INFINITELY-MANY-SPECIES LOTKA-VOLTERRA EQUATIONS ARISING FROM SYSTEMS OF COALESCING MASSES

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Abstract

We consider nonlinear, probability measure–valued dynamical systems that generalise those classical Lotka–Volterra equations in which, to use ecological terminology, the total size of a finite number of populations of interacting species is conserved. In our generalisation there is a "different species" at each point of an arbitrary measurable space.

Such infinitely–many–species analogues of the classical Lotka–Volterra equations appear as hydrodynamic–type limits of stochastic systems of randomly coalescing masses related to those that have been used to model physical and chemical processes of agglomeration, coagulation and condensation.

One natural instance of our generalisation has closed form solutions, including a family of solutions that exhibit soliton–like behaviour. The large time asymptotics of other classes of examples can be completely described using analogues of Lyapunov function techniques. Moreover, there are conserved quantities in the form of relative entropies that generalise those found by Volterra in the classical case. Finally, each solution has a series expansion as a time-varying, geometric mixture of a fixed sequence of probability measures. The existence of this expansion is related to the fact that the system is in martingale problem duality with a function–valued Markov process.

Running head: LOTKA-VOLTERRA EQUATIONS.
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1 INTRODUCTION.

Lotka–Volterra systems of ordinary differential equations were introduced in [16] and [28] as deterministic models for the growth of populations of several species in the presence of competition and predation. The general form of the equations for n species is

$$\dot{x}_i(t) = x_i(t)(a_i + \sum_{j=1}^{n} b_{ij} x_j(t)), \quad 1 \le i \le n, \tag{1}$$

for some n-vector (a_i) and some $n \times n$-matrix (b_{ij}).

Besides being the basis for a huge literature in ecology (see [22] for a recent work with an extensive bibliography), these equations have also been used to model such diverse phenomena as proportions of reagents in certain chemical reactions [16], mode coupling of waves in laser physics [14] and plasma physics [15], and interactions of gases in a background host medium [17]. They are also closely related to the dynamical games that appear in quantitative evolutionary theory [26].

Moreover, as relatively tractable nonlinear dynamical systems, they have been of considerable purely mathematical interest. For example, they provide simple examples of nonlinear lattices with soliton solutions (cf. [21] and Chapter 2 of [27]), and attempts to analyse them have led to developments in the theory of nonassociative algebras (cf. [20] and the references therein).

When $(a_i) = 0$ and (b_{ij}) is skew–symmetric (that is, $b_{ij} = -b_{ji}$ for all i, j), any solution of (1) with initial value in the simplex $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \ge 0, \ldots, x_n \ge 0, \sum_{i=1}^n x_i = 1\}$ will remain in this region. Therefore, if in this case we put $\mu_t(A) = \sum_{i \in A} x_i(t)$ for $A \subseteq \{1, \ldots, n\}$, then μ_t is a probability measure on $\{1, \ldots, n\}$. Define $\Phi : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \mathbb{R}$ by $\Phi(i, j) = b_{ij}$. We can rewrite (1) as

$$\frac{d}{dt} \int \mu_t(dy) f(y) = \int \int \mu_t(dy) \mu_t(dz)\, f(y) \Phi(y, z), \tag{2}$$

where f ranges over the functions from $\{1, \ldots, n\}$ into $\mathbb{R}$. Equation (2) still makes sense if we: replace $\{1, \ldots, n\}$ by any set E with a σ-field $\mathcal{E}$ defined on it, let Φ be a bounded $\mathcal{E} \times \mathcal{E}$-measurable function with the property $\Phi(y, z) = -\Phi(z, y)$, and look for solutions $t \to \mu_t$ that take values in the set of probability measures on $(E, \mathcal{E})$. It will be this extension of the classical Lotka–Volterra system to (possibly) "infinitely many species" that we consider here.

