The top ten things that math probability says about the real world

David Aldous

3 February 2012

1. Everyday perception of chance

The mathematical probability we learn in the classroom seems to have little connection with our experience of chance and uncertainty outside the classroom.

It was easy to write that sentence. Is it true?

Here are some Chapter and section titles from a textbook.

How do people think about chance in everyday life?

There are many ways one might study that question, for example by searching blogs to examine casual usage of specific words or phrases. I will show results of examining a sample of queries submitted to the search engine Bing containing the phrase "chance of" or "probability of". We manually examined about 1,000 queries, retained those where the user was seeking to discover the chance of something, and sorted these 675 retained queries into 66 groups (each containing about 10 queries) of queries on some similar topic. I then chose one representative query from each of these groups. All 66 are on my web page – below I show 30 of them, to indicate the range and frequency of topics that occur in such searches.

Can you guess which topics appear most often?

Do you think they will have much connection with typical textbook topics?

Query: chance of pregnancy on pill
Query: how to improve chance of getting pregnant
Query: chance of getting pregnant at age 41
Query: chance of getting pregnant while breastfeeding

Query: can you increase your chance of having a girl
Query: if twins run in my family what's my chance of having them?

Query: does a father having diabetes mean his children have a 50% chance of getting diabetes
Query: chance of siblings both having autism

Query: chance of miscarriage after seeing good fetal movement heartbeat at 10 weeks
Query: chance of bleeding with placenta previa
Query: any chance of vaginal delivery if first birth was ceaserian

Query: probability of having an adverse reaction to amoxicillin
Query: does hypothyroid in women increase chance of liver cancer?
Query: does progesterone increase chance of breast cancer
Query: which treatment has the least chance of prostate cancer recurring?
Query: what is the chance of relapse in a low risk acute lymphoblastic lukemia patient
Query: chance of getting a brain tumor

Query: probability of flopping a set with pocket pair in poker
Query: does a ring of wealth affect the chance of the dragon pickaxe drop in runescape?

Query: chance of surviving severe head injury

Query: chance of snow in Austin Texas
Query: is there chance of flood in Saint Charles, Illinois today?

Query: calculate my chance of getting to university of washington
Query: what are the chance of becoming a golf professional

Query: chance of closing airports in Mexico because of swine flu

Query: any chance of incentive packages for government employees to retire

Query: chance of children of divorce being divorced
Query: chance of food spoiling if left out over night

Query: what does it mean 50/50 chance of living 5 years
Query: probability of life and evolution

In my course at Berkeley I give 20 lectures on very different topics relating to Probability. I asked students to say (if they had an opinion) whether I should re-use the topic next time. Here are the results – the number who said Yes minus the number who said No.

I was intrigued to see that the less mathematical classes were generally more popular than the more mathematical ones. This dispels the possibility that we had brainwashed our students into thinking that only quantitative data is real; but keeps open the possibility that they find mathematics difficult and are relieved not to face it.

(22) Psychology of probability: predictable irrationality **[5]**
(18) Global economic risks **[4]**
(17) Everyday perception of chance **[1]**
(16) Luck
(16) Science fiction meets science
(14) Risk to individuals: perception and reality
(13) Probability and algorithms.
(13) Game theory.
(13) Coincidences and paradoxes.
(11) So what do I do in my own research? (spatial networks)
(10) Stock Market investment, as gambling on a favorable game **[2]**
(10) Mixing and sorting
(9) Tipping points and phase transitions
(9) Size-biasing, regression effect and dust-to-dust phenomena
(6) Prediction markets, fair games and martingales
(6) Branching processes, advantageous mutations and epidemics
(5) Toy models of social networks
(4) The local uniformity principle
(2) Coding and entropy **[3]**
(-5) From neutral alleles to diversity statistics.

2. Stock Market investment, as gambling on a favorable game

The Kelly criterion marks the borderline between aggressive and insane investing.

