Route-Length Efficiency in Spatial Transportation Networks

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Abstract

In networks such as inter-city road or rail networks, a natural notion of "efficiency" is that the route length $\ell(i,j)$ between typical cities i,j be not much longer than straight line distance $d(i,j)$. Quantify inefficiency via a function $R(s)$ = average of $\frac{\ell(i,j)}{d(i,j)} - 1$ over city-pairs with $d(i,j) \approx s$. Then define $R^* = \max_s R(s)$ as a single numerical statistic to measure ineffecency. Standardize the unit of length so that the density of cities is 1 per unit area, and write θ = (total network length)/(number of cities) for normalized network length per city. Thus a real network is summarized as a point (θ, R^*). Now consider all possible networks on the given set of real cities. For each normalized length θ there is some optimal network whose inefficiency is the smallest value $R^*(\theta)$ possible. We formulate a *sharp curve conjecture* about properties of the (theoretical) curve $R^*(\theta)$ and the real data point (R^*, θ) for typical real networks. We give preliminary study of some data, collected and analyzed as an Undergraduate Research Project in Spring 2007.

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1 Introduction

A network to be built over a short time period can be designed to be optimal in some way, to maximize some "benefit" subject to a given "cost". On the other hand, most of the familiar large scale physical networks for transportation and communication (roads, railroads, gas pipelines, electricity grid, telephone and internet) were built and expanded ("evolved") over a long period, so one cannot expect them to be exactly optimal under today's circumstances. Such considerations prompt a general mathematical question

> how does the structure of optimal networks depend on the criteria for optimality?

and an empirical question

> do real-world evolved networks of a given type resemble optimal networks for some type-specific optimality criterion?

In this paper we consider the latter question in a very simple setting, envisaged as a road or rail network, where "cost" is total network length and "benefit" relates to shortness of within-network routes. We emphasize that we are considering networks with physical links in two-dimensional space, rather than non-spatial networks such as WWW links or social networks that have attracted much attention under the name *complex networks* [1, 5, 6]. We also remark that certain criteria lead to optimal spatial networks being trees [4] but these criteria are not appropriate for transportation networks.

To compare networks with different numbers n of cities and in spatial regions of different areas A, it is convenient to normalize by setting

$$\text{standard unit of length } = \sqrt{A/n}$$

so that the density of cities is 1 per unit area. Then (from now on we measure distances in standard units) we can define

$$\theta := n^{-1} \times \text{ (total length of network)}$$

as normalized network length per city, and we regard θ as the "cost" of the network. Write $\ell(i,j)$ for the route length between vertices i and j, that is the length of shortest route within the network from i to j. So $\ell(i,j) \geq d(i,j) =$ Euclidean distance from i to j. To formulate a single statistic to measure the "inefficiency" of the network in providing short routes, one's first thought is to use the overall average

$$R_{\text{overall}} = \text{ave}_{i,j} \left(\frac{\ell(i,j)}{d(i,j)} - 1 \right) \tag{1}$$

or some weighted average (section 2.1.1)[1]. But for reasons explained in section 2.2 we will use a different statistic R^*, and the introduction of this statistic is our methodological innovation. Given a real network on n cities, we draw a scatter diagram of the $\binom{n}{2}$ points $(d(i,j), r(i,j) := \frac{\ell(i,j)}{d(i,j)} - 1)$ from each city-pair (i,j). See Figure 1.