The only example of an infinitely-many-species instance of (2) that we are aware of in the literature is in [11]. There $(E, \mathcal{E})$ is the circle with the usual Borel σ-field and $\Phi(y, z) = 1_{[y-\pi, y]}(z) - 1_{[y, y+\pi]}(z)$. An infinite collection of conserved quantities are found for this system. These conserved quantities are given a geometric probability interpretation in [12].

Our interest in (2) arose from the study of stochastic processes called *coalescents* in [8]. These processes models the evolution of a collection of objects, each with a certain "mass", through a random process of binary mergers. When two objects with masses a and b merge, they form a new object with mass $a + b$. Note that the total mass is conserved and we can take it to be 1.

Processes of this form were introduced in [19] and [18] and they have been used in a number of chemical and physical contexts to model situations in which there is aggregation or coagulation (cf. [9, 25, 2]). They also arise in the study of the combinatorics of random forests (cf. [23]).

One way of representing the states of a coalescent process is to label the objects with elements of the measurable space E and regard them as atomic probability measures. More specifically, we code a state consisting of n objects with masses $x_1, \ldots, x_n$ and labels $e_1, \ldots, e_n$ as the atomic probability measure $\sum_{k=1}^n x_k \delta_{e_k}$.

The connection with (2) arises when one considers time–homogeneous, Markov coalescent processes that make jumps of the form

$$\sum_k x_k \delta_{e_k} \longrightarrow \sum_{k \notin \{i,j\}} x_k \delta_{e_k} + (x_i + x_j)\delta_{e_i} + 0.\delta_{e_j}$$

at rate , $(e_i, e_j)x_i$ for some function , $: E \times E \to \mathbb{R}_+$. That is, the two objects with labels e_i and e_j are merged into one with label e_i at rate , $(e_i, e_j)x_i$. When , $\equiv 1$ this is just a representation of the *additive coalescent* of [8]; two objects of mass a and b merge at rate $a + b$.

Let $(Z^n)_{n\in\mathbb{N}}$ be a sequence of such processes with $Z_0^n = \zeta^n \in \mathcal{P}$. A special case of Theorem 11 below is that if E is a locally compact metric space, , is continuous, and ζ^n converges in probability in the topology of weak convergence of probability measures on E to a diffuse probability measure ρ as $n \to \infty$, then Z^n converges in probability (in the corresponding Skorohod topology on the space of càdlàg probability measure–valued functions) to a deterministic function μ that satisfies (2) with $\Phi(y, z) =$, $(y, z) -$, (z, y).

The plan of the rest of the paper is as follows. In Section 2 we give a general existence and uniqueness result for (2) and establish some general properties of the solution such as continuity in the initial value. Moreover, we exhibit a general class of conserved quantities in the form of relative entropies. These conserved quantities generalise those presented in [28].

In Section 3 we examine a number of special cases. We give a natural, nontrivial example that can be solved in closed form and show that this equation has a family of soliton–like solutions. We also study the large time asymptotics of some other broad classes of interesting examples.

We establish the convergence of a system of coalescing masses with labels in a locally compact metric space to a solution of (2) in Section 4 and study the evolution of a "tagged" mass.

Finally, in Section 6 we show that (2) is dual (in the sense of duality of martingale problems) to a certain function–valued Markov process, and use this relation to obtain a series solution of (2) as a time-varying, geometric mixture of a fixed sequence of probability measures. This expansion appears to be new even in the finitely-many-species case.

2 EXISTENCE, UNIQUENESS AND GENERALITIES.

We begin by introducing some notation. Let $(E, \mathcal{E})$ be a measurable space. Let $b\mathcal{E}$ (respectively, $bp\mathcal{E}$) denote collection of bounded (respectively, bounded positive) $\mathcal{E}$-measurable functions. For $f \in b\mathcal{E}$ put $\|f\| = \sup\{|f(x)| : x \in E\}$.