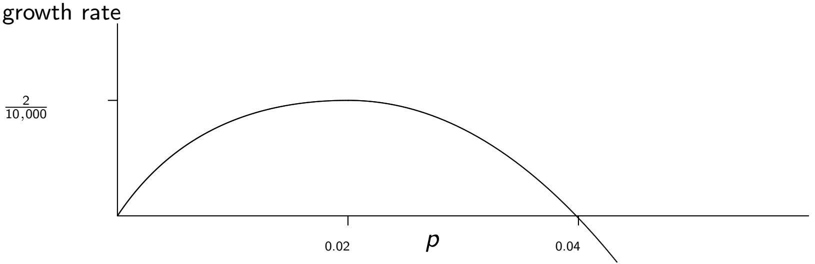
Background: if you're worth $500,000 then it's irrational to be risk-averse for small amounts – should regard "gaining $250" and "losing $250" as equal-but-opposite. But it's rational to be risk-averse for $250,000.

In fact people are **Predictably Irrational** (title of recent Dan Ariely book, and item 7 on our list) in such matters, but . . . . . .

I focus on **long-term** investment. Imagine you inherit a sum of money at age 25 and you resolve to invest it and not start spending it until age 65. We envisage the following setting.
(i) You always have a choice between a safe investment (pays interest, no risk) and a variety of risky investments. You know the probabilities of the outcomes of these investments. [of course in reality you don't know probabilities – unlike casino games – so have to use your best guess instead].
(ii) Fixed time period – imagine a year, could be month or a day – at end your take you gains/losses and start again with whatever you've got at that time ("rebalancing").

The Kelly criterion gives you an explicit rule for how to divide your investments to maximize long-term growth rate.

To illustrate, imagine day-trading scheme with stocks based on some statistical non-randomness; within one day
51% chance to double money; 49% chance to lose all money.
Looks good – expected gain 2% per day – but don't want to risk all your money on one day. Instead use strategy: bet fixed proportion p of money each day. Theory says: long-term growth rate, depends on p, but in an unexpected way.



Optimal strategy: bet $p = 2\%$ of your capital each day; this provides growth rate $\frac{2}{10,000}$ per day, which (250 trading days per year) becomes 5% per year.

The numbers above depended on hypothetical assumptions. But the conceptual point is completely general. We are not assuming you can predict the future, just that you can assess future **probabilities** correctly. Provided there is **some** risky investment whose expected payoff is greater than the risk-free payoff, the **Kelly criterion** is a formula that tells you how to divide your portfolio between different investments.

There's one remarkable consequence of using this strategy. To get the maximum possible long-term growth rate, using "100% Kelly strategy", you must accept a short-term risk, of the specific form

50% chance that at some time your wealth will drop to only 50% of your initial wealth.

And 10% – 10% too! Of course, if not comfortable with this level of risk, you can use "partial Kelly strategy" combining with risk-free assets.

This story is told in the popular book **Fortune's Formula** by William Poundstone. Maybe nothing in this story seems intellectually remarkable, but in fact something is. Consider an analogy: the light speed barrier.

[Common sense says objects can be stationary or move slowly or move fast or move very fast, and that there should be no theoretical limit to speed – but physics says in fact you can't go faster than the speed of light. And that's a very non-obvious fact.]

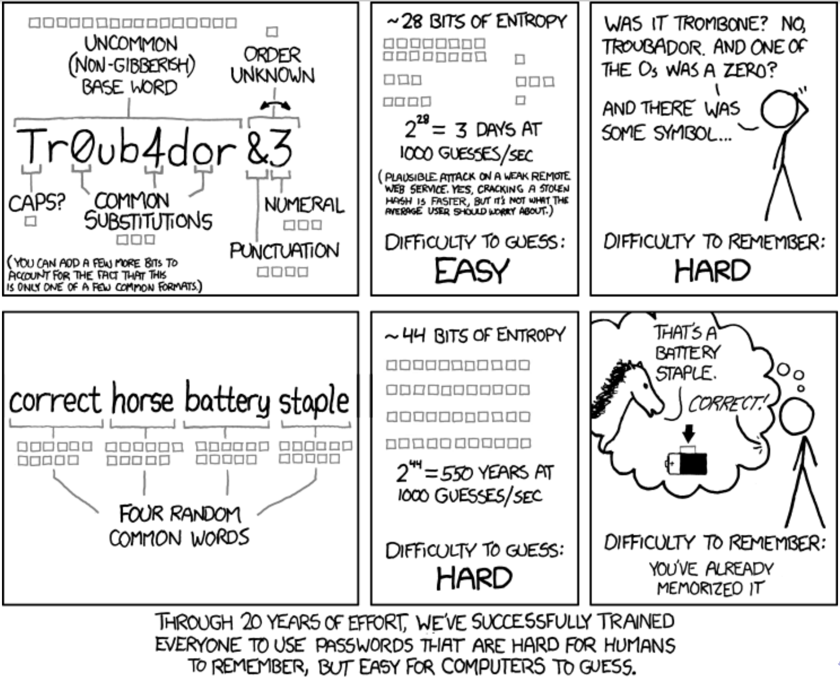
Similarly, we know there are risk-free investments with low return; by taking a little risk (**risk** here equals short-term fluctuations) we can get higher low-term reward. Common sense says this risk-reward trade-off spectrum continues forever. But in fact it doesn't. As a math fact, you can't get a higher long-term growth rate than you get from the "100% Kelly strategy".

