

# Uniform Multicommodity Flow through the Complete Graph with Random Edge-capacities

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## Abstract

Give random capacities  $C$  to the edges of the complete  $n$ -vertex graph. Consider the maximum flow  $\Phi_n$  that can be simultaneously routed between each source-destination pair. We prove that  $\Phi_n \rightarrow \phi$  in probability where the limit constant  $\phi$  depends on the distribution of  $C$  in a simple way, and that asymptotically one need use only one- and two-step routes. The proof uses a reduction to a random graph problem.

## 1 Introduction

This paper is part of a project studying optimal flows through random networks, where a network has both a graph structure and extra structure such as capacities and costs on edges, and where we are in the “multicommodity flow” setting with simultaneous flows between each source-destination pair. Possible models span a broad spectrum from realistic to mathematically tractable, and at the latter end are models based on the complete graph. Including study of such models within a project is natural both for mathematical completeness and for comparison purposes.

Consider first the setting of an arbitrary finite connected undirected graph  $G$ . Let  $\phi > 0$ . A flow of volume  $\phi/2$  between vertex  $v$  and vertex  $w$  has net out-flow =  $\phi/2$  at  $v$ , net out-flow =  $-\phi/2$  at  $w$ , and zero net out-flow at other vertices. For such a flow write  $f_{v,w}(e) \geq 0$  for the absolute value of the flow volume across an undirected edge  $e$ . Suppose we have such a flow simultaneously for each ordered pair  $(v,w)$  with  $w \neq v$ ; call this collection a *uniform flow* of volume  $\phi$  and write  $f(e) := \sum_{(v,w)} f_{v,w}(e)$  for the combined volume of flow across the undirected edge  $e$ . Suppose now we are given capacities  $C(e)$  for

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edges  $e$ . Then the *maximum uniform flow volume* (MUFV) is defined to be the largest  $\phi$  such that there exists a uniform flow of volume  $\phi$  which satisfies the capacity constraints

$$f(e) \leq C(e) \quad \forall e. \quad (1)$$

One modeling paradigm, seeking to combine the spatial inhomogeneity of real networks with mathematical tractability, is to consider some standard family  $G_n$  of  $n$ -vertex graphs, and to assume the edge-capacities  $C(e)$  are random (specifically, are i.i.d. copies of a reference r.v.  $C$ ). Now the MUFV is a r.v.  $\Phi_n$ , and one can seek to study its  $n \rightarrow \infty$  behavior.

Apparently, and somewhat surprisingly, such questions have not been studied before. There is literature [7, 5, 2, 8] on flows with a single source-destination pair and on flows from the top to the bottom of a square, but these fall into the one-commodity setting of the max-flow min-cut theorem, rather than our multicommodity setting.

In this paper we consider the complete graph; a similar problem on the  $m \times m$  square grid was studied by very different methods in [1]. An interesting observation is that in both these models the limit constants for  $\Phi_n$  depend on the distribution of  $C$  (not just on its expectation  $EC$ ), but for rather different reasons in the two models. An intermediate model is the cube  $\{0, 1\}^d$ , and here we conjecture that the limit constant does depend only on  $EC$  when  $C$  is bounded away from zero.

## 1.1 Statement of results

Consider the complete  $n$ -vertex graph whose edges  $e$  have independent random capacities  $C(e)$  whose common distribution satisfies

$$0 < EC < \infty. \quad (2)$$

Note that the function

$$\phi \rightarrow 2E \max(\phi - C, 0) - E \max(C - \phi, 0)$$

is continuous and strictly increasing from  $-EC$  to  $\infty$  as  $\phi$  increases from 0 to  $\infty$ , and so we can define a constant  $0 < \phi_* < \infty$  as the unique solution of

$$E \max(C - \phi_*, 0) = 2E \max(\phi_* - C, 0). \quad (3)$$

**Theorem 1.** *Under assumption (2) the MUFV  $\Phi_n$  satisfies  $\Phi_n \rightarrow \phi_*$  in probability as  $n \rightarrow \infty$ .*

