# Short-length routes in low-cost networks *via* Poisson line patterns

(joint work with David Aldous)

Wilfrid Kendall

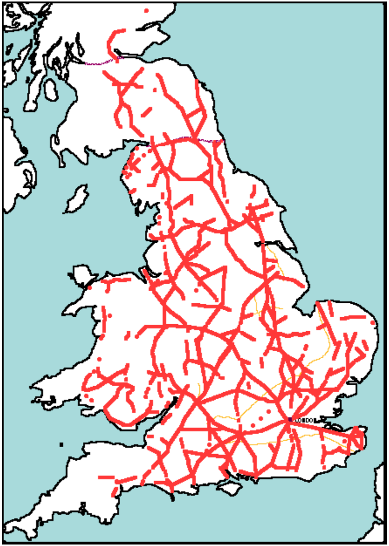
w.s.kendall@warwick.ac.uk

"Stochastic processes and algorithms" workshop, Hausdorff Research Institute for Mathematics
Bonn, 3-7 September 2007

7 September 2007

THE UNIVERSITY OF WARWICK

# An ancient optimization problem



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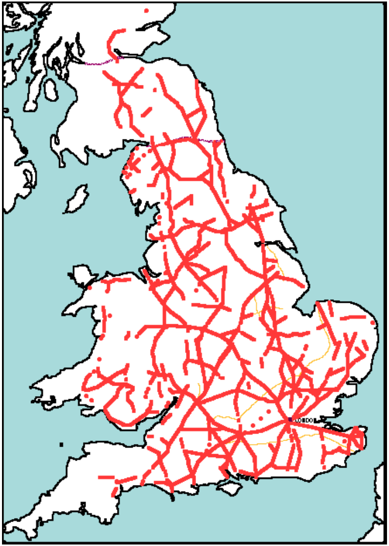

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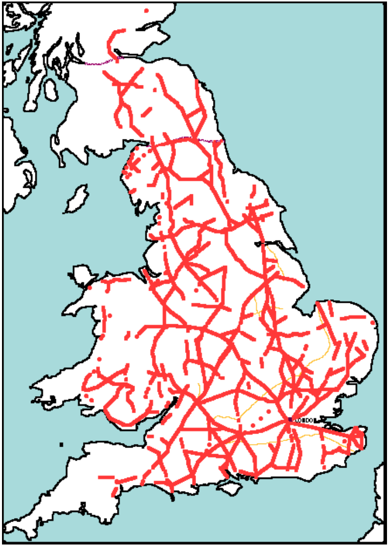

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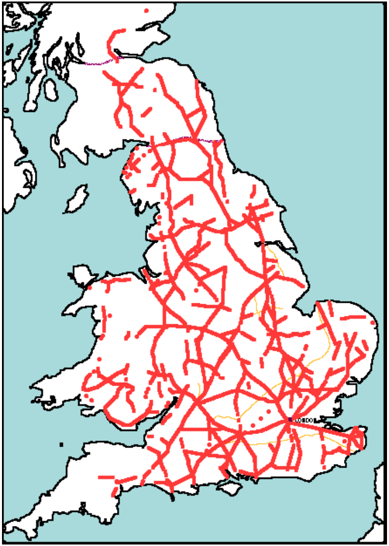

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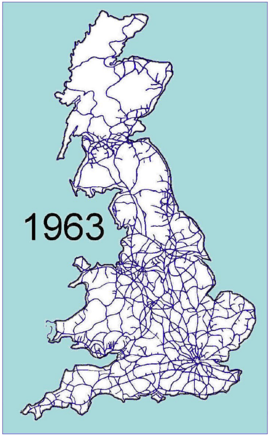
Pro optimo quod faciendum est?