3. Coding and entropy

5. Coding for secrecy is essentially the same as coding for efficient communication or storage.

The fact that most letter strings JQTOMXDW KKYSC have no meaning is what makes most simple letter substitution ciphers easy to break. In a hypothetical language in which every "monkeys on typewriters" string had a meaning, a letter substitution cipher would be impossible to break, because each of the 26 x 25 x 24 x .... x 2 possible decodings might be the true message.

Now if you want to transmit or store information efficiently, you want every string to be possible as a coded string of some message (otherwise you're wasting an opportunity) and indeed you ideally want every string to be **equally likely** as the coded string of some message. This is "coding for efficiency", but with such an ideal public code one could just apply a private letter substitution cipher and get an unbreakable "code for secrecy".

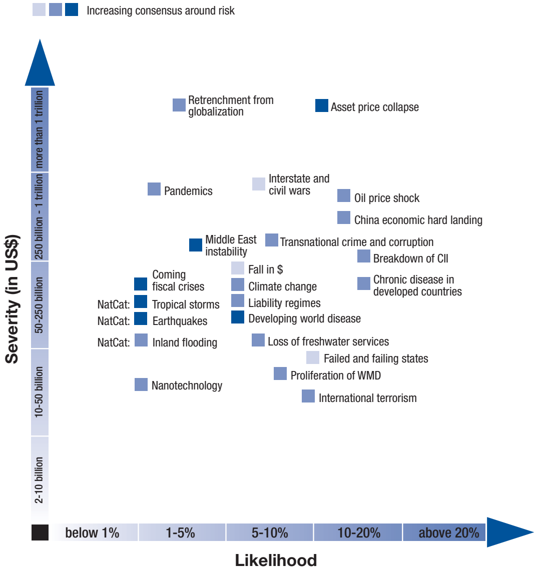

UNCOMMON (NON-GIBBERISH) BASE WORD
ORDER UNKNOWN
Tr0ub4dor &3
CAPS?
COMMON SUBSTITUTIONS
NUMERAL
PUNCTUATION
(YOU CAN ADD A FEW MORE BITS TO ACCOUNT FOR THE FACT THAT THIS IS ONLY ONE OF A FEW COMMON FORMATS.)
~28 BITS OF ENTROPY
2^{28} = 3 DAYS AT 1000 GUESSES/SEC
(PLAUSIBLE ATTACK ON A WEAK REMOTE WEB SERVICE. YES, CRACKING A STOLEN HASH IS FASTER, BUT IT'S NOT WHAT THE AVERAGE USER SHOULD WORRY ABOUT.)
DIFFICULTY TO GUESS: EASY
WAS IT TROMBONE? NO, TROUBADOR. AND ONE OF THE Os WAS A ZERO?
AND THERE WAS SOME SYMBOL...
DIFFICULTY TO REMEMBER: HARD
correct horse battery staple
FOUR RANDOM COMMON WORDS
~44 BITS OF ENTROPY
2^{44} = 550 YEARS AT 1000 GUESSES/SEC
DIFFICULTY TO GUESS: HARD
THAT'S A BATTERY STAPLE.
CORRECT!
DIFFICULTY TO REMEMBER: YOU'VE ALREADY MEMORIZED IT
THROUGH 20 YEARS OF EFFORT, WE'VE SUCCESSFULLY TRAINED EVERYONE TO USE PASSWORDS THAT ARE HARD FOR HUMANS TO REMEMBER, BUT EASY FOR COMPUTERS TO GUESS.

