Notes on project in progress. All the simulatioms are by Weijian Han.

Waves in a spatial queue and Brownian scaling

Long queues of people. In classical models of queues, randomness enters via assumed randomness of arrival and service times. (xxx cite) We consider a setting where randomness enters in a conceptually quite different way. Imagine joining the end of a long line – at airport security, or for the first showing of a popular movie. The people at the front are being served in a regular way. What if you're the 100th person in line? If people were robots, each time one person departed, everyone might move forwards exactly the same distance, so the times between your successive moves would be the times between the departures. But in our everyday experience, what actually happens is that you move forwards less frequently but by a larger distance than the inter-person distance. In other words, each departure creates a "wave" of movement of the people near the front of the queue, but often this wave dies out before reaching the end of the line. There is a rather obvious qualitative explanation. Humans have a "comfort zone" – we want to stand neither too close to, nor too far behind, the person in front of us; so when that person moves, we move only if we need to move in order to remain in that comfort zone.

Figure 1 shows a schematic. People are lined up left-to-right at time 0. At time 1 the head person in stage 0 has been served, the next person has moved to the front of the line, the next two people have moved forward but the other people have not moved. At time 2 the next person has moved to the front, the next seven people have moved forward, but the eighth has not moved.

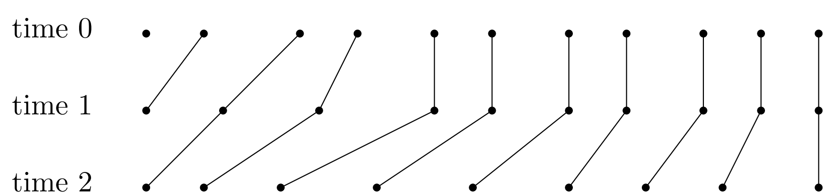

Figure 1. Schematic for a waiting line, with the person being served at the left end. The • indicates the center of a person.

We will study a simple stochastic model for this phenomenon, seeking to understand quantitative behavior of the model. Curiously, we do not know previous work on this natural-looking model.

The queue model. There is a single parameter $c > 1$. In words, the description is very simple. The "comfort zone" is the interval $[1, c]$ of distance. To the qualitative description earlier we add the rule

> For sucessive people $(i-1, i)$, if $i-1$ moves to a position at distance more than c ahead of i, then i moves forward, to the position at a distance ξ behind $i-1$, where ξ is uniform random on $[1, c]$.

To say this more formally, at each discrete time $t = 0, 1, 2, \ldots$ there are people at an infinite set of positions $0 = X_t(0) < X_t(1) < X_t(2) < \ldots$ on the real half-line, these positions satisfying

$$1 \leq X_t(i) - X_t(i-1) \leq c \text{ for each } t \geq 0 \text{ and each } i \geq 1.$$

Given $\mathbf{X}_t = (X_t(0), X_t(1), \ldots)$, take $\xi_t(1), \xi_t(2), \ldots$ independent uniform on $[1, c]$ and construct $\mathbf{X}_{t+1}$ by

$X_{t+1}(0) = 0$

if $X_t(2) \leq c$ then set $X_{t+1}(i) = X_t(i+1), i \geq 1$ and the construction is finished; otherwise set $X_{t+1}(1) = \xi_t(1)$ and continue as follows;

if $X_t(3) - X_{t+1}(1) \leq c$ then set $X_{t+1}(i) = X_t(i+1), i \geq 2$ and the construction is finished; otherwise set $X_{t+1}(2) = X_{t+1}(1) + \xi_t(2)$ and continue;

....

Note that in this notation, i is the current rank of a person; so the successive positions of a particular individual are $X_t(i), X_{t+1}(i-1), X_{t+2}(i-2), \ldots$. Note also that the case where the comfort zone is $[c_1, c_2]$ can be reduced to our case $[1, c]$ by scaling.

Remarks on the model. The actual phenomenon we will study – the "waves" of motion – is unaffected by adding more realistic structure such as random service times, or non-instantaneous moves of the people in line; our results just depend on the model for where people stand relative to the person in front. That is clear *a priori*. Even more emerges (at last intuitively) from the asymptotic analysis: we don't require the same "comfort zone" distribution for each person, just that for a given person their successive moves involve IID distances ξ.

A too-simple heuristic. We start with a simple heuristic for visualizing a limit process; this turns out to be only partly correct but points the way to the correct picture.

