Stat206: Random Graphs and Complex Networks

Lecture 19: The Giant Component in the Just-Supercritical Regime

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19.1 Section 5.4 of [1]

19.2 Excess of a component

Consider a graph on m vertices. This graph has $\geq m-1$ edges, with exactly $m-1$ edges iff the graph is a tree. For a component C define:

$$\text{excess}(C) = (\text{\# of edges}) - (\text{\# of vertices} - 1) \geq 0$$

For $\mathcal{G}(n, p = 1/n)$ there are many large components all with size $\sim n^{2/3}$ with similar coefficients that may change from realization to realization (see Figure 1).

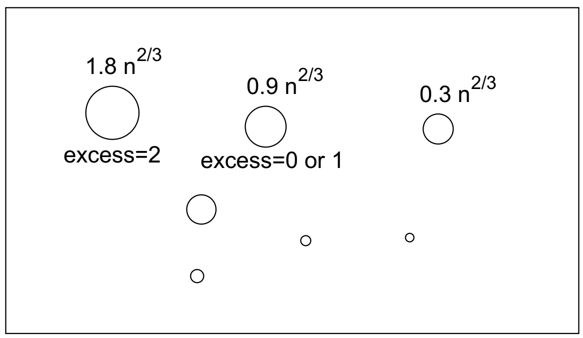

Figure 1: Large components in $\mathcal{G}(n, p = 1/n)$.

In order to see something quantitatively different, consider $\mathcal{G}(n, p = 1/n + 100/n^{4/3})$. In this case many large components merge together giving rise to a very large component with huge excess (see Figure 2).

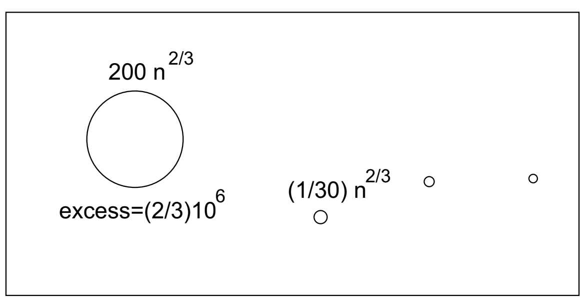

Figure 2: Large components in $\mathcal{G}(n, p = 1/n + 100/n^{4/3})$.

This behavior can be analyzed using the Brownian motion picture that we derived previously.

19.3 Core and Kernel of a component

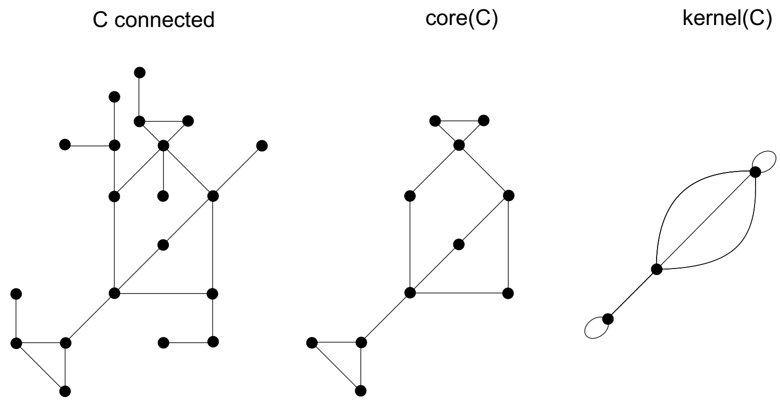

Figure 3: Core and kernel of a connected graph C.

Consider a connected component C of a graph. Define:

$$\text{core}(C) = \text{maximal subgraph with all vertex-degrees } \geq 2$$

which can be obtained from the original component by repeatedly deleting degree-1 vertices.

We also define kernel(C) as the subgraph obtained from core(C) by collapsing every path of degree-2 vertices to a single edge (See Figure 4). Note that kernel(C) may have loops and multiple edges.

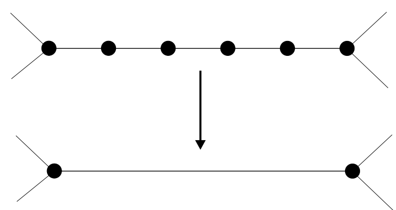

Figure 4: Collapsing a path of degree-2 vertices to a single edge.

19.4 Study of the giant component in the just-supercritical regime

Recall that for a GWBP with Poisson($1+\delta$) offspring

$$P(\text{non-extinction}) = \text{solution of equation} \approx 2\delta$$

for δ small.

We'll study an Erdös-Renyi random graph $\mathcal{G}(n, M\text{edges})$ with $M = \binom{n}{2}p$, which corresponds to $\mathcal{G}(n,p)$ with $p = 2M/n^2$. We will focus on the regime $M = n/2 + s$ with $n^{2/3} \ll s \ll n$, which corresponds to $np = 2M/n = 1 + 2s/n$.