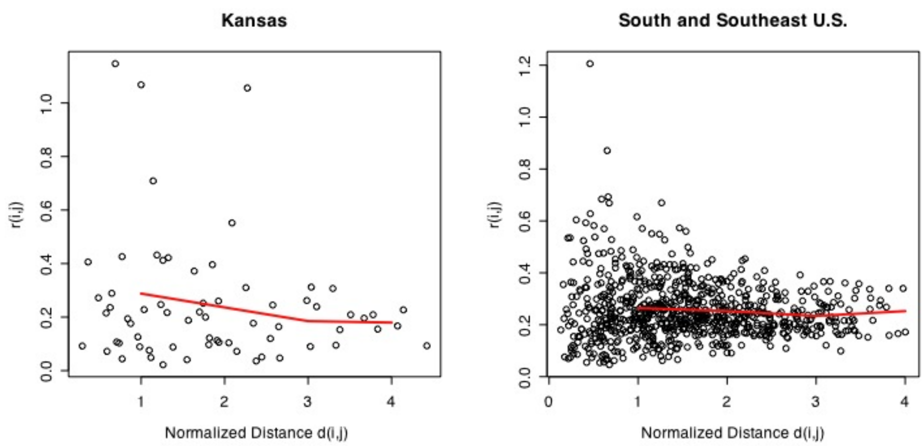

Figure 1. Scatter diagrams and estimators of $R(s)$ for 12 cities in Kansas and for 40 cities in the U.S. South and Southeast (see section 2.1.4). The values of θ are 1.8 (Kansas) and 1.2 (South and Southeast).

Applying a nonlinear regression estimator to this data gives an estimate of the function

$$R(s) = \frac{\text{(mean route-length between points at distance } s \text{ apart)}}{s} - 1.$$

The summary statistic we propose is, conceptually,

$$R^* := \max_s R(s).$$

That is, instead of averaging over all distance-scales we take the "worst case" inter-city distance to measure the network's inefficiency. In practice,

[1]We defer many technical comments to the Appendix, section 2

instead of using a computer package nonlinear regression estimator we did the more naive procedure of averaging over city-pairs whose distance apart (in standard units) was the same when rounded to the closest integer. That is, calculate for small $k = 1, 2, 3, 4, 5, \ldots$

$$\widetilde{R}(k) := \mathrm{ave}\{r(i,j);\ k - \tfrac{1}{2} \leq d(i,j) < k + \tfrac{1}{2}\}$$

and use

$$R^* := \max_k \widetilde{R}(k). \tag{2}$$

Thus we propose the summary statistics (θ, R^*) as descriptors of the total length and the inefficiency of a real network. One could now start collecting data on real networks, but to tackle our initial question

> do real-world evolved networks of a given type resemble optimal networks for some type-specific optimality criterion?

we want first to speculate about optimal networks. That is, we take a real network, note the real city positions and then consider hypothetical networks linking these cities. Each possible network has some summary statistics (θ, R^*), and so we can define $R^*(\theta)$ as the minimum possible value of R^* for a given θ. So each configuration of cities determines a function $\theta \to R^*(\theta)$ which indicates the inefficiency of optimal networks.

The Sharp Curve Conjecture. *Take a typical configuration of real-world city positions. Then the function $\theta \to R^*(\theta)$ representing inefficiency of optimal networks of different lengths has a* sharp curve *somewhere around $(2, 0.2)$, as in Figure 2. Moreover, if the real network is (intuitively) efficient, then its position (*a priori *an arbitrary point $\bullet$ above the curve) will usually be near the sharp curve.*