Consider a spatial interval of fixed length, say $[x \pm 10]$, as x becomes large. Suppose that when a wave reaches this region, the individuals move forward by more than 20 units. Then after the move the position of new people in this interval will be approximately a realization of the stationary renewal process associated with ξ, independent of the pre-wave configuration. Because this configuration is unchanged until the next wave arrives, we see the approximation

> over the interval $[x \pm 10]$ for large x, a wave replaces one realization of the stationary renewal process by an independent realization.

Consider the person nearest to the right of x before a wave. In the wave he moves some distance D. The positions, relative to this same person, of the j'th successive people, before and after the wave, will be independent partial sum processes $(S_j = \sum_{i=1}^{j} \xi_i, j \geq 0)$ and $(S'_j = \sum_{i=1}^{j} \xi'_i, j \geq 0)$. The wave will continue to move some number J of people, where J is approximately the first j such that $S_j - S'_j = -D$. So in terms of the mean-zero random walk $(\tilde{S}_j = \sum_{i=1}^{j} (\xi_i - \xi'_i, j \geq 0)$, J is approximately the first hitting time on $-D$. This is essentailly saying that the way that waves overshoot position x is the same as the way that positive excursions of the mean-zero random walk overshoot x, and suggests the following picture for the $x \to \infty$ asymptotics:

(i) Waves reach x with probability proportional to $x^{-1/2}$

(ii) The distance moved in such a wave by a person near x has Rayleigh distribution with mean proportional to $x^{1/2}$

(iiii) Successive waves reach x at the times of a Poisson process of rate proportional to $x^{-1/2}$, and the distances moved are i.i.d.

(Here the Rayleigh distribution arises at the terminal state of Brownian meander).

Simulations. In these we set $c = 3.0$, maintained a line of 10,000 people, and ran for 10,000 steps.

Figure 1 shows the number of waves involving more than i people. This appears to follow a power law $i^{-\gamma}$ with $\gamma \approx 0.56$, reasonably consistent with (i). Figure 2 shows, for the 5000'th person, the proportion of times $G(x)$ that a move distance is $> x$. Consistent with (ii), this is a close fit to (smooth curve) the Rayleigh distribution $\exp(-ax^2)$ for the best-fit value of a.

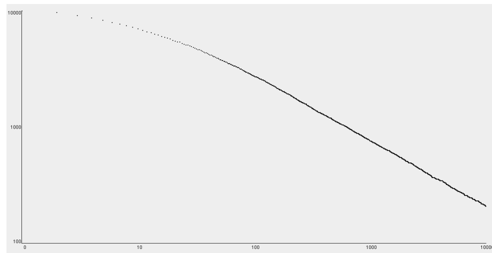

Figure 1. log-log plot of relative number of waves involving more than i people.

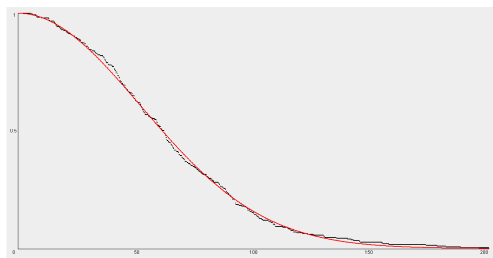

Figure 2. Complementary distribution function of move distances.

Next we considered inter-move times (the time between successive waves reaching the 5,000'th person). Figure 3 (left) shows $H(y)$ = proportion of inter-move times $> y$, and Figure 3 (right) is a scatter diagram showing sucessive inter-move times. The former fits not the exponential distribution predicted by (iii) but a Rayleigh distribution, and the second has correlation $= -0.2$ rather than the predicted 0.

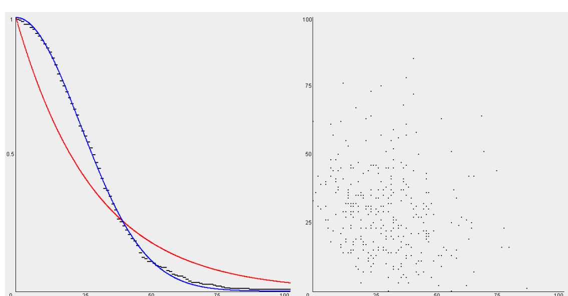

Figure 3. Inter-move times: complementary distribution function and serial correlation.

The correct picture? Figure 3 indicates that our previous guess (i)-(iii) for asymptotic behavior was over-simplistic. Here is what we believe is the correct asymptotic picture.