Write $\mathcal{P}$ for the space of probability measures on $(E, \mathcal{E})$. For $\xi \in \mathcal{P}$ and $f \in b\mathcal{E}$ put $\xi f = \int \xi(dx)\, f(x)$, and for $F \in b(\mathcal{E} \times \mathcal{E})$ put $\xi F(x) = \int \xi(dy)\, F(x, y)$. Write $\mathcal{M}$ for the space of functions $\mu : \mathbb{R}_+ \to \mathcal{P}$ such that the function $t \mapsto \mu_t f$ from $\mathbb{R}_+$ to $\mathbb{R}$ is Borel measurable for all $f \in b\mathcal{E}$.

Given $\rho \in \mathcal{P}$ and $\Phi \in b(\mathcal{E} \times \mathcal{E})$, say that $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ if

$$\mu_t f = \rho f + \int_0^t \mu_s(f.\mu_s \Phi)\, ds \tag{3}$$

for all $f \in b\mathcal{E}$ and all $t \in \mathbb{R}_+$.

Define the total variation metric δ on $\mathcal{P}$ by

$$\delta(\xi, \zeta) = \sup\{|\xi f - \zeta f| \;:\; f \in b\mathcal{E},\ \|f\| \leq 1\}.$$

Say that $F \in b(\mathcal{E} \times \mathcal{E})$ is a *product–simple* function if there exists a finite partition $A_1, \ldots, A_m \in \mathcal{E}$ of E such that $F = \sum_{i,j} c_{ij} 1_{A_i \times A_j}$ for some collection of constants c_{ij}, $1 \leq i, j \leq m$.

Say that $F \in b(\mathcal{E} \times \mathcal{E})$ is *skew–symmetric* if $F(x, y) = -F(y, x)$ for all x, y. Note that if $G \in bp(\mathcal{E} \times \mathcal{E})$, then $\hat{G}$ defined by $\hat{G}(x, y) = G(x, y) - G(y, x)$ is skew–symmetric. Moreover, by taking $G = \frac{1}{2}(F + \|F\|)$ we see that every skew–symmetric F arises this way.

THEOREM 1 *(i) If $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ and $\nu \in \mathcal{M}$ solves $\mathtt{LV}(\pi, \Phi)$, then*

$$\delta(\mu_t, \nu_t) \leq \delta(\rho, \pi) e^{2\|\Phi\| t}.$$

In particular, there is at most one solution to $\mathtt{LV}(\rho, \Phi)$.

(ii) If $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$, then $t \mapsto \mu_t$ is δ-continuous.

(iii) If $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$, then μ_t and ρ are equivalent measures for all $t \in \mathbb{R}_+$, and

$$e^{-\|\Phi\| t} \leq \frac{d\mu_t}{d\rho} \leq e^{\|\Phi\| t}.$$

(iv) If $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ and $\nu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Psi)$, then

$$\delta(\mu_t, \nu_t) \leq \frac{(e^{(\|\Phi\|+\|\Psi\|)t} - 1)e^{(\|\Phi\|+\|\Psi\|)t}}{\|\Phi\| + \|\Psi\|} \int \int \rho(dx)\rho(dy)\, |\Phi(x,y) - \Psi(x,y)|.$$

(v) Suppose that $\Phi = \sum_{1 \leq i,j \leq m} c_{ij} 1_{A_i \times A_j}$ is product–simple. For $\rho \in \mathcal{P}$ define an $m \times m$-matrix (b_{ij}) by $b_{ij} = c_{ij}\rho(A_i)\rho(A_j)$. Then $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ if and only if the system of equations

$$h_i(t) = \rho(A_i) + \int_0^t h_i(s) \sum_{j=1}^m b_{ij} h_j(s)\, ds, \quad 1 \leq i \leq m,$$

has a solution and $\mu_t(dx) = (\sum_i h_i(t) 1_{A_i}(x))\rho(dx)$.

(vi) If Φ is skew–symmetric, then there exists $\mu \in \mathcal{M}$ that solves $\mathtt{LV}(\rho, \Phi)$.