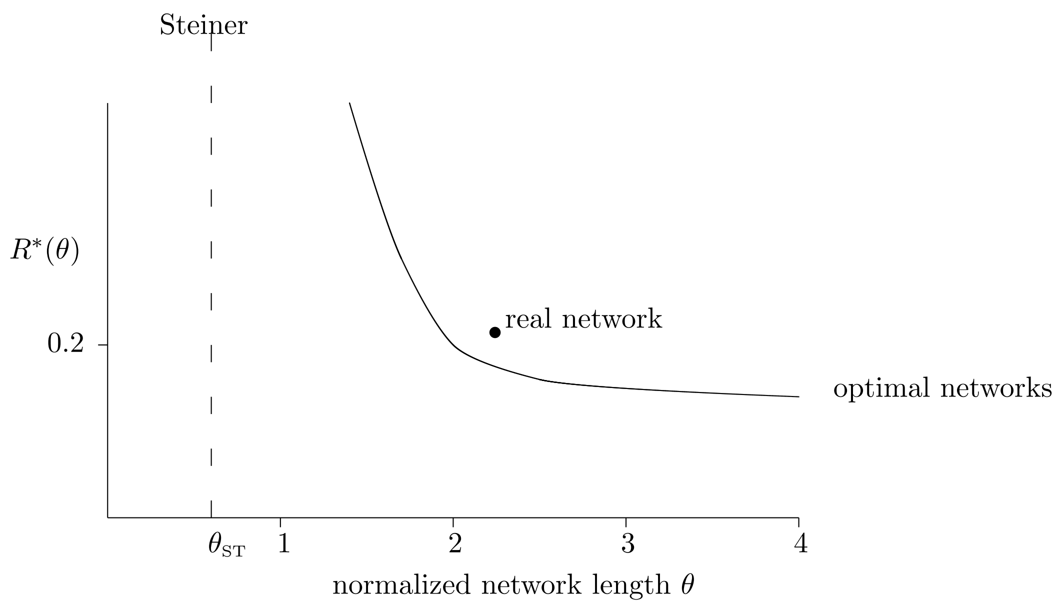

Figure 2. Conjectured qualitative form of the relationship between inefficiency of real network and inefficiencies of optimal networks on the same cities, in the case where the real network seems efficient in an intuitive sense.

Because the formulation of this conjecture is the central point of the paper, let us elaborate in more detail. The minimum length network connecting the cities is by definition the *Steiner tree*, which will typically have normalized length θ_{ST} around $0.5 - 0.6$. The Steiner tree itself is very inefficient as a transportation network – $R^*(\theta_{\mathrm{ST}})$ will increase to ∞ as n increases – but we are conjecturing that as θ increases the efficiency of the optimal network improves rapidly up to some point but thereafter only slowly improves. In other words our conjecture is that a "law of diminishing returns" arises rather abruptly. Note that to make $R^*(\theta) = 0$ we need the network to be the complete graph, which will have normalized length of order $n^{3/2}$; we are interested in "realistic" networks whose normalized length is order 1.

Saying that that a real network is close to optimal is saying that its representing point (θ, R^*) is only just above the curve. Moreover, however the network might have evolved historically, if today it appears efficient in some intuitive sense (i.e. users do not complain about unnecessarily long routes) then for obvious economic reasons one expects the network to be near the sharp curve.

The final aspect of the conjecture – that the sharp curve is typically near the particular position $(2, 0.2)$ – is perhaps the most remarkable. One reason is our preliminary analysis of data sets such as the two shown in Figure 1. One can also compare with the artificial setting of cities arranged at the vertices of the square grid. For this network, $\theta = 2$ and (suitably smoothed - see section 2.1.3) $R(s) \approx 0.28$. This regular configuration of cities is not very representative for many reasons . We expect $R^*(\theta)$ to be smaller for irregular configurations of cities than for regular configurations, because it is easier to construct a few good routes through cities that happen to be nearby or on nearly straight lines.

For purely theoretical analysis, the natural model for city positions is the Poisson point process. One can show by abstract arguments that in the $n \to \infty$ limit there is a well-defined deterministic function $R_{\mathrm{Poi}}(\theta)$. We expect this function to be qualitatively like the curve in Figure 2. But we do not have any quantitative information about $R_{\mathrm{Poi}}(\theta)$, and its study is a challenging problem.

1.1 Shape of the curve $R(s)$

Related conjectures concern the shape of the function $R(s)$ for a typical efficient real network. Figure 3 illustrates our conjecture. For a real network, which has some value θ of normalized length, we conjecture (see section 2.2) that $R(s)$ typically increases rapidly to a maximum roughly near $s = 2$, and then decreases slowly. The (appropriately defined) optimum network with the same length has (we conjecture) qualitatively similar but slightly smaller values of $R(\cdot)$. Implicit in the sharp curve conjecture is that increasing normalized network length will (for an optimal network) only slightly decrease $R(\cdot)$, whereas decreasing normalized network length will make $R(\cdot)$ considerably larger. The values obtained from the two data sets in Figure 1 are not particularly supportive of this conjecture.