Define $V_n(s) = \#$ of vertices in giant component. Then we have:

$$\begin{aligned} E[V_n(s)] &= nP(\text{v in giant component}) = nP(\text{non-extinction in GWBP, offspring Poisson}(np = 1+2s/n)) \\ &= n2\frac{2s}{n} = 4s \end{aligned}$$

Therefore $E[V_n(s)] = 4s$ and $V_n(s) \approx 4s$ in probability (by a Chebyshev's type argument).

Write $\mathcal{E}_n(s)$ = excess in giant component. Adding an edge increases $\mathcal{E}_n(s)$ by 1 when both ends are in the giant component (otherwise the excess does not change). At the edge $n/2 + i$, chance $\mathcal{E}_n(\cdot)$ increases $= (4i/n)^2 = 16i^2/n^2$. Then

$$E[\mathcal{E}_n(s)] = \sum_{i=1}^{s} \frac{16i^2}{n^2} = \frac{16s^3}{3n^2}$$

and again $\mathcal{E}_n(s) \approx \frac{16s^3}{3n^2}$ in probability.

Given that the giant component (GC) has $k \approx 4s$ vertices and $k + l \approx k + \frac{16s^3}{3n^2}$ edges, the giant component is the random graph $\mathcal{C}(k,l)$ uniform on all $c(k,l)$ connected graphs with k vertices and $k+l$ edges.

Study core of GC $\approx$ core of $\mathcal{C}(k,l)$. A formula for l and k/l both large:

$$c(k,l) \sim \left(\frac{e}{12l}\right)^{l/2} k^{k+(3l-1)/2}$$

One can argue (omit) that # of edges in core $\mathcal{C}(k,l) \approx$ # of edges of $\mathcal{C}(k,l)$ whose removal won't disconnect $\mathcal{C}(k,l)$. This gives a trick to calculate the size (# of vertices) of the core. Consider k vertices, $k+l$ edges, one edge marked * whose removal will not disconnect the core. Then,

$$c(k,l) \times (\text{size of core } \mathcal{C}(k,l)) = c(k,l-1) \times (\text{\# of ways to add an edge to a } \mathcal{C}(k,l-1))$$

where the last factor equals $\binom{k}{2} - (k+l-1) \approx k^2/2$. It follows that

$$\text{size of core } \mathcal{C}(k,l) = \frac{k^2 c(k,l-1)}{2c(k,l)} \sim \sqrt{3kl}$$

and

$$\text{size of core of GC of } \mathcal{G}(n,M) \sim \sqrt{\frac{3 \cdot 4s16s^3}{3n^2}} = \frac{8s^2}{n} \approx \text{\# degree-2 vertices in core}$$

One can check that only a small fraction of the total number of vertices are in the core.

Adding 1 excess edge to GC adds 1 edge to kernel and increases sum of core-degrees by 2.

$$\sum_{v \in \text{core}} (d_{\text{core}}(v) - 2) = 2\mathcal{E}_n(s) = \frac{32s^3}{3n^2} \approx \text{\# of degree-3 vertices}$$

We expect most of the sum to come from degree-3 vertices.

How is a degree-4 vertex of the core created ?

When an edge arrives between some vertex in a tree component of GC rooted at v (degree-3 vertex in core) and some other vertex in GC,

$$\text{chance edge } \frac{n}{2} + i \text{ creates a degree-4 vertex of core } \cong (\text{\# deg-3 core vertices})$$

$$\text{x(ave. size of tree in GC rooted at a core vertex)x(size of GC)} \frac{1}{\binom{n}{2}} = \frac{32i^2}{3n^2} \times \frac{(\text{size of GC})^2}{(\text{size of core})} \times \frac{2}{n^2}$$

$$= \frac{32i^2}{3n^2} \times \frac{(4i)^2}{8i^2/n} \times \frac{2}{n^2} = \frac{128i^3}{3n^3}$$

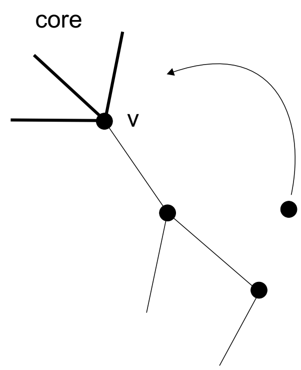

Figure 5: Creation of a degree-4 vertex in core.

It follows that

$$\text{mean \# of deg-4 vertices in core at time } s = \sum_{i=1}^{s} \frac{128 i^3}{3n^3} = \frac{32 s^4}{3n^3}$$

so they first appear at $s \approx n^{3/4}$.

References

[1] Svante Janson, Tomasz Luczak, Andrzej Rucinski, *Random Graphs*, Wiley, 2000.