# Modern variants

British Railway network before Beeching



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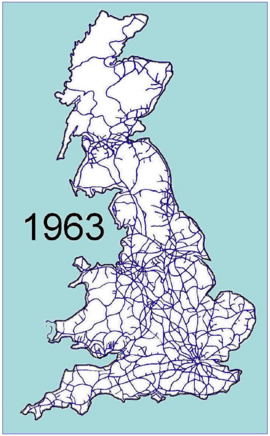


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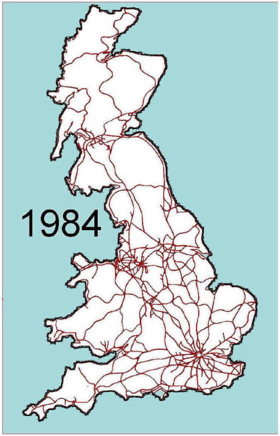

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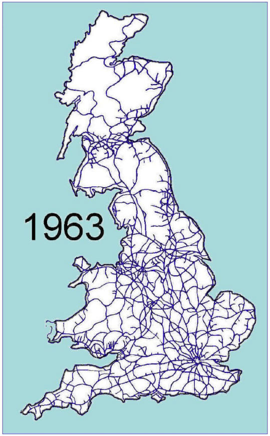


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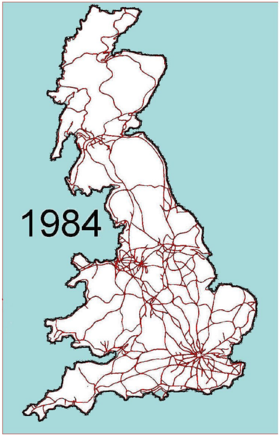
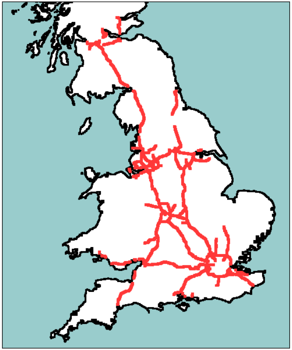



UK Motorways:



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(minimized by laying tarmac for complete graph).

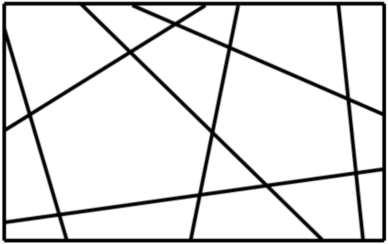
Aldous and K. (2007) provide answers for the following

**Question**

Consider a configuration $x^{(N)}$ of $N$ cities in $[0, \sqrt{N}]^2$ as above, and a well-chosen connecting network $G = G(x^{(N)})$. How does large-$N$ trade-off between $\text{len}(G)$ and $\text{average}(G)$ behave?

And how clever do we have to be to get a good trade-off?

# Today's focus (I)



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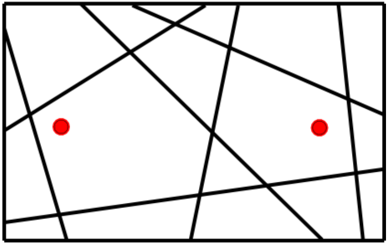

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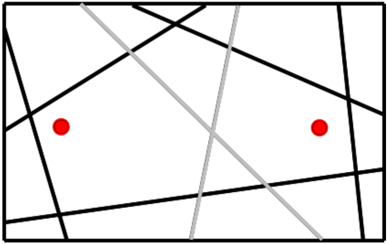

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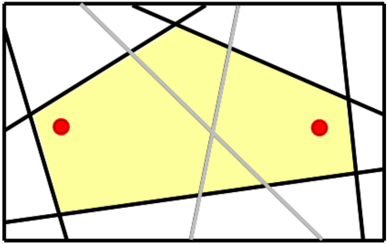
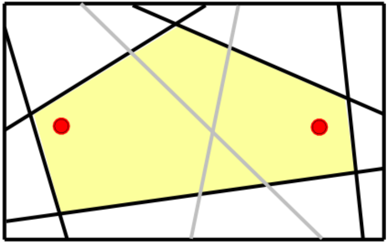


- Idealize the road network as a low-intensity invariant Poisson line process $\Pi_1$.
- Pick two cities $x$ and $y$ at distance $n$ units apart. Remove lines separating the cities and identify the cell $\mathcal{C}_{x,y}$ which then contains the two cities.

# Today's focus (II)



- Upper-bound the "network distance" between the two cities by the mean semi-perimeter of this cell, $\frac{1}{2}\,\mathbb{E}\left[\text{len}\,\partial\mathcal{C}_{x,y}\right]$.