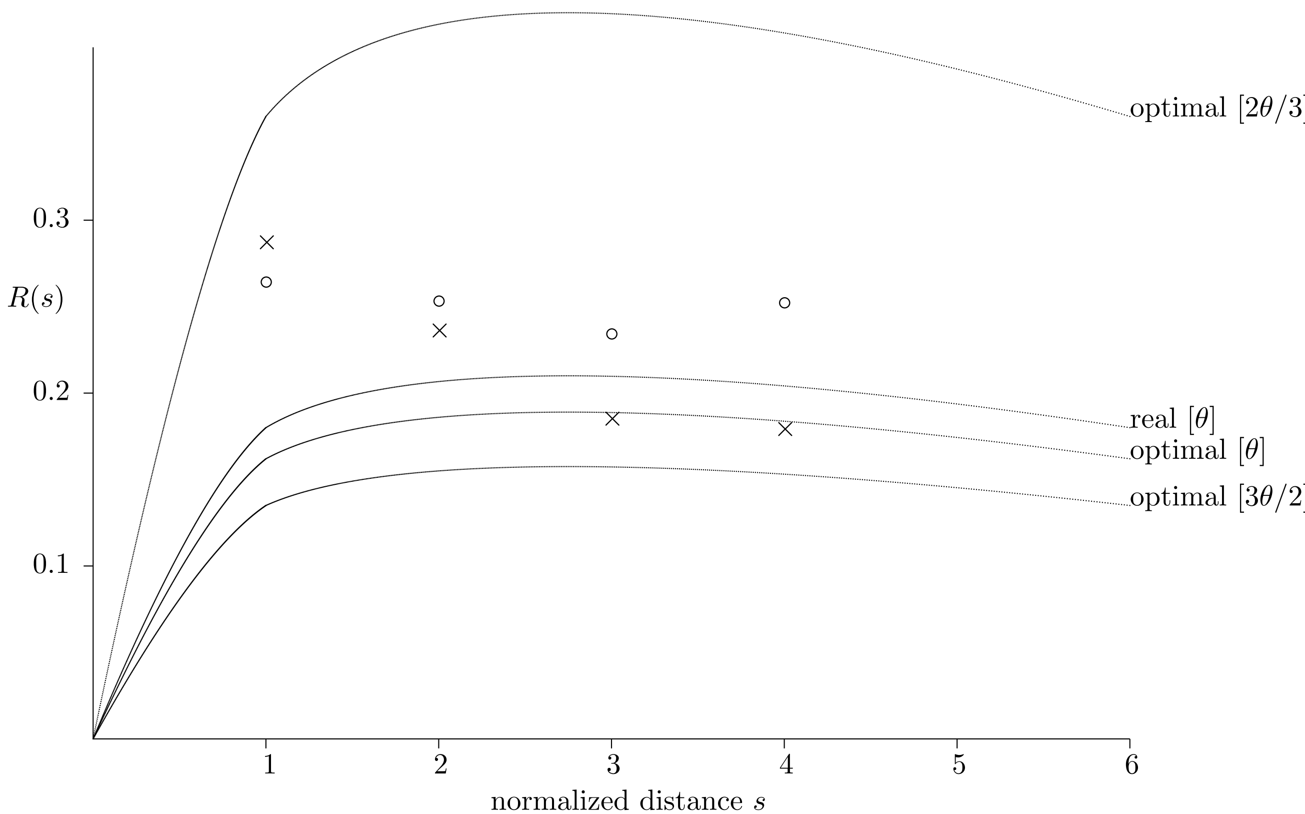

Figure 3. Conjectured qualitative form of $R(s)$ for a typical real network and for hypothetical optimal networks on the same cities. The points $\times$ and $\circ$ are values from the Figure 1 data for Kansas ($\times$) and South and Southeast U.S. ($\circ$).

