

4.6 A first look at finance: the long term and the Kelly criterion

xxx intro: overview of finance and risk

Notions of *long term* versus *short term* play an important role in this topic, so let's start with a brief discussion. In everyday language, a job which will only last six months is a short term job; someone who has worked for a company for fifteen years is a long term employee. Joining a softball team for a summer is a short term commitment; raising children is a long term commitment. We judge these matters relative to human lifetime; *long term* means some noticeable fraction of a lifetime.

Turning to money matters, consider the difference between simple interest and compound interest. Table xxx compares the value, after increasing numbers of years, of an initial \$1,000 earning 7% interest.

year	0	4	8	12	16	20
simple interest	1000	1,280	1,560	1,840	2,120	2,400
compound interest	1000	1,311	1,718	2,252	2,952	3,870

Table xxx. Effect of 7% interest, compounded annually.

One of several possible notions of *long term* in financial matters is “the time span over which compounding has a noticeable effect”. Rather arbitrarily interpreting “noticeable effect” as “10% more” and taking the 7% interest rate, this suggests taking 8 years as the cut-off for *long term*. Being about 10% of a human lifetime, this fortuitously matches reasonably well the “noticeable fraction of a lifetime” criterion above. And indeed in matters pertaining to individuals, financial or otherwise, most writers use a cut-off between 5 and 10 years for “long term”.

The theme of this section is the nature of compounding when gains and losses are unpredictable. Recall the relevant arithmetic is multiplication not addition: a 20% gain followed by a 20% loss combine to a 4% loss, because $1.2 \times 0.8 = 0.96$.

In class I start as follows. Suppose you invest \$1,000 today in the stock market, more precisely in the S&P500 index (via a fund with very low expenses). What do you guess the investment will be worth in 10 years? In 2008 the student guesses⁶ were

\$800, \$1,000, \$1,600, \$1,700, \$1,800, \$2,200, \$2,500.

I have a deck of cards on which are pasted the annual total returns of the S&P500 index over each of the 52 years 1956 through 2007. I say “let's suppose the annual returns over the next ten years are statistically like random years from the past; we can track our hypothetical investment value over the next ten years by shuffling and dealing ten cards”. Doing this once in the 2008 class, the hypothetical investment grew from \$1,000 to \$1,839.

⁶on September 17, 2008; the index closed at 1,156, down 4.7% on that day.

Let us make explicit the type of model used implicitly above. A “return” $x = 0.2$ or $x = -0.2$ in a year means a 20% gain or a 20% loss.

The multiplicative model. Write X_i for the return in year i . Suppose the (X_i) are i.i.d. random variables. Then the value Y_n of your investment at the end of year n is

$$Y_n = Y_0 \prod_{i=1}^n (1 + X_i) \quad (4.6)$$

where Y_0 is your initial investment.

To analyze this model we take logs and divide by n :

$$n^{-1} \log Y_n = n^{-1} \log Y_0 + n^{-1} \sum_{i=1}^n \log(1 + X_i)$$

and the law of large numbers says that as $n \rightarrow \infty$ the right side converges to $E \log(1 + X)$. For a non-random growth rate r per year, we would have $Y_n = Y_0(1 + r)^n$ and therefore $n^{-1} \log Y_n \rightarrow \log(1 + r)$. Matching the two cases by setting $E \log(1 + X) = \log(1 + r)$ gives the conclusion

In the model above, the long term growth rate is

$$\exp(E \log(1 + X)) - 1. \quad (4.7)$$

The main impact of this result is that what matters about the random return X isn't precisely the mean EX . Other aspects deserve careful thought, and one's first thoughts might be

- Is the model realistic for stock market investing?
- The phrase *long term* here refers to the applicability of the law of averages as an approximation to finite time behavior – this is a third meaning of the phrase, distinct from the two previous meanings.

We'll discuss those issues later (xxx) but let's jump to the main point

The Kelly Criterion. Given a range of possible investment portfolios, that is a range of ways to allocate money to different risky or safe assets, where way α will produce return X_α , choose the way α that maximizes the long term growth rate (4.7), that is the way that maximizes $E \log(1 + X_\alpha)$.

Setting aside one's reservations about using this criterion in the real world, let us show some of the mathematics that can be done using the criterion.

4.6.1 Mathematics: one risky and one safe asset

Formula (4.7) applies if we invest all our money in “a stock” (meaning an index fund, and assuming the multiplicative model for stock market returns). But

suppose there's a risk-free alternative investment (a “bond”, in the usual terminology) that pays a fixed interest rate r . Suppose we choose some number $0 \leq p \leq 1$ and at the start of each year we invest a proportion p of our total “investment portfolio” in the stock market, and the remaining proportion $1 - p$ in the bond. In this case formula (4.6) becomes

$$Y_n = Y_0 \prod_{i=1}^n (1 + X_i^*)$$

where $X_i^* = pX_i + (1 - p)r$. The long term growth rate is now a function of p :

$$\text{growth}(p) = \exp(E \log(1 + pX + (1 - p)r)) - 1. \quad (4.8)$$

The Kelly criterion says: choose p to maximize $\text{growth}(p)$. Let's see two examples. In the first X is large, and we end up with p small; in the second X will be small, and we end up with large p . In these two examples we take the time unit to be 1 day instead of 1 year (which doesn't affect math formulas).

