1 Introduction

xxx the model is "made up" to be mathematically interesting. Use "cities" as convenient language, not literally.

The model At each time $t = 1, 2, 3, \ldots$ there are cities at positions x_i in the unit square $[0, 1]^2$, with populations $N_i(t) \geq 1$, the total population being $\sum_i N_i(t) = t$. The model has three parameters

$$0 < c_0 < \infty, \quad 0 < \alpha < \infty, \quad \beta > 0$$

(though we focus on the case $0 < \alpha < 1$) which are used to define a function

$$I_0(n, r) = c_0 n^\alpha r^{-\beta} \tag{1}$$

interpreted as the "influence" of a city of population n at distance r from the city. For a position $y \in [0, 1]^2$ define

$$I(y, t) = \max_i I_0(N_i(t), |y - x_i|) = c_0 \max_i N_i^\alpha(t)|y - x_i|^{-\beta} \tag{2}$$

and then define the *sphere of influence* of city i to be

$$\mathcal{S}(i, t) = \{y : I_0(N_i(t), |y - x_i|) = I(y, t)\} \tag{3}$$

where the convention about boundaries between cities' spheres is unimportant. At time 1 there is a single city of population 1 at a uniform random point of $[0, 1]^2$. The general evolution rule is:

At time $t + 1$ an immigrant arrives at a uniform random position U in $[0, 1]^2$, and either
(i) (with probability $1/(1 + I(U, t))$) founds a new city at position U with population 1;
or (ii) (with probability $I(U, t)/(1 + I(U, t))$) joins the city i for which $U \in \mathcal{S}(i, t)$, thereby increasing its population to $N_i(t + 1) = N_i(t) + 1$.

Remarks If city populations were equal then the partition into spheres of influence is just the usual Voronoi tessellation [10]; so in general it is a form of weighted Voronoi tessellation.

Comments on model The two qualitative features of the model are
the growth rate of a city depends on its size and on the sizes and distances of other cities
some stochastic rule for founding of new cities.
One could imagine any different rules to formalize these features; while there is no necessary connection between the two features, we are using a "slick" formulation in which both are xxx via the same influence function.

Heuristic analysis Take $0 < \alpha < 1$. We study the order of magnitude of several quantities. Suppose there are $M(t)$ cities at time t, and suppose their populations are mostly the same order of magnitude. Then typical city population $N^*(t) \approx t/M(t)$ and distance $R^*(t)$ from typical point to nearest city is $\approx M^{-1/2}(t)$, so the typical value $I^*(t)$ of the influence function $I(y, t)$ is $\approx (\frac{t}{M(t)})^\alpha \ M^{\beta/2}(t) = t^\alpha M^{-\alpha+\beta/2}(t)$. The chance that a new arrival founds a new city is about $1/I^*(t)$, so we get an equation

$$\tfrac{dM}{dt} \approx \tfrac{1}{I^*(t)} \approx t^{-\alpha} M^{-\beta/2+\alpha}. \tag{4}$$

This has solution

$$M(t) \approx t^{\theta}, \quad \theta = \frac{1-\alpha}{1-\alpha+\beta/2}$$

obtained from solving $\theta - 1 = -\alpha + \theta(\alpha - \beta/2)$. Note that the typical influence is therefore

$$I^*(t) \approx (dM(t)/dt)^{-1} \approx t^{1-\theta}; \quad 1-\theta = \tfrac{\beta}{2-2\alpha+\beta} \tag{5}$$

and the typical distance to nearest city is

$$R^*(t) \approx M^{-1/2}(t) \approx t^{-\theta/2}; \quad \theta/2 = \tfrac{1-\alpha}{2-2\alpha+\beta} \tag{6}$$

and the typical city population size is

$$N^*(t) \approx \tfrac{t}{M(t)} \approx t^{1-\theta}; \quad 1-\theta = \tfrac{\beta}{2-2\alpha+\beta}. \tag{7}$$

The heuristic argument above rests upon an intuitive picture of the qualitative behavior of the process, that for large t and a typical position y
(a) most different cities' populations are the same order of magnitude
(b) y is in the sphere of influence of some *nearby* city
(c) a city newly founded at t will grow, in time δt, to some population which is $\varepsilon(\delta)$ times the typical time-t city population.

Call this the *balanced growth scenario*. But one can imagine an alternative picture, the *unbalanced growth scenario*, in which
(d) y is in the sphere of influence of some city A at distance r which is much larger than the distance to nearby cities
(e) the nearby cities' populations are a smaller order of magnitude than city A's, and their spheres of influence are surrounded by that of city A.