Global economic risks

How good are risk estimates?

A cynical view of retrospective analysis of the late-2000s worldwide financial crisis is that commentators say either "no-one saw it coming" or "I saw it coming", depending on whether they can exhibit evidence of the latter! Is such cynicism justified?

Each year since 2006 the OECD has produced a "global risks" report for the World Economic Forum annual meeting in Davos. The 2007 report, written around January 2007 (at which time there were concerns about the worldwide boom in house prices, and some concerns about U.S. subprime mortgages, but nothing dramatic had happened in other markets) gave, as in other years, a list of "'core global risks", summarized using the next graphic. The horizontal axis shows "likelihood" and the vertical axis shows economic effect.


Increasing consensus around risk
Severity (in US$)
more than 1 trillion
250 billion - 1 trillion
50-250 billion
10-50 billion
2-10 billion
Retrenchment from globalization
Asset price collapse
Pandemics
Interstate and civil wars
Oil price shock
China economic hard landing
Middle East instability
Transnational crime and corruption
Breakdown of CII
Fall in $
Coming fiscal crises
Climate change
Chronic disease in developed countries
NatCat: Tropical storms
Liability regimes
NatCat: Earthquakes
Developing world disease
NatCat: Inland flooding
Loss of freshwater services
Failed and failing states
Proliferation of WMD
Nanotechnology
International terrorism
below 1%
1-5%
5-10%
10-20%
above 20%
Likelihood

Definining risk as "likelihood" multiplied by "economic effect", the 5 most serious risks (as assessed in 2008) were

Asset price collapse
Oil price shock
China economic hard landing
Inter-state and civil wars
Breakdown of civil informational infrastructure

The entry "asset price collapse" was defined as

"A collapse of real and financial asset prices leads to the destruction of wealth, deleveraging, reduced household spending and impaired aggregate demand."

Given that these 5 risks were assessed to have 10-20% likelihood and that one of them occured with even more than predicted severity, this OECD assessment is actually as good as one could hope for. Note that the "oil price shock" assessed as 2nd most serious did almost occur in 2008 but was overtaken by the asset price collapse and did not have the severe impact predicted – next graphic.

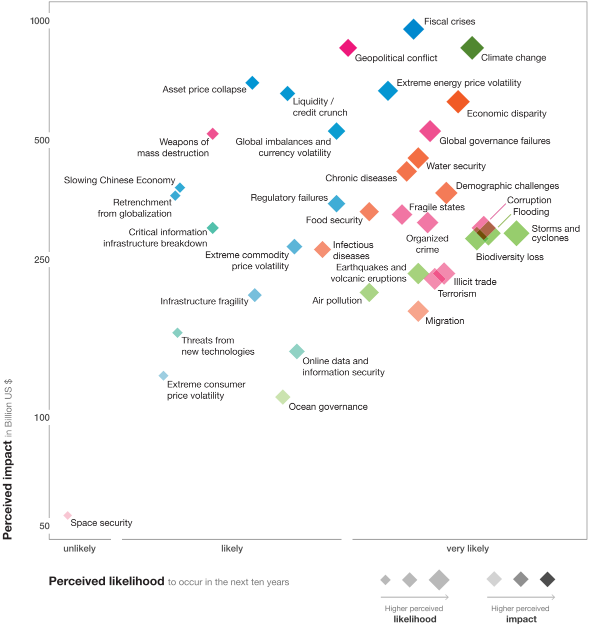

Crude oil, Brent price chart
133.9
110.8
87.7
64.7
41.6
Sep-06
Mar-07
Sep-07
Mar-08
Sep-08
Mar-09
Sep-09
Mar-10
Sep-10
Mar-11
Sep-11
($/bbl)

What's my point? Interesting aspects of the future are uncertain, so whatever forecasting/prediction you do, say it is probability terms.