Proof. (i) We have

$$|\mu_t f - \nu_t f| \leq |\rho f - \pi f| + \int_0^t \mu_s(f|\mu_s\Phi - \nu_s\Phi|)\, ds + \int_0^t |\mu_s(f.\nu_s\Phi) - \nu_s(f.\nu_s\Phi)|\, ds.$$

Therefore,

$$\delta(\mu_t, \nu_t) \leq \delta(\rho, \pi) + 2\|\Phi\| \int_0^t \delta(\mu_s, \nu_s),$$

and Gronwall's lemma gives the desired result.
(ii) Clear.

(iii) Consider first the upper bound. Take $f \in bp\mathcal{E}$ and observe that

$$\mu_t f \le \rho f + \|\Phi\| \int_0^t \mu_s f \, ds,$$

and hence $\mu_t f \le e^{\|\Phi\| t} \rho f$ by Gronwall's lemma.

Now consider the lower bound. Again take $f \in bp\mathcal{E}$. It follows from part (ii) that $t \mapsto \mu_t f$ is continuously differentiable with $d(\mu_t f)/dt \ge -\|\Phi\| \mu_t f$. Hence, $t \mapsto (\mu_t f)^{-1}$ is continuously differentiable on $[0, \tau[$, where $\tau = \{t \ge 0 : \mu_t f = 0\}$. Moreover,

$$\frac{d}{dt}(\mu_t f)^{-1} = -(\mu_t f)^{-2} \frac{d}{dt}(\mu_t f) \le \|\Phi\| (\mu_t f)^{-1}, \quad t \in [0, \tau[.$$

Thus,

$$(\mu_t f)^{-1} \le (\rho f)^{-1} + \|\Phi\| \int_0^t (\mu_s f)^{-1} \, ds, \quad t \in [0, \tau[,$$

and Gronwall's lemma gives $(\mu_t f)^{-1} \le e^{\|\Phi\| t} (\rho f)^{-1}$, $t \in [0, \tau[$. From the continuity of $t \mapsto \mu_t f$ we get $\mu_t f \ge e^{-\|\Phi\| t} \rho f$ for all t, as required.

(iv) Applying part (iii), we have for $f \in b\mathcal{E}$ with $\|f\| \le 1$ that

$$\begin{aligned}
|\mu_t f - \nu_t f| &\le \int_0^t |\mu_s(f.\mu_s\Phi) - \mu_s(f.\nu_s\Phi)| \, ds + \int_0^t |\mu_s(f.\nu_s\Phi) - \mu_s(f.\nu_s\Psi)| \, ds \\
&\quad + \int_0^t |\mu_s(f.\nu_s\Psi) - \nu_s(f.\nu_s\Psi)| \, ds \\
&\le \int_0^t \mu_s |\mu_s\Phi - \nu_s\Phi| \, ds + \int_0^t \mu_s \nu_s |\Phi - \Psi| \, ds \\
&\quad + \int_0^t |\mu_s(f.\nu_s\Psi) - \nu_s(f.\nu_s\Psi)| \, ds \\
&\le \|\Phi\| \int_0^t \delta(\mu_s, \nu_s) \, ds \\
&\quad + \int_0^t [e^{\|\Phi\| s} e^{\|\Psi\| s} \int \int \rho(dx)\rho(dy) |\Phi(x,y) - \Psi(x,y)|] \, ds \\
&\quad + \|\Psi\| \int_0^t \delta(\mu_s, \nu_s) \, ds, \\
&= \frac{e^{(\|\Phi\| + \|\Psi\|)t} - 1}{\|\Phi\| + \|\Psi\|} \int \int \rho(dx)\rho(dy) \, |\Phi(x,y) - \Psi(x,y)| \\
&\quad + (\|\Phi\| + \|\Psi\|) \int_0^t \delta(\mu_s, \nu_s) \, ds,
\end{aligned}$$

and another application of Gronwall's lemma gives the desired inequality.
(v) Suppose that $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$. Put $h_i(t) = \mu_t(A_i)/\rho(A_i)$, $1 \leq i \leq m$, then it is straightforward to check that $h_1, \dots, h_m$ solve the required system of equations. Define $\bar{\mu} \in \mathcal{M}$ by

$$\bar{\mu}_t(dx) = \sum_i h_i(t) 1_{A_i}(x))\rho(dx),$$

then $\bar{\mu}$ also solves $\mathtt{LV}(\rho, \Phi)$, and so $\mu = \bar{\mu}$ by part (i). The converse is obvious.
(vi) See Section 5.