To investigate these scenarios we try a self-consistency check. Consider a city founded at time t, and consider $N(s)$ = population of this city at time s after founding, looked at over a relatively short time period $0 < s < \frac{1}{100}t$, say. The radius $r(s)$ of its sphere of influence satisfies

$$N^{\alpha}(s) r^{-\beta}(s) \approx I^*(t) \approx t^{1-\theta}.$$

The rate of population growth is proportional to area of sphere of influence, so

$$\frac{dN(s)}{ds} \approx r^2(s) \approx t^{-2(1-\theta)/\beta}\ N^{2\alpha/\beta}(s); \quad N(0) = 1.$$

We now have two cases.

Case 1. $\beta < 2\alpha$.
The solution of $dy(s)/ds = y^{2\alpha/\beta}(s)$ explodes in finite time s, but stays bounded for some small time, and so $N(s)$ stays bounded for some time s of order $t^{2(1-\theta)/\beta}$. But the assumption $\beta < 2\alpha$ implies $2(1-\theta)/\beta = \frac{1}{1-\alpha+\beta/2} > 1$ implying that $N(\frac{1}{100}t)$ is bounded, in contradiction to behavior (c) above.

Case 2. $\beta > 2\alpha$.
Here the solution is

$$N(s) \approx t^{\xi}(s + t^{-\xi/\phi})^{\phi}; \quad \phi = \frac{\beta}{\beta - 2\alpha}, \quad \xi = \frac{2(\theta-1)}{\beta-2\alpha}.$$

Here $-\xi/\phi$ works out to be $\frac{1}{1-\alpha+\beta/2} < 1$ and so $N(\frac{1}{100}t)$ is order $t^{\xi+\phi}$. Then (somewhat magically?) a calculation shows $\xi + \phi = 1 - \theta$, consistent with behavior (c) above.

So the conclusion of these heuristics is that we predict (recall $0 < \alpha < 1$)

> the balanced growth scenario holds when $\beta > 2\alpha$
> the unbalanced growth scenario holds when $\beta < 2\alpha$.

One can *a posteriori* see a conceptually simpler distinction between the two cases. Consider a city founded at time t. If the area of its sphere of influence upon founding is $> 1/t$ then it will tend to grow faster (proportional to size) than average, while if this area is smaller than $1/t$ it will tend to grow slower. But to calculate this area one needs some information about the behavior of the process, so we do not see any simple *a priori* heuristic that these two cases are $\beta < 2\alpha$ and $\beta > 2\alpha$.

In the unbalanced case we envisage that most of the population is in one city, or in a small number of cities, and so the analogs of (5 - 7) are

$$N^*(t) = t^{1-o(1)}; \quad R^*(t) = t^{-o(1)}; \quad I^*(t) = t^{1-o(1)}.$$

Note that these exponents are therefore discontinuous as (α, β) cross the boundary between the balanced and unbalanced regions.

In the case $\alpha > 1$ we expect unbalanced growth (the solution of $dy(s)/ds = y^{2\alpha/\beta}(s)$ explodes in finite time s). The case $\alpha = 1$ can be considered *critical*, in that growth rate is proportional to population size. This case is loosely analogous to other models (see Gibrat's law (section 1.2) and the Chinese restaurant processs (section 1.2).

To summarize: heuristics suggest the parameter space should divide into three regions:

$0 < \alpha < 1$ and $\beta > 2\alpha$	balanced growth
$\alpha > 1$ or $\beta < 2\alpha$	unbalanced growth
$\alpha = 1$ and $\beta > 2\alpha$	critical growth

and this guides our theoretical development.

1.1 Methodology

Our methodology is to seek statistics $S(t)$ of the process for which we can control (upper or lower bound) the incremental conditional expectation $E(S(t+1) - S(t)|\mathcal{F}(t))$ and thereby upper or lower bound the growth rate of $S(t)$ by martingale methods. Conceptually we imagine $I(y, t)$ as a time-varying "random environment" and study how the environment affects growth of city population and distance to nearest city, and how these changes feed back into changing the environment. The fact that $I(y, t)$ is monotone in t is technically helpful.

xxx in balanced case, end up with a bunch of inequalities between rigorously-defined upper and lower exponents for growth rates. In other cases, scattered results.

xxx state results!

As described in section 6.4 there is a certain associated dynamical system whose study might lead to stronger results. xxx future work!