Example

Know a family has 2 children, and know at least one child is a girl. What is the chance the other child is a girl?

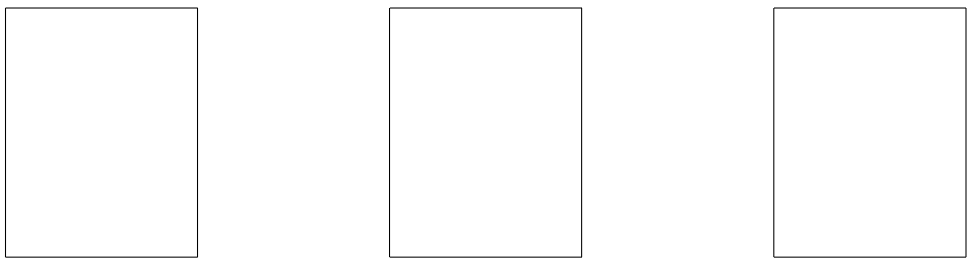
Can't answer; it depends on **how** we got this information.

(1) Computer list story; chance = 2/3

(2) Park story; chance = 1/2

The Monty Hall problem

A prize is hidden behind one of three doors.



Contestant names a door. Monty opens a different door, and shows the prize is not there, and offers contestant the option to switch choice of door. Should contestant accept?

if refuse : $P(\text{prize}) = 1/3$
if switch : $P(\text{prize}) = 2/3$.

This analysis works because Monty always makes the offer. If you play, and Monty is replaced by his evil twin brother who makes the offer only when the contestant first chooses the correct door, the "switch" strategy would never work!

Events A and B are **independent** if:

knowing whether A occured <u>does not change</u> the probability of B.

Mathematically, can say in two equivalent ways:

$$\begin{aligned} P(B|A) &= P(B) \\ P(B \cap A) &= P(B) \times P(A). \end{aligned}$$

Important to distinguish independence from **mutually exclusive** which would say $B \cap A$ is empty.

Example. Deal 2 cards from deck

A first card is Ace
C second card is Ace

$P(C|A) = \frac{3}{51}$
$P(C) = \frac{4}{52}$ (last class).
So A and C are **dependent**.

Example. Throw 2 dice

A first die lands 1
B second die shows larger number than first die
C both dice show same number

$$P(B|A) = \tfrac{5}{6} \quad P(B) = ? = \tfrac{15}{36} \text{ by counting}$$

so A and B **dependent**.

$$P(C|A) = \tfrac{1}{6} \quad P(C) = \tfrac{6}{36} = \tfrac{1}{6}$$

so A and C **independent**.

Note 1: here B and C are mutually exclusive.

Note 2: writing $B' =$ "second die shows smaller number than first die " we have

$P(B') = P(B)$ by symmetry
$P(B \cup B') = P(C^c) = 1 - P(C) = \frac{5}{6}$
giving a "non-counting" argument that
$P(B) = \frac{5}{12}$.

Example. Deal 1 card from deck

A card is Ace
S card is Spade

$$P(A) = \tfrac{4}{52} \quad P(S) = \tfrac{13}{52} \quad P(A \cap S) = \tfrac{1}{52}.$$

Here $P(A \cap S) = P(A) \times P(S)$ so **independent**.

Conceptual point.
(a) In a fully-specified math model, two events are either dependent or independent; can be checked by calculation.
(b) Often we use independence as an **assumption** in making a model. For instance we **assume** that different die throws give independent results. Most probability models one encounters in engineering or science have some assumption of "bottom level" independence; but one needs to be careful about which other events within the model are independent.

(silly) Example. Throw 2 dice. If sum is at least 7 I show you the dice; if not, I don't.

A: I show you first die lands 1
B: I show you second die lands 1

$$P(A) = \frac{1}{36}, \quad P(B) = \frac{1}{36}, \quad P(A \cap B) = 0$$

so A and B **dependent**.

Conceptual point. This illustrates a subtle point: being told by a truthful person that "A happened" is not (for probability/statistics purposes) exactly the same as "knowing A happened".

Systems of components

Logic diagrams: system works if there is some path left-to-right which passes only though working components.

Assume components work/fail independently,

$$P(C_i \text{ works }) = p_i, \quad P(C_i \text{ fails }) = 1 - p_i.$$

Note in practice the independence assumption is usually unrealistic.

Math question: calculate $P($ system works $)$ in terms of the numbers p_i and the network structure.

Example: "in series".

$$P(\text{ system works }) = p_1 p_2 p_3.$$

Example: "in parallel".

$$P(\text{ system fails }) = (1-p_1)(1-p_2)(1-p_3).$$

$$P(\text{ system works }) = 1-(1-p_1)(1-p_2)(1-p_3).$$

Example:

We could write out all 16 combinations; instead let's condition on whether or not C_1 works.

$$\begin{aligned} P(\text{system works}) &= P(\text{system works}|C_1 \text{ works})P(C_1 \text{ works}) \\ &+ P(\text{system works}|C_1 \text{ fails})P(C_1 \text{ fails}) \end{aligned}$$

Example: Deal 4 cards. What is chance we get exactly one Spade?

event	1st	2nd	3rd	4th
F_1	S	N	N	N
F_2	N	S	N	N
F_3				
F_4	N	N	N	S

$$P(F_1) = \frac{13}{52} \times \frac{39}{51} \times \frac{38}{50} \times \frac{37}{49}$$

$$P(F_1) = P(F_2) = P(F_3) = P(F_4)$$

$P(\text{exactly one Spade}) = P(F_1 \text{ or } F_2 \text{ or } F_3 \text{ or } F_4))$

$= P(F_1)+P(F_2)+P(F_3)+P(F_4) = 4\times P(F_1) \approx 44\%$.

Example: Deal 4 cards. What is chance we get one card of each suit?

event	1st	2nd	3rd	4th
A_1	C	D	H	S
A_2	C	D	S	H
.	.	.	.	.
.	.	.	.	.

$$P(A_1) = \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49}$$

$$P(A_1) = P(A_2) = \ldots$$

Number of possible orders $= 4 \times 3 \times 2 \times 1 = 24 = 4!$

$P(\text{one card of each suit}) = 24 \times P(A_1) \approx 10.5\%$.