarrow{p} \theta^*$ as $n \to +\infty$.

(c) Consider the estimator of θ given by $\delta(X) = \frac{2}{n} \sum_{i=1}^n X_i$. Is it unbiased? Is it admissible under squared error loss? Justify your answers.

Now suppose that we view the parameter as a random variable Θ, and assume a *Pareto prior density* of the form

$$\lambda(\theta) \quad = \quad \gamma \beta^\gamma \theta^{-\gamma-1} \mathbb{I}[\beta \leq \theta], \qquad \text{for all } \theta > 0,$$

where $\beta > 0$ and $\gamma > 2$ are fixed.

(d) Compute the *prior* mean of the random variable Θ.

(e) Compute the posterior distribution of Θ conditioned on $X = (X_1, \ldots, X_n)$.

(f) Compute the Bayes estimate of Θ under quadratic loss. *Hint:* New calculation may not be required given previous parts to the question.

Problem 12.2
Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, \theta^2)$ where $\theta > 0$ is an unknown parameter. Given the sample mean $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$, define

$$\widehat{\theta}_n \quad = \quad \bar{X}_n \left(1 + \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 - n\bar{X}_n^2}{3 \sum_{i=1}^n (X_i - \bar{X}_n)^2} \right).$$

(a) Show that $\widehat{\theta}_n \xrightarrow{p} \theta$ as $n \to +\infty$.

(b) Compute the Cramér-Rao lower bound for any unbiased estimator of θ. (Assume that all necessary regularity conditions are satisfied).

(c) Compute the asymptotic distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$. Is this estimator asymptotically efficient?

(d) Does there exist an unbiased estimator that achieves the lower bound in (b)? Why or why not?

Problem 12.3

Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with density function

$$p(x;\theta) \quad = \quad \exp\left[-\lambda(x-\mu)\right] \quad \text{for} \quad x \geq \mu,$$

and set $\theta = (\lambda, \mu)$.

(a) Prove that $T = (\sum_{i=1}^{n} X_i,\ X_{(1)})$ is sufficient for $\theta = (\lambda, \mu)$. (Recall that $X_{(1)}$ denotes $\min\{X_1, \ldots, X_n\}$.)

(b) Compute the maximum likelihood estimate $\hat{\theta}_n = (\hat{\lambda}_n, \hat{\mu}_n)$. Is the information inequality applicable to this model? If not, show why not. If yes, compute the bound, and determine whether the MLE is asymptotically efficient.

(c) Show that the distribution of $S_n(\lambda) = 2\lambda \sum_{i=1}^{n} \left(X_i - X_{(1)}\right)$ is χ^2 with $d = 2(n-1)$ degrees of freedom. Use this result to specify an exact $(1-\alpha)\%$-confidence interval for λ.

(d) You are stuck on a desert island, with only coconuts and a normal quantile table (i.e., for each $\beta \in (0,1)$, the value z_β such that $\beta = \mathbb{P}[Z \geq z_\beta]$ where $Z \sim N(0,1)$). Use asymptotic theory as applied to $S_n(\lambda)$, and your normal quantile table to construct an approximate $(1-\alpha)\%$ confidence interval for λ, valid for large n. (You may also eat your coconuts to obtain extra energy for speeding up the calculation.)

Problem 12.4

For some fixed integer $r > 1$, suppose that the discrete random variable $X \in \{0, 1, 2, \ldots\}$ has the renormalized negative binomial PMF

$$p(x;\theta) \quad = \quad \binom{r+x-1}{x} \theta^x \, (1+\theta)^{-(r+x)}, \quad \text{for} \quad x = 0, 1, 2, \ldots$$

and $\theta \in \Theta = (0, \infty)$. Note that $\mathbb{E}_\theta[X] = r\theta$ by definition.

(a) Find the maximum likelihood estimator of θ.

Now consider the loss function $L(\theta, \delta) = \frac{(\theta-\delta)^2}{\theta(1+\theta)}$ with decision space $\mathcal{A} = [0, \infty)$.

(b) Show that $\delta_0(x) = x/r$ has constant risk under the given loss function. For $r > 1$, show that δ_0 is a generalized Bayes estimator (i.e., a Bayes estimator based on some prior over θ, possibly improper).

(c) Show that the prior distribution $\pi(\theta \mid a, b) \propto \theta^{a-1}(1+\theta)^{-(a+b)}$ is conjugate for $p(x;\theta)$, and compute the Bayes estimator (as a function of a, b) under the given loss. (*Hint:* First try to write the Bayes estimator in terms of quantities of the form $\int_0^\infty \theta^c (1+\theta)^d d\theta$ for suitable c, d. The transform $\theta = q/(1-q)$ could be useful in evaluating.)

(d) Show that $\delta_0(x) = x/r$ is a minimax estimator for $r > 1$.

(e) Now suppose that we are interested in testing hypothesis $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$. Specify the generalized LRT for this problem.