The intuition is very simple. Suppose we wish to route a uniform flow of volume  $\phi$ . First route as much flow as possible across the direct edge, that is route volume  $\min(\phi, C_{v,w})$  across edge  $\{v, w\}$ . This leaves an unsatisfied demand for a volume  $\max(\phi - C_{v,w}, 0)$  of flow. Now the mean surplus capacity per edge is  $E \max(C - \phi, 0)$ . We try to route the unsatisfied demand via 2-step paths with surplus capacity; for this to work it seems evidently necessary that

$$E \max(C - \phi, 0) \geq 2E \max(\phi - C, 0).$$

Conversely, this should be sufficient because the set of edges with surplus capacity forms a dense random graph which should be sufficiently well-connected to permit the desired 2-step paths.

The “necessary” part is indeed easy to formalize (Lemma 2), and we do this in Section 2. We prove the converse (Lemma 5) in Section 3.1; the proof uses a reduction to a result (Proposition 3) on random coloured graphs which we prove in section 3.2.

Finally, we mention a possible connection between our setting and the more elaborate setting of *dynamic routing* (of e.g. phone calls) on a complete network. Section 2.1 of [4] analyzes the throughput of a model in which it is assumed that calls use either a one-link or two-link route, having previously commented

We shall occasionally mention the possibility that a call might be connected along a path of more than two links, but ... this possibility is rarely of interest and we shall exclude it from our formal development.

Our results suggest the possibility of proving that asymptotically one cannot improve throughput by using such longer paths.

## 2 The upper bound

The upper bound in Theorem 1 is provided by

**Lemma 2.**  $\lim_n P(\Phi_n \geq \phi) = 0, \quad \phi > \phi_*$ .

*Proof.* Fix a realization of the edge-capacities. Suppose a uniform flow of volume  $\rho$  exists. For an edge  $(v, w)$

$$\begin{aligned} \sum_e (f_{v,w}(e) + f_{w,v}(e)) &\geq \rho \text{ if } C(v, w) \geq \rho \\ &\geq C(v, w) + 2(\rho - C(v, w)) \text{ if } C(v, w) \leq \rho \end{aligned}$$

because in the latter case volume of at least  $\rho - C(v, w)$  must use at least a 2-step route. Combining the two cases,

$$\sum_e (f_{v,w}(e) + f_{w,v}(e)) \geq \min(\rho, C(v, w)) + 2 \max(\rho - C(v, w), 0).$$

Summing over edges  $e' = (v, w)$  and using the capacity constraint (1),

$$\sum_e C(e) \geq \sum_{e'} (\min(\rho, C(e')) + 2 \max(\rho - C(e'), 0)).$$

Dividing by  $\binom{n}{2}$  and recalling we supposed that the uniform flow exists, we have shown

$$Q_n := \frac{1}{\binom{n}{2}} \sum_{e'} C(e') - \frac{1}{\binom{n}{2}} \sum_{e'} (\min(\rho, C(e')) + 2 \max(\rho - C(e'), 0)) \geq 0 \text{ on } \{\Phi_n \geq \phi\}.$$

But as  $n \rightarrow \infty$  the quantity  $Q_n$  converges in probability to

$$q := EC - (E \min(\rho, C) + 2E \max(\rho - C, 0)) = E \max(C - \rho, 0) - 2E \max(\rho - C, 0).$$

If  $\rho > \rho_*$  then  $q < 0$  and hence we must have  $\lim_n P(\Phi_n \geq \phi) = 0$ .  $\square$

### 3 The reduction argument

#### 3.1 The reduction

We will use a reduction to the following ‘‘random graph’’ result. To motivate this reduction, consider the case where the edge-capacity  $C$  takes only values  $\{0, 1, 2\}$  and where we seek to route a uniform flow of volume 1. Then traffic across capacity-0 edges (colored scarlet, say) needs to be routed through two capacity-2 edges (colored blue, say). Colors are mnemonics for *smaller* and *bigger* capacity.