As a preliminary, consider the σ-finite Ito measure Ψ on positive Brownian excursions, but extend such excursions to functions $\zeta(x), 0 \leq x < \infty$ by continuing as Brownian motion. Consider a deterministic continuous function $h(x)$ with $h(0) = 0$. Given $\zeta(\cdot)$ define

$$\tau(\zeta, h) = \inf\{x > 0 : \zeta(x) \leq h(x)\}$$

If $\tau(\zeta, h) > 0$ write $[\zeta; h]$ for the function obtained from h by replacing $h(\cdot)$ by $\zeta(\cdot)$ over the interval $0 \leq x \leq \tau(\zeta, h)$.

Note that the property

$$\Psi\{\zeta : \ \tau(\zeta, h) > \delta\} < \infty \quad \forall \delta > 0$$

may or may not hold, depending on the behavior of $h(x)$ as $x \to 0$. Let $\mathcal{H}$ be the set of "good" h for which this property holds.

We seek to define a process whose state space is $C_0[0, \infty)$, the space of continuous functions $h : [0, \infty) \to \mathbb{R}$ with $h(0) = 0$. Write this process as $H(t, x)$, so $x \to H(t, x)$ is the state at time t. Here is the informal description of the process dynamics, which involve two mechanisms.

Associated with Ψ is the space-time Poisson process of (extended) Brownian excursions – paths ζ appear at random times. When ζ appears at t, if $\tau(\zeta, H(t, \cdot)) > 0$ then replace $H(t, \cdot)$ by $[\zeta, H(t, \cdot)]$.

Secondly, there is a determinstic drift downwards at rate 1:

$$\tfrac{d}{dt} H(t, x) = -1$$

Call this process **the continuous model**. Of course, from a rigorous viewpoint it is not obvious that such a process exists. But some heuristic self-consistency arguments make it plausible that there is a time-invariant distribution $(H^*(x), 0 \leq x < \infty)$ which is (space-)invariant under Brownian scaling. Qualitatively, Brownian scaling suggests that the intensity measure ν for the lengths τ of the replacement segments will be $\nu\{\tau > x\} \propto x^{-1/2}$. Consider the process $t \to H(t, x)$ for fixed x. It decreases at rate 1 until a segment with $\tau > t$ appears, and because the appearance rate is order $x^{-1/2}$ we expect the values of $H(t, x)$ to stay in the range $[0 \pm O(x^{1/2})]$.

To relate this continuous model to the queue model, let $N(s, y)$ be the number of people in interval $[0, y]$ at time s, in the queue model. Fix a large distance L and set

$$H_L(t, x) = \frac{N(L^{1/2} t, xL) - xL/\mu}{L^{1/2}}$$

where $\mu = (1+c)/2$ is the mean of ξ. (xxx scale involves s.d. too). The motivation for considering the continuous process H is as the presumed $L \to \infty$ limit of this process H_L.

The project. So for a rigorous treatment we would like:

- For the queue model, can we prove directly (e.g. via the natural coupling) that it converges in distribution to a unique stationary distribution?
- For the continuous model, give a rigorous construction, and prove it has a stationary distribution $(H^*(x), 0 \le x < \infty)$.
- Prove the stationary distribution is (space-)invariant under Brownian scaling.
- Any explicit formulas for aspects of the continuous model?
- Prove weak convergence of the queue model (represented as H_L) to the continuous model H under approporiate initial conditions.

Bottom line conclusions. The actual bottom line, regarding the queue model, should be the Brownian scaling properties.

– the chance that the wave reaches at least as far as the m'th person in line scales as $m^{-1/2}$, or equivalently that for the m'th person the frequency of moves scales as $m^{-1/2}$ and the typical distance moved forward (when the person does move) scales as $m^{1/2}$.

More simulations. The figure shows (left) a realization of

$$N(t) = \text{number of people in } [0, 10,000]$$

where $c = 3$, so $\mu = 2$. Of course $N9t)$ usually just decreases by 1 each step, but when a wave hits it will jump up. The right figure shows

$$H(i) = \text{proportion of times } t \text{ over } 0 \le t \le 10,000 \text{ that } N(t) \ge i.$$

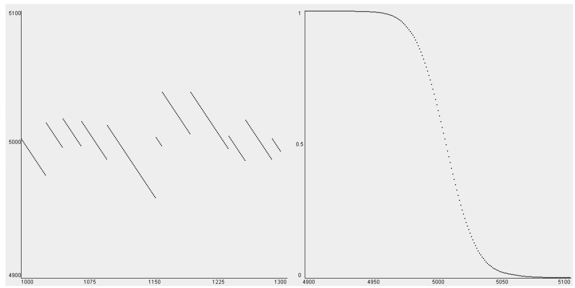
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Figure 4. xxx