□

PROPOSITION 2 *Suppose that E is a compact metric space with Borel σ-field $\mathcal{E}$ and metric r, and $\Phi \in C(E \times E)$. There exists a metric Δ on $\mathcal{P}$ such that Δ induces the topology of weak convergence, and, if $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ and $\nu \in \mathcal{M}$ solves $\mathtt{LV}(\pi, \Phi)$, then $\Delta(\mu_t, \nu_t) \leq \Delta(\rho, \pi) e^{2(\|\Phi\|+1)t}$.*

Proof. For $\alpha > 0$ put $w(\alpha) = \sup\{|\Phi(x,y) - \Phi(x',y')| : r(x,x') \leq \alpha, r(y,y') \leq \alpha\}$. Write $\mathcal{D} \subseteq C(E)$ for the class of functions $f \in bC(E)$ such that $\|f\| \leq 1$ and, for all $\alpha > 0$, $\sup\{|f(x) - f(x')| : r(x,x') \leq \alpha\} \leq w(\alpha) + \alpha$.

For $\xi, \zeta \in \mathcal{P}$ put $\Delta(\xi, \zeta) = \sup\{|\xi f - \zeta f| : f \in \mathcal{D}\}$. Since $\mathcal{D}$ is compact in $C(E)$ (by the Arzela-Ascoli theorem) and the linear span of $\mathcal{D}$ is dense in $C(E)$, it follows that the metric Δ induces the topology of weak convergence.

We have for $f \in \mathcal{D}$ that

$$|\mu_t f - \nu_t f| \leq |\rho f - \pi f| + \int_0^t \mu_s(f|\mu_s\Phi - \nu_s\Phi|)\, ds + \int_0^t |\mu_s(f.\nu_s\Phi) - \nu_s(f.\nu_s\Phi)|\, ds.$$

Note that $(\|\Phi\| + 1)^{-1}\Phi(x, \cdot) \in \mathcal{D}$ for all $x \in \mathcal{E}$ and $(\|\Phi\| + 1)^{-1} f \,.\, \xi\Phi \in \mathcal{D}$ for all $f \in \mathcal{D}$ and $\xi \in \mathcal{P}$. Thus

$$\Delta(\mu_t, \nu_t) \leq \Delta(\rho, \pi) + 2(\|\Phi\| + 1) \int_0^t \Delta(\mu_s, \nu_s),$$

and an application of Gronwall's lemma completes the proof.

□

For the next result we need to recall a notion from information theory. If $\xi, \zeta \in \mathcal{P}$, then ξ and ζ are said to have *well-defined relative entropy* if ξ and ζ are equivalent measures and $\log(d\zeta/d\xi) \in L^1(\xi)$. The quantity $D(\xi\|\zeta) = -\xi(\log(d\zeta/d\xi))$ is called the relative entropy or Kullback-Leibler divergence of ξ and ζ.

PROPOSITION 3 *Suppose that Φ is skew–symmetric and $\lambda \in \mathcal{P}$ is such that $\lambda\Phi = 0$, λ-a.e.*

(i) If $\nu \in \mathcal{M}$ is the solution of $\mathtt{LV}(\lambda, \Phi)$*, then $\nu_t = \lambda$ for all $t \in \mathbb{R}_+$.*

(ii) If $\rho \in \mathcal{P}$ is such that the relative entropy $D(\lambda\|\rho)$ is well-defined and $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$*, then the relative entropy $D(\lambda\|\mu_t)$ is well-defined and equals $D(\lambda\|\rho)$ for all $t \in \mathbb{R}_+$.*

Proof. (i) This is obvious.
(ii) We know from part (iii) of Theorem 1 that μ_t and λ are equivalent with

$$e^{-\|\Phi\|t}\frac{d\rho}{d\lambda} \le \frac{d\mu_t}{d\lambda} \le e^{\|\Phi\|t}\frac{d\rho}{d\lambda}.$$

Thus $D(\lambda\|\mu_t)$ is well-defined.