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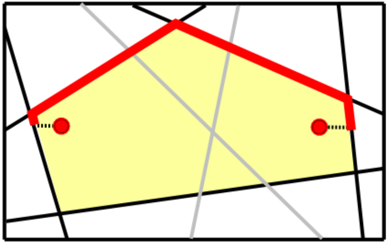

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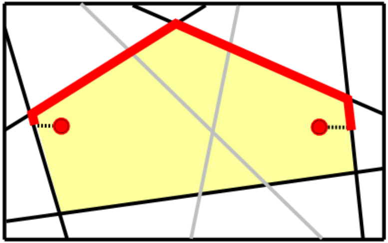

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- Aldous and K. (2007) show how to apply this to resolve our Question, and how to use other methods from stochastic geometry to show that the resolution is nearly optimal.

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- Calculate $\pi$ by dropping a needle randomly on a ruled plane and counting mean proportion of hits,

  or (dually)
- (H. Steinhaus) compute length of a *regularizable* curve by counting mean number of hits of curve by a unit-intensity invariant Poisson line process.

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**Angles** Generate a planar line process from a unit-intensity Poisson point process on a reference line $\ell$, by constructing lines through the points whose angles $\theta \in (0, \pi)$ to $\ell$ are independent with density $\frac{1}{2} \sin \theta$.

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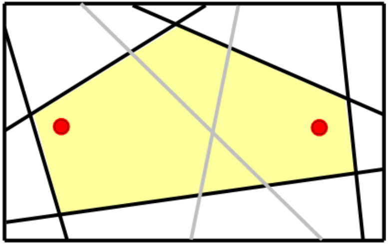
# The key construction



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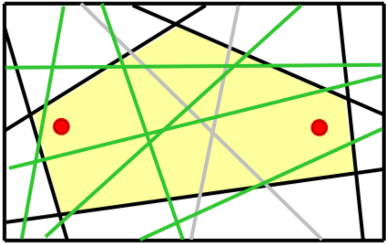

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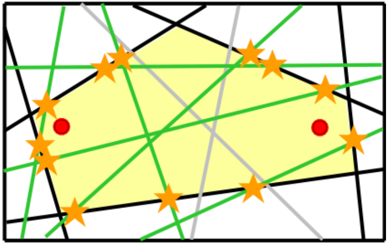

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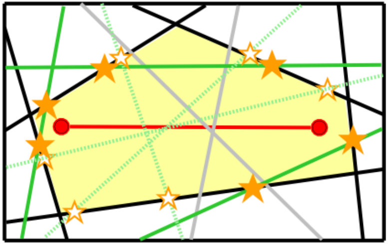

For simplicity, renormalize to a unit-intensity line process.

- Compute mean length of $\partial\mathcal{C}_{x,y}$ by use of independent unit-intensity invariant Poisson line process $\Pi_2$, and determine the mean number of hits.
- It is convenient to form $\Pi_2^*$ by deleting from $\Pi_2$ those lines separating $x$ from $y$. (Mean number of hits: $2|x-y| = 2n$.)

# Some stochastic geometry (I)

- We have

$$\mathbb{E}\left[\operatorname{len}\partial\mathcal{C}_{x,y}\right] - 2|x-y| \quad = \quad \mathbb{E}\left[\#\left(\Pi_2^* \cap \partial\mathcal{C}_{x,y}\right)\right] .$$

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- This is the total intensity of the intersection point process $\Pi_1^* \cap \Pi_2^*$ thinned by removing $z \in \Pi_1^* \cap \Pi_2^*$ when $z$ is separated from both $x$ and $y$ by $\Pi_1^*$.

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- We can appeal to a variant of Slivynak's theorem: the retention probability for $z$ is the probability that no line of $\Pi_1^*$ hits both of segments $\overline{xz}$ and $\overline{yz}$, namely

$$\exp\left(-\tfrac{1}{2}\left(|x-z|+|y-z|-|x-y|\right)\right) \;=\; \exp\left(-\tfrac{1}{2}\left(\eta-n\right)\right) \;,$$

where $\eta = |x-z|+|y-z|$.

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- The intensity of $\Pi_1^* \cap \Pi_2^*$ is obtained by careful computation of the probability that the intersection lines of a point of $\Pi_1 \cap \Pi_2$ do not hit $\overline{xy}$, using the "Angle" construction from above. The resulting intensity is:

$$\frac{\pi}{2} \times \frac{\alpha - \sin\alpha}{\pi} \quad = \quad \frac{\alpha - \sin\alpha}{2},$$

where $\alpha$ is the exterior angle of the triangle $\Delta xyz$ at $z$.