**Proposition 3.** *Fix  $0 < p_s < p_b/2$  with  $p_s + p_b \leq 1$ . Randomly colour the edges of the complete  $n$ -vertex graph as blue (probability  $p_b$ ) or scarlet (probability  $p_s$ ) or neither (probability  $1 - p_b - p_s$ ). Then whp there exists a collection of edge-disjoint triangles, each triangle having one scarlet edge and two blue edges, such that every scarlet edge is in some triangle.*

We defer the proof of this proposition to the next subsection, and show here how to deduce the lower bound in Theorem 1, stated as Lemma 5. Note that the condition  $\rho < \rho_*$  is equivalent to

$$r := \frac{E \max(\rho - C, 0)}{E \max(C - \rho, 0)} < \frac{1}{2}. \quad (4)$$

We first prove a version of the result for integer capacities and demands, and then use this to deal with the general case in Lemma 5.

**Lemma 4.** *Suppose  $C$  is integer-valued and bounded, and suppose  $\rho$  is an integer satisfying (4). Then, with high probability, we can construct flows of volume  $\rho$  between every pair of vertices such that the capacity constraint (1) holds.*

*Proof.* Let  $M$  be an upper bound for  $C$ . We construct  $M$  separate flow problems  $\mathcal{P}_1, \dots, \mathcal{P}_M$  of the following type. Each problem  $\mathcal{P}_i$  will be encoded by an  $n$ -vertex graph with scarlet and blue edges. A scarlet edge  $vw$  indicates a demand of 1 between  $v$  and  $w$ , while blue edges have capacity 1 and demand 0. The absence of an edge indicates that demand and capacity are both 0.

We construct the problems as follows. For each edge  $vw$  with  $C(vw) \geq \rho$  we choose (uniformly at random) a subset  $I \subset \{1, \dots, M\}$  of size  $C(vw) - \rho$  and insert a blue edge between  $v$  and  $w$  in  $\mathcal{P}_i$  for each  $i \in I$ . For each edge  $vw$  with  $C(vw) < \rho$  we choose (uniformly at random) a subset  $I \subset \{1, \dots, M\}$  of size  $\rho - C(vw)$  and insert a scarlet edge between  $v$  and  $w$  in  $\mathcal{P}_i$  for each  $i \in I$ .

Note that the instances  $\mathcal{P}_i$  and  $\mathcal{P}_j$  may be dependent, but the edges inside any instance  $\mathcal{P}_i$  are present (and coloured) independently.

By (4) the hypothesis of Proposition 3 is satisfied, so with high probability, we can find for each instance  $\mathcal{P}_i$  a collection of edge-disjoint triangles with two scarlet edges and one blue edges covering all scarlet edges. For each scarlet edge in one of the triangles, route unit flow between its end-vertices by using the two blue edges in the triangle. Taking the sum of these flows over all  $M$  instances, we establish the lemma.  $\square$

**Lemma 5.** *Assume (2) and let  $\phi < \phi_*$ . Then  $\lim_n P(\Phi_n \geq \phi) = 1$ .*

*Proof.* Let  $p_0 = P(C > 0)$  and choose  $c_0 > 0$  such that  $P(C \geq c_0) \geq p_0/2$ . Define

$$C_k = \max\{\min(2^{-k} \lfloor C2^k - 1 \rfloor, k), 0\}$$

for  $k$  sufficiently large that  $2^{-k} < c_0$ . So  $0 \leq C_k \leq \max\{C - 2^{-k}, 0\}$ . Define  $\rho_k$  as the largest multiple of  $2^{-k}$  for which

$$E \max(C_k - \rho_k, 0) > 2E \max(\rho_k - C_k, 0).$$

It is easy to check that  $\rho_k \uparrow \rho_*$  as  $k \rightarrow \infty$ . Thus it is sufficient to show that, for each fixed large  $k$ ,

$$P(\text{uniform flow of volume } \rho_k \text{ exists}) \rightarrow 1 \text{ as } n \rightarrow \infty. \quad (5)$$

But by applying Lemma 4 to the integer-valued quantities  $2^k C_k$  and  $2^k \rho_k$ , then rescaling, we find that we can construct flows of volume  $\rho_k$  between every pair such that the total flow volume  $f(e)$  satisfies the capacity constraints  $f(e) \leq C_k(e) \forall e$ .  $\square$

### 3.2 Proof of Proposition 3

Given nonnegative reals  $p_1, \dots, p_k$  with  $\sum_{i=1}^k p_i \leq 1$ , we write  $\mathcal{G}(n; p_1, \dots, p_k)$  for the probability space of edge-coloured graphs on  $n$  vertices, obtained as follows: for each pair of vertices independently we have an edge of colour  $i$  with probability  $p_i$ , and no edge with probability  $1 - \sum_{i=1}^k p_i$ . Proposition 3 follows immediately from the following result (give scarlet edges colour 1, and blue edges colour 2 or colour 3 with probability 1/2 each).