First consider the case when there exist constants $0 < c \le C < \infty$ such that $c \le d\rho/d\lambda \le C$ and hence

$$ce^{-\|\Phi\|t} \le \frac{d\mu_t}{d\lambda} \le Ce^{\|\Phi\|t}.$$

It is straightforward to check for each $T \ge 0$ that

$$\lim_{\delta\downarrow 0} \sup_{0\le t\le T} \sup_{|h|\le\delta} \left\| \frac{1}{h}\left[\frac{d\mu_{t+h}}{d\lambda} - \frac{d\mu_t}{d\lambda}\right] - \frac{d\mu_t}{d\lambda}.\mu_t\Phi \right\|_{L^1(\lambda)} = 0,$$

and thence

$$\lim_{\delta\downarrow 0} \sup_{0\le t\le T} \sup_{|h|\le\delta} \left\| \frac{1}{h}\left[\log\left(\frac{d\mu_{t+h}}{d\lambda}\right) - \log\left(\frac{d\mu_t}{d\lambda}\right)\right] - \mu_t\Phi \right\|_{L^1(\lambda)} = 0.$$

Thus, $D(\lambda\|\mu_t)$ is continuously differentiable with derivative $\lambda\mu_t\Phi = 0$, by assumption, and hence $D(\lambda\|\mu_t) = D(\lambda\|\rho)$.

Now suppose that ρ is any probability measure such that $D(\lambda\|\rho)$ is well-defined. For $n \in \mathbb{N}$ define a probability measure ρ^n equivalent to λ by

$$\frac{d\rho^n}{d\lambda}(x) = \left(\frac{d\rho}{d\lambda}(x) \vee n^{-1}\right) \wedge n \Bigg/ \left[\int \lambda(dy)\left(\frac{d\rho}{d\lambda}(y) \vee n^{-1}\right) \wedge n\right].$$

Write $\mu^n \in \mathcal{M}$ for the solution of $\mathtt{LV}(\rho^n, \Phi)$. We have from above that $D(\lambda \| \mu_t^n) = D(\lambda \| \rho^n)$. By part (i) of Theorem 1, $d\mu_t^n / d\lambda$ converges in λ-measure to $d\mu_t / d\lambda$ for all $t \in \mathbb{R}$. Moreover, by part (iii) of Theorem 1,

$$\begin{aligned} \left| \log\left(\frac{d\mu_t^n}{d\lambda}\right) \right| &\le \left| \log\left(\frac{d\rho^n}{d\lambda}\right) \right| + \|\Phi\| t \\ &\le \left| \log\left(\frac{d\rho}{d\lambda}\right) \right| + \left| \log\left(\int \lambda(dy) \left(\frac{d\rho}{d\lambda}(y) \vee n^{-1} \right) \wedge n \right) \right| + \|\Phi\| t. \end{aligned}$$

Therefore, by dominated convergence,

$$D(\lambda \| \mu_t) = \lim_{n \to \infty} D(\lambda \| \mu_t^n) = \lim_{n \to \infty} D(\lambda \| \rho^n) = D(\lambda \| \rho),$$

as required.

□

The function $\xi \mapsto D(\lambda \| \xi)$ with λ satisfying the conditions of Proposition 3 provides what is variously called an invariant, conserved quantity, first integral or constant of motion. In the classical case when E is finite, this type of invariant can be found in [28]. There is a substantial literature on invariants for the general Lotka–Volterra system (1) (see, for example, [5, 3, 24, 4]). However, if we take $(a_i) = 0$ and (b_{ij}) to be skew–symmetric as we have done here, then this body of work does not produce any invariants beyond those found in [28] unless one imposes further conditions on (b_{ij}).