# Mean perimeter length as a double integral

## Theorem

$$\mathbb{E}\left[\operatorname{len}\partial\mathcal{C}_{x,y}\right] - 2|x-y| \quad = \quad \binom{\text{intersection}}{\text{intensity}} \times \iint_{\mathbb{R}^2} \begin{pmatrix}\text{intersection at } z\text{:}\\ \text{lines don't hit } \overline{xy}\end{pmatrix} \begin{pmatrix}\text{retention:}\\ z \text{ not sep from } \overline{xy}\end{pmatrix} \mathrm{d}\, z$$

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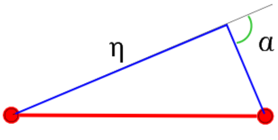
- Note that $\alpha = \alpha(z)$ and $\eta = \eta(z)$ both depend on $z$.



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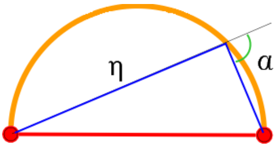


- Fixed $\alpha$: locus of $z$ is circle.

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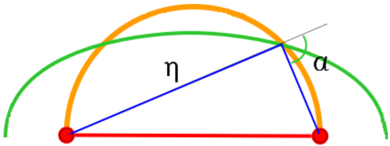


- Fixed $\alpha$: locus of $z$ is circle.
- Fixed $\eta$: locus of $z$ is ellipse.

# Asymptotics

## Theorem

Careful asymptotics for $n \to \infty$ show that

$$\mathbb{E}\left[\tfrac{1}{2}\operatorname{len}\partial\mathcal{C}_{x,y}\right] = n + \tfrac{1}{4}\iint_{\mathbb{R}^2}(\alpha - \sin\alpha)\exp\left(-\tfrac{1}{2}(\eta - n)\right)\mathrm{d}\,z \approx n + \frac{4}{3}\left(\log n + \gamma + \frac{5}{3}\right)$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascharoni constant.

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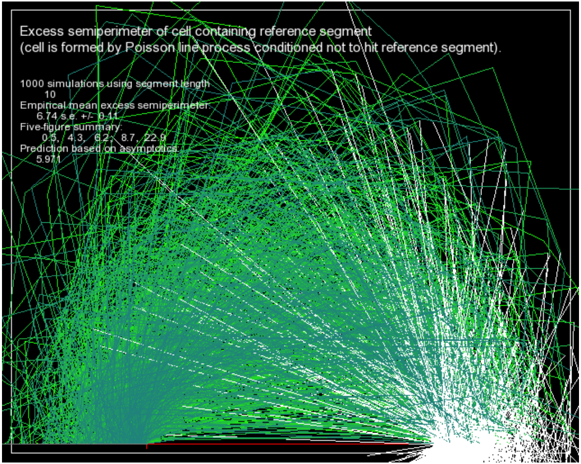
Thus a unit-intensity invariant Poisson line process is within $O(\log n)$ of providing connections which are as efficient as Euclidean connections.

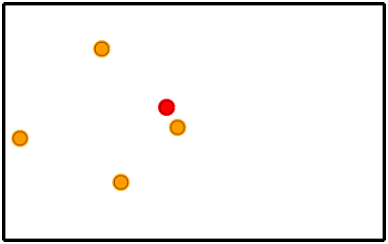
# Simulations (example)




1000 simulations at $n = 10$: average 6.74, s.e. 0.41, asymptotic 5.971.

Vertical exaggeration: $\sqrt{n}$

# Illustration of the final construction



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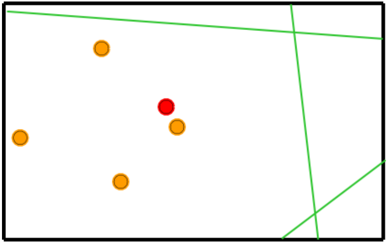

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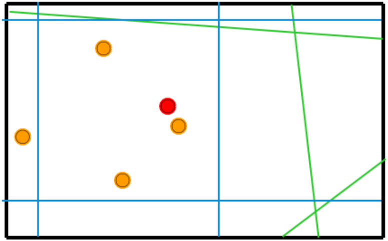