With this in mind let us look at the corresponding graphic from the 2011 report. Now the most serious risks (for the next 10 years) are assessed as

Climate change
Fiscal crises
Economic disparity
Geopolitical conflict
Extreme energy price volatility

Figure 1 | Global Risks Landscape 2011:
Perception data from the World Economic Forum's Global Risks Survey



5. Psychology of probability: predictable irrationality

Much psychology research since 1980 (Amos Tversky et al) involves experiments on "decisions under uncertainty". Here's a famous example: decisions can be strongly affected by how information is presented.

Imagine a rare disease is breaking out in some community. if nothing is done, 600 people will die. There are two possible programs. To some subjects you describe the alternatives as
(A) will save 200 people
(B) will save everyone with chance $1/3$ and save no-one with chance $2/3$

to others you describe the alternatives as
(C) 400 people will die
(D) no-one will die with chance $1/3$; 600 people will die with chance $2/3$.

Given "A or B" choice, most people choose A.
Given "C or D" choice, most people choose D.

In my undergraduate course, students do course projects, and one option is to repeat some classic experiment. Here's a fun example.

- Subjects: college educated, non-quantitative majors.
- Equipment: bingo balls (1 – 75) and 10 Monopoly $500 bills.
- Draw balls one at a time; subject has to bet $500 on whether next ball will be higher or lower than last ball; prompt subject to talk (recorded) about thought process. Repeat for 5 bets.
- Say: we're doing this one last time; this time you have option to bet all your money. Prompt talk.

What is the point of this experiment?

In first part, everyone "plays the odds" – behaves and explains rationally: if this ball is 43 then more likely that next ball is less than 43, so bet that way.

Point: what **explanations** do people give for their choice (last stage) of whether or not to bet all their money. In our experiments, about 50-50 split between

- risk-aversion; good or poor chances to win
- feeling (or have been) lucky or unlucky.

Conclusion: even when "primed" to think rationally, people have innate tendency to revert to "luck" explanations.

Wrap-up: probability in fantasy and reality

Most textbook examples and questions are either "just maths" – X's and Y's – or unrealistic little stories, for example

a. A student must choose exactly two out of three electives: art, French, and mathematics. He chooses art with probability $5/8$, French with probability $5/8$, and art and French together with probability $1/4$. What is the probability that he chooses mathematics? What is the probability that he chooses either art or French?

b. A restaurant offers apple and blueberry pies and stocks an equal number of each kind of pie. Each day ten customers request pie. They choose, with equal probabilities, one of the two kinds of pie. How many pieces of each kind of pie should the owner provide so that the probability is about .95 that each customer gets the pie of his or her own choice?

c. Take a stick of unit length and break it into two pieces, choosing the break point at random. Now break the longer of the two pieces at a random point. What is the probability that the three pieces can be used to form a triangle?

d. Suppose you toss a dart at a circular target of radius 10 inches. Given that the dart lands in the upper half of the target, find the probability that

1. it lands in the right half of the target.
2. its distance from the center is less than 5 inches.
3. its distance from the center is greater than 5 inches.
4. it lands within 5 inches of the point $(0, 5)$.

e. You are in a casino and confronted by two slot machines. Each machine pays off either 1 dollar or nothing. The probability that the first machine pays off a dollar is x and that the second machine pays off a dollar is y. We assume that x and y are random numbers chosen independently from the interval $[0, 1]$ and unknown to you. You are permitted to make a series of ten plays, each time choosing one machine or the other. How should you choose to maximize the number of times that you win?

(22) Psychology of probability: predictable irrationality **[5]**
(18) Global economic risks **[4]**
(17) Everyday perception of chance **[1]**
(16) Luck
(16) Science fiction meets science
(14) Risk to individuals: perception and reality
(13) Probability and algorithms.
(13) Game theory.
(13) Coincidences and paradoxes.
(11) So what do I do in my own research? (spatial networks)
(10) Stock Market investment, as gambling on a favorable game **[2]**
(10) Mixing and sorting
(9) Tipping points and phase transitions
(9) Size-biasing, regression effect and dust-to-dust phenomena
(6) Prediction markets, fair games and martingales
(6) Branching processes, advantageous mutations and epidemics
(5) Toy models of social networks
(4) The local uniformity principle
(2) Coding and entropy **[3]**
(-5) From neutral alleles to diversity statistics.