Some comments are in order regarding the existence of $\lambda \in \mathcal{P}$ with the property $\lambda\Phi = 0$, λ-a.e., for Φ skew–symmetric. Of course, any point mass has the property, but there are other, nontrivial, examples. When E is a finite set, we can think of Φ as the pay-off matrix in a finite, symmetric, two-person, zero-sum game. The fundamental result on the solution of such games guarantees that in this case there is $\lambda \in \mathcal{P}$ such that $\int \lambda(dx)\, \Phi(x,y) \ge 0$ for all y and, equivalently, $\int \lambda(dy)\, \Phi(x,y) = \lambda\Phi(x) \le 0$ for all x (cf. [13]). In particular, $\int \lambda(dx)\lambda(dy)\, \Phi(x,y) = 0$ and we must in fact have $\lambda\Phi = 0$, λ-a.e. More generally, when E is a compact metric space and Φ is continuous, a straightforward compactness argument using the finite case shows that there still exists λ such that $\lambda\Phi(x) \le 0$ for all x and hence $\lambda\Phi = 0$, λ-a.e. Clearly, the measure λ is a point mass at some $y \in \mathcal{E}$ if and only if $\Phi(x,y) \le 0$ for all $x \in \mathcal{E}$. Another situation in which a suitable, nontrivial λ arises is when E is a compact group and Φ is such that $\Phi(zx, zy) = \Phi(x,y)$ for all $x, y, z \in E$, for in this case we can take λ to be Haar measure.

We end this section with the following result on building up solutions of $\mathtt{LV}(\rho, \Phi)$ using a product construction. The proof is, of course, immediate from the product rule for differentiation.

PROPOSITION 4 *Suppose that $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ and $\nu \in \mathcal{M}$ solves $\mathtt{LV}(\pi, \Psi)$. Then $t \mapsto \mu_t \otimes \nu_t$ solves $\mathtt{LV}(\rho \otimes \pi, \Theta)$, where $\Theta((x_1, x_2), (y_1, y_2)) = \Phi(x_1, y_1) + \Psi(x_2, y_2)$.*

3 SOME SPECIAL CASES.

As the class of equations $\mathtt{LV}(\rho, \Phi)$ contains the classical Lotka–Volterra systems that preserve total mass, we cannot, in general, expect closed-form solutions. However, there is one natural and nontrivial special case in which $\mathtt{LV}(\rho, \Phi)$ can be explicitly solved.

PROPOSITION 5 *Suppose that $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ with $E = \mathbb{R}$, $\mathcal{E}$ the usual Borel σ-field, and $\Phi(x, y) = 1_{\{x \le y\}} - 1_{\{y \le x\}}$. Then*

$$\mu_t([z, \infty[) = \frac{\rho([z, \infty[)e^{-t}}{1 - \rho([z, \infty[)(1 - e^{-t})}.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt}\mu_t([z, \infty[) &= \mu_t(1_{[z,\infty[}.\mu_t \Phi) \\
&= (\mu_t \otimes \mu_t)(\{(x, y) : z \le x, x \le y\}) - (\mu_t \otimes \mu_t)(\{(x, y) : z \le x, y \le x\}) \\
&= (\mu_t \otimes \mu_t)(\{(x, y) : z \le x, x \le y\}) - (\mu_t \otimes \mu_t)(\{(x, y) : z \le y, x \le y\}) \\
&= -(\mu_t \otimes \mu_t)(\{(x, y) : x < z \le y\}) \\
&= -\mu_t([z, \infty[)(1 - \mu_t([z, \infty[)).
\end{aligned}$$

The ordinary differential equation $g'(t) = -g(t)(1 - g(t))$ (sometimes called the epidemic or logistic equation) has the solution $g(t) = (1 + ce^t)^{-1}$ with c a constant, and the result follows.