Example: pure gambling. Imagine a hypothetical bet which is slightly favorable. Suppose each day we can place a bet of any size s ; we will either gain s (with probability $0.5 + \delta$) or lose s (with probability $0.5 - \delta$), independently for different days (here δ is assumed small). Take $r = 0$ for the moment. What proportion p of our portfolio do we want to bet each day? Here, for small δ ,

$$\begin{aligned} E \log(1 + pX) &= \left(\frac{1}{2} + \delta\right) \log(1 + p) + \left(\frac{1}{2} - \delta\right) \log(1 - p) \\ &\approx \left(\frac{1}{2} + \delta\right)(p - p^2/2) + \left(\frac{1}{2} - \delta\right)(-p - p^2/2) \\ &= 2\delta p - p^2/2. \end{aligned}$$

Thus the asymptotic growth rate is approximately the quadratic function of p

$$G(p) = 2\delta p - p^2/2 \quad (4.9)$$

shown in Figure xxx. The Kelly criterion says to choose $p = 2\delta$ and then your long term growth rate will be $\approx 2\delta^2$.

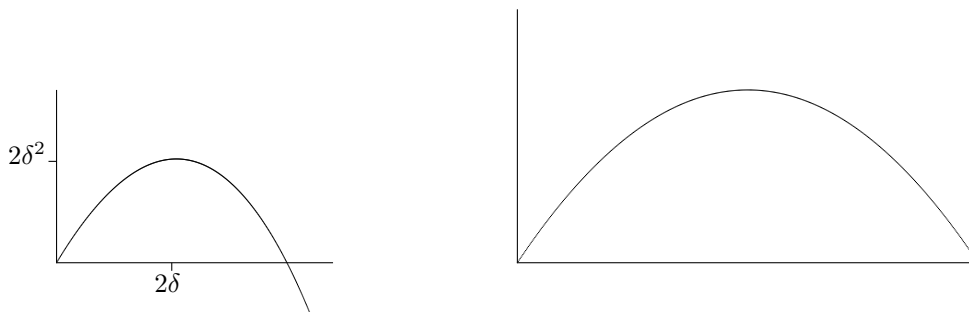


Figure xxx. Schematics. The left diagram shows the growth rate $G(p)$ at (4.9). The right diagram xxx.

Now recall that we simplified by taking $r = 0$; when $r > 0$, the fact that a proportion $1 - p \approx 1$ of the portfolio not put at risk each day can earn interest, brings up the optimal growth rate to $r + 2\delta^2$; the quantity $2\delta^2$ represents the *extra* growth one can get by exploiting the favorable gambling opportunity.

To give a more concrete mental picture, suppose $\delta = 1\%$. The model matches either of the two following hypothetical scenarios.

(a) To attract customers, a casino offers (once a day) an opportunity to make a roulette-type bet with a 51% chance of winning.

(b) You have done a statistical analysis of day-to-day correlations in some corner of the stock market and have convinced yourself that a certain strategy (buying a portfolio at the start of a day, and selling it at the end) replicates the kind of favorable bet in (a).

In either scenario, the quantity $2\delta^2 = 2/10,000$ is the “extra” long term growth rate available by taking advantage of the risky opportunity. Note this growth rate is much smaller than 2% “expected gain” on one bet. On the other hand we are working “per day”, and in the stock market case there are about 250 days in a year, so the growth rate becomes about 5% per year; recalling this is “5% above the risk-free interest rate”, it seems a rewarding outcome. But if δ were instead 0.5% then the extra growth rate becomes $1\frac{1}{4}\%$, and (taking into account transaction costs and our work) the strategy hardly seems worth the effort.

xxx other interesting features: can lose;

Example: small X I first give a nicer algebraic way of dealing with the interest rate r . Set

$$X = r + (1 + r)X^*$$

and interpret $X^* = (X - r)/(1 + r)$ as “return relative to interest rate”. Then a couple of lines of algebra let us rewrite (4.8) as

$$\text{growth}(p) = (1 + r) \exp(E \log(1 + pX^*)) - 1 \quad (4.10)$$

and the optimization problem now doesn’t involve any r . If we imagine the stock market on a daily time-scale and suppose changes X^* are small, with mean μ and variance σ^2 , then we can use the series approximation

$$\log(1 + pX^*) \approx pX^* - \frac{1}{2}(pX^*)^2$$

to calculate

$$E \log(1 + pX^*) \approx p\mu - \frac{1}{2}p^2(\mu^2 + \sigma^2) \approx p\mu - \frac{1}{2}p^2\sigma^2$$

(the latter because μ and σ^2 are of the same order – see below xxx – so μ^2 is of smaller order than σ^2). So the Kelly criterion says: choose p to maximize $p\mu - \frac{1}{2}p^2\sigma^2$, that is choose

$$p = \mu/\sigma^2. \quad (4.11)$$

This is another remarkable formula, and (xxx again keeping in mind there are issues with how applicable it is to real investing) let us discuss some of its mathematical implications.

1. The formula is (as it should be) *time-scale free*. That is, if $\mu_{day}, \mu_{year}, \sigma_{day}^2, \sigma_{year}^2$ are the means and variances over a N -day year, then (because compounding has negligible effect) $\mu_{year} \approx N\mu_{day}$ and $\sigma_{year}^2 \approx N\sigma_{day}^2$, so we get the same value for μ/σ^2 whether we work in days or years.

2. Even though we introduced xxx by stating that $0 \leq p \leq 1$, the model and its analysis make sense outside that range. Economic theory and experience both say that the case $\mu < 0$ doesn't happen (investors are risk averse and so would buy no stock; this would cause the current price of stock to drop), but if it did then the formula $p = \mu/\sigma^2 < 0$ says that not only should we invest 100% of our wealth in the bond, but also we should "sell short" (i.e. borrow) stock and invest the proceeds in the bond.

3. More interesting is the case $p > 1$. Typical values given for the S&P500 index (xxx refs: ifa?) are $\mu = 5.6\%$ and $\sigma = 20\%$, in which case the Kelly criterion says to invest a proportion $p = 140\%$ of your wealth in the stock market, i.e. to borrow money (at fixed interest rate r) and invest your own and the borrowed money in the stock market.