**Lemma 6.** *Fix  $\delta > 0$ , and suppose that  $p_1, p_2, p_3 \geq 0$  with sum at most one satisfy  $p_1 + \delta \leq \min\{p_2, p_3\}$ . Then for  $G \in \mathcal{G}(n; p_1, p_2, p_3)$  there is whp a collection  $\mathcal{T}$  of edge-disjoint triangles such that every triangle in  $\mathcal{T}$  contains one edge of each colour and every edge of colour 1 is contained in some triangle in  $\mathcal{T}$ .*

*Proof.* The proof will go in two steps: we begin by setting aside a subset of the edges of colours 2 and 3, and use the remainder to cover most of the edges of colour 1. We then use the edges we have set aside to cover the remaining edges of colour 1.

Let  $G \in \mathcal{G}(n; p_1, p_2, p_3)$ . We define edge-disjoint subgraphs  $G_1$  and  $G_2$  of  $G$  with  $V(G_1) = V(G_2) = V(G)$  as follows:  $G_1$  contains all edges of colour 1; each edge of colour  $i > 1$  is placed in  $G_1$  with probability  $p_1/p_i$  and in  $G_2$  with probability  $\delta/p_i$  (and is discarded with the remaining probability  $(p_i - p_1 - \delta)/p_i \geq 0$ ). Then  $G_1$  has distribution  $\mathcal{G}(n; p_1, p_1, p_1)$  and  $G_2$  has distribution  $\mathcal{G}(n; 0, \delta, \delta)$ .

We begin with  $G_1$ , and try to cover edges of colour 1 with multicoloured triangles. We first partition  $V = V(G_1)$  into a number of sets. Recall that a *Steiner triple system* on a vertex set  $U$  is a collection  $\mathcal{S}$  of triples in  $U$  such that every pair of elements of  $U$  is contained in exactly one triple from  $\mathcal{S}$ . Steiner triple systems exist whenever  $|U| \equiv 1$  or  $3 \pmod{6}$ . So let  $t \sim \sqrt{n}$  be of form  $6k + 1$  and let  $V_1, \dots, V_t$  be a partition of  $V$  into  $t$  sets of size  $\lfloor n/t \rfloor$  or  $\lceil n/t \rceil$ . Let  $\mathcal{S}$  be a Steiner triple system on  $[t] = \{1, \dots, t\}$ , so  $\mathcal{S}$  contains  $\binom{t}{2}/3$  triples. For each element  $S = \{i, j, k\}$  of  $\mathcal{S}$ , we consider the corresponding tripartite subgraph  $G_S$  of  $G_1$  with tripartition  $(V_i, V_j, V_k)$  that contains all edges from  $G_1$  joining these vertex classes. The graphs  $\{G_S : S \in \mathcal{S}\}$  are independent (disjoint) tripartite random graphs, and every edge of  $G_1$  is contained either in some vertex class  $V_i$  or in exactly one of the  $G_S$ .

Consider a fixed  $S = \{\alpha_1, \alpha_2, \alpha_3\} \in \mathcal{S}$ . The corresponding subgraph  $G_S$  has vertex classes  $V_{\alpha_1}, V_{\alpha_2}$  and  $V_{\alpha_3}$ . We decompose  $G_S$  into subgraphs  $G_S^1, G_S^2$  and  $G_S^3$  where, for  $i, j = 1, 2, 3$ ,  $G_S^i$  has all edges from  $G_S$  with colour  $j$  that join  $V_{\alpha_{i+j}}$  and  $V_{\alpha_{i+j+1}}$  (all subscripts taken modulo 3). Thus each  $G_S^i$  has edges of different colours between different pairs of vertex classes, and in particular all triangles in  $G_S^i$  have edges of three different colours. Furthermore, if we ignore colours, then  $G_S^i$  is a random tripartite graph with vertex classes  $V_{\alpha_1}, V_{\alpha_2}, V_{\alpha_3}$  and edge probability  $p_1$ .