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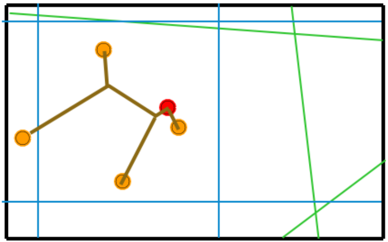

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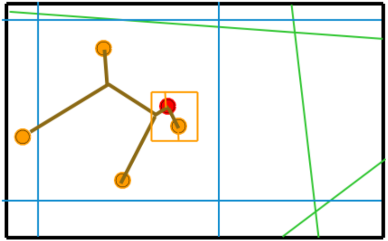

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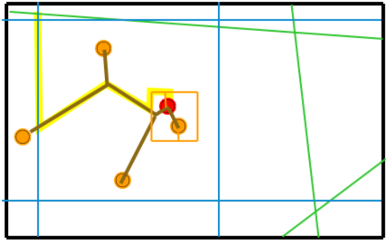
2. a rectangular grid at a moderately large length scale;
3. the Steiner tree $\mathrm{ST}(x^{(N)})$;
4. a few boxes from a grid at a small length scale, to avoid potential “hot-spots” where cities are close (boxes are connected to the cities).

# Illustration of the final construction



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# The result

**Theorem**

For any configuration $x^{(N)}$ in square side $\sqrt{N}$ and for any sequence $w_N \to \infty$ there are connecting networks $G_N$ such that:

$$\begin{aligned} \mathrm{len}(G_N) &= \mathrm{len}(\mathrm{ST}(x^{(N)})) + o(N) \\ \mathrm{average}(G_N) &= \frac{1}{N(N-1)} \sum\sum_{i \neq j} \|x_i - x_j\| + o(w_N \log N) \end{aligned}$$

The sequence $\{w_N\}$ can tend to infinity arbitrarily slowly.

# A complementary result

## Theorem

**Given** a configuration of $N$ cities in $[0, \sqrt{N}]^2$ which is $L_N = o(\sqrt{\log N})$-equidistributed: random choice $X_N$ of city can be coupled to uniformly random point $Y_N$ so that

$$\mathbb{E}\left[\min\left\{1, \frac{|X_N - Y_N|}{L_N}\right\}\right] \quad \longrightarrow \quad 0\,;$$

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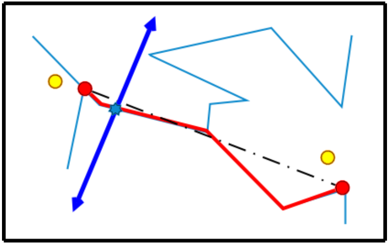
**then** any connecting network $G_N$ with length bounded above by a multiple of $N$ connects the cities with average connection length exceeding average Euclidean connection length by at least $\Omega(\sqrt{\log N})$ .

# Sketch of proof



Use tension between two facts:

(a) efficient connection of a random pair of cities forces a path which is almost parallel to the Euclidean path, *and*

(b) the coupling means such a random pair is almost an independent uniform draw from $[0, \sqrt{N}]^2$ (equidistribution), so a random perpendicular to the Euclidean path is almost a uniformly random line.

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QUESTIONS?

## Bibliography

This is a rich hypertext bibliography. Journals are linked to their homepages, and stable URL links (as provided for example by JSTOR or Project Euclid ) have been added where known. Access to such URLs is not universal: in case of difficulty you should check whether you are registered (directly or indirectly) with the relevant provider. In the case of preprints, icons , , , linking to homepage locations are inserted where available: note that these are less stable than journal links!.

Aldous, D. J. and W. S. K. (2007).
Short-length routes in low-cost networks via Poisson line patterns.
Research Report 451, , and http://arxiv.org/abs/math.PR/0701140 arXiv

Steele, J. M. (1997).
*Probability theory and combinatorial optimization*, Volume 69 of *CBMS-NSF Regional Conference Series in Applied Mathematics*.
Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Stoyan, D., W. S. K., and J. Mecke (1995).
*Stochastic geometry and its applications* (Second ed.).
Chichester: John Wiley & Sons.
(First edition in 1987 joint with Akademie Verlag, Berlin).