1.2 Continuing work

As well as collecting data on other networks to calculate the summary statistics (θ, R^*), we need to devise algorithms for actually calculating $R^*(\theta)$ for a given configuration of cities. As a generalization of the Steiner tree problem, this is NP-hard, so we seek heuristic algorithms. During the Spring 2006 project we experimented with several different algorithms, but none seemed sufficiently accurate for investigating our sharp curve conjecture, so we will continue devising algorithms as a future project.

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2 Appendix

2.1 Technical asides

2.1.1 Weighted averages

In analyzing real networks it would be natural to use *weighted* averages over city-pairs (i, j). One could weight by
(i) empirical traffic intensity
or (ii) some function of population sizes n_i, n_j and distance $d(i, j)$; for instance by $n_i n_j d^{-2}(i, j)$.
An incidental advantage of using R^* is in (ii) that one doesn't need to worry about the distance factor.

2.1.2 Close city pairs

When $d(i, j)$ is small, the value of $r(i, j)$ is sensitive to exact details of where in a city we measure from. For this reason it seems best to ignore pairs for which $d(i, j)$ is too small, and our naive procedure (2) automatically ignores pairs with $d(i, j) < \frac{1}{2}$.

2.1.3 The grid configuration

On the square grid network, for the origin i and for a vertex j at polar coordinates (s, θ) with $0 < \theta < \pi/2$ we have $r(i, j) = \cos\theta + \sin\theta - 1$. Thus for large s, by averaging over θ we get a "smoothed" value of $R(s)$ as

$$\tfrac{2}{\pi} \int_0^{\pi/2} 2 \sin\theta \ d\theta \quad - 1 = \tfrac{4}{\pi} - 1 \approx 0.28.$$

2.1.4 South and Southeast U.S. data

The "South and Southeast U.S." data used cities in Georgia, Texas, Louisiana, Alabama, South Carolina, North Carolina, Tennessee, Arkansas, Mississippi, Florida, Oklahoma.

2.2 Why use R^* instead of R_{overall}?

Our reason for using R^* instead of the simpler statistic R_{overall} at (1) is that the latter does not scale as one would expect as n grows, making it hard to compare different sized networks. In fact the following counter-intuitive result is given in [2]. Fix $\varepsilon > 0$ and suppose we have n-city configurations such that the normalized lengths of their Steiner trees tends to some limit θ_{Steiner}. Then we can construct networks of normalized length $\theta_{\text{Steiner}} + \varepsilon$ for which we have $R_{\text{overall}} \to 0$ as $n \to \infty$. Thus for large n the curve in Figure 2, if we used R_{overall} instead of R^*, would approximate a step function at θ_{Steiner} instead of a continuous curve.

This also motivates our conjecture in Figure 3 about the shape of $R(s)$, because for large n and any $\theta > \theta_{\text{Steiner}}$ we can construct networks with $R(s)$ becoming small for large s. On the other hand, the expected number of cities at distance between 1.5 and 2.5 from a given city is $\pi(2.5)^2 - \pi(1.5)^2 = 4\pi \approx 13$ and for reasonable values of θ one cannot have straight roads to all these neighbors, implying that $R(2)$ is intrinsically bounded away from zero. As s gets larger the ability of a network designer to include a low density of long straight roads becomes more relevant in allowing $R(s)$ to be made small.

2.3 The Poisson-Delaunay network

For theoretical study, a natural network to consider is the *Delaunay triangulation* [7]. Consider the Poisson point process model for city positions. In this case a simple calculation gives $\theta = 2$; indeed by considering a city at position x at distance r from the origin 0, the network contains the segment $0x$ iff the disc with diameter ox contains no other city, so

$$\theta = \int_0^\infty \tfrac{r}{2} \times \exp(-\pi(r/2)^2) \times 2\pi r \ dr.$$

Simulations in [3] show that for large s, $R(s) \approx 0.05$, and it would be interesting to simulation the curve $R(s)$ accurately over (say) $0 \leq s \leq 5$ to determine whether the behavior conjectured in Figure 3 (that $R(s)$ increases for s less than some $s_0 \approx 2$, and then decreases) holds.