Now fix  $i$ , and consider  $G_S^i$ . Given vertices  $x, y$  in different classes, the number  $X$  of  $x$ - $y$  paths of length 2 in  $G_S^i$  has distribution  $B(r, p_1^2)$ , where  $r \sim n/t \sim \sqrt{n}$  is the size of the third vertex class. By Chernoff's inequality, the probability that  $|X - rp_1^2| \geq n^{1/2}/\ln n$  is at most  $\exp(-\Omega(n^{1/2}/\ln^2 n))$ . It follows that with failure probability  $O(\exp(-n^{1/3}))$ , every edge of  $G_S^i$  is contained in  $(1+o(1))p_1^2 n/t$  triangles. Thus, giving each triangle in  $G_S^i$  weight  $(1-o(1))t/np_1^2$ , we obtain a fractional triangle-packing of size  $(1+o(1))e(G_S^i)/3$ . It follows from Theorem 1 of Haxell and Rödl [6] (or using arguments of Frankl and Rödl [3]) that there is a (proper) triangle-packing of size  $(1+o(1))e(G_S^i)/3$ . In particular, the triangle-packing covers all but at most  $o(n^2/t^2)$  edges from  $G_S^i$ .

We therefore have, with probability  $1 - o(1)$ , for every  $S$  and  $i$ , a triangle-packing that covers all but at most  $o(n^2/t^2)$  edges of  $G_S^i$ . Let  $\mathcal{T}_1$  be the union of these triangle-packings, and let  $H$  be the subgraph of  $G_1$  that remains after removing all the triangles in  $\mathcal{T}_1$ .

We claim that, with probability  $1 - o(1)$ ,  $H$  has maximum degree at most  $o(n)$ . Consider a vertex  $v$ , say  $v \in V_i$ . The edges inside  $V_i$  only contribute  $O(\sqrt{n})$  to the degree of  $v$ , so it is enough to consider edges between vertex classes. Since  $i$  belongs to  $(t-1)/2 \sim \sqrt{n}/2$  triples from  $\mathcal{S}$ ,  $v$  belongs to  $(1+o(1))\sqrt{n}/2$  subgraphs  $G_S$ . These subgraphs and their packings are independent: each

subgraph contains at most  $2\lceil n/t \rceil \sim 2\sqrt{n}$  edges that remain incident with  $v$  in  $H$ , and an expected  $o(n/t) \sim o(\sqrt{n})$  such edges. It follows by (for instance) the Azuma-Hoeffding inequality that with probability  $1 - o(1)$ , we have  $\Delta(H) = o(n)$ . Let us choose  $n$  large enough that we can assume  $\Delta(H) < \delta^2 n/10$ .

Finally, we use the edges from  $G_2$  to cover the remaining edges of colour 1 in  $H$ . Let us orient the edges of  $G_2$  at random. Then (easily, by Chernoff's inequality) with probability  $1 - o(1)$ , for every ordered pair  $(x, y)$  of vertices there are at least  $\delta^2 n/5$  oriented paths  $xzy$  in  $G_2$  such that  $xz$  has colour 2 and  $zy$  has colour 3. We now choose triangles greedily: for each edge  $xy$  of colour 1 (taking edges in arbitrary order and with arbitrary orientation), we pick an oriented path  $xzy$  in  $G_2$  (with  $xz$  of colour 2 and  $zy$  of colour 3). Since we have previously used at most  $\Delta(H) - 1$  edges out of  $x$  and at most  $\Delta(H) - 1$  edges into  $y$ , and there are at least  $2\Delta(H)$  directed paths to choose from, there is at least one path edge-disjoint from all previous choices. This enables us to cover all colour 1 edges of  $H$ ; adding the resulting triangles to  $\mathcal{T}_1$  gives our desired collection  $\mathcal{T}$  of triangles.  $\square$

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