□

Note that in Proposition 5 the probability measure μ_t is the geometric mixture $\sum_{n=1}^{\infty}(1 - e^{-t})^{n-1}e^{-t}\nu_n$, where ν_n is the probability measure defined by $\nu_n([z, \infty[) = \rho([z, \infty[)^n$. A corresponding, albeit less explicit, representation will be given in Section 6 for the general case.

The following result shows that the equation considered in Proposition 5 has an infinite family of soliton–like solutions. The proof is a simple calculation.

COROLLARY 6 *Suppose that $\mu \in \mathcal{M}$ solves $\mathtt{LV}(\rho, \Phi)$ with $E = \mathbb{R}$, $\mathcal{E}$ the usual Borel σ-field, and $\Phi(x, y) = 1_{\{x \le y\}} - 1_{\{y \le x\}}$.*

(i) Let $\rho \in \mathcal{P}$ be defined by $\rho([z,\infty[) = (1+ae^{z/b})^{-1}$, where $a, b > 0$. Then $\mu_t = \rho(\cdot + bt)$ for all $t \in \mathbb{R}_+$. That is, μ_t is the push–forward of ρ by the map $z \mapsto z - bt$.

(ii) Let $\rho \in \mathcal{P}$ be such that

$$\lim_{z \to -\infty} \frac{1 - \rho([z,\infty[)}{ae^{z/b}} = 1,$$

where $a, b > 0$. Then $\mu_t(\cdot - bt)$ (the push–forward of μ_t by the map $z \mapsto z + bt$) converges weakly as $t \to \infty$ to $\xi \in \mathcal{P}$ defined by $\xi([z,\infty[) = (1 + ae^{z/b})^{-1}$.

In other special cases it is possible to use Lyapunov functions (that is, real–valued functions on $\mathcal{P}$ with the property that their composition with a solution of $\mathtt{LV}(\rho, \Phi)$ is monotone) to derive information about the large time asymptotics of the solution. The proofs of the next three results are examples of this technique.

PROPOSITION 7 *Suppose that μ is a solution of $\mathtt{LV}(\rho, \Phi)$, where $E = \mathbb{R}$, $\mathcal{E}$ the usual Borel σ-field, $\operatorname{supp} \rho$ bounded below, and Φ is continuous and skew–symmetric with $\Phi(x, y) > 0$ when $x < y$, $x, y \in \operatorname{supp} \rho$. Then μ_t converges weakly as $t \to \infty$ to the point mass at $\min \operatorname{supp} \rho$.*

Proof. By part (iii) of Theorem 1, each probability measure μ_t has the same support as ρ.

Note that for $\xi \in \mathcal{P}$ with support contained in that of ρ and $f \in b\mathcal{E}$

$$\begin{aligned} \xi(f.\xi\Phi) &= \int_{\{x<y\}} \xi(dx)\xi(dy)\, f(x)\Phi(x, y) + \int_{\{x>y\}} \xi(dx)\xi(dy)\, f(x)\Phi(x, y) \\ &= \int_{\{x<y\}} \xi(dx)\xi(dy)\, [f(x)\Phi(x, y) + f(y)\Phi(y, x)] \\ &= \int_{\{x<y\}} \xi(dx)\xi(dy)\, [f(x) - f(y)]\Phi(x, y). \end{aligned} \tag{4}$$

Let $a = \min \operatorname{supp} \rho$. By (4), we have for each nondecreasing function $f \in b\mathcal{E}$ that $\frac{d}{dt}\mu_t f \le 0$, and hence $t \mapsto \mu_t f$ is nonincreasing. One consequence of this is that $\lim_{t\to\infty} \mu_t f$ exists for all nondecreasing $f \in b\mathcal{E}$. Another is that $\mu_t([c,\infty[) \le \rho([c,\infty[)$ for all $c \in \mathbb{R}$, and this taken with the fact that $\mu_t(]{-\infty}, a[) = 0$ implies that the family of probability measures $\{\mu_t\}_{t\in\mathbb{R}_+}$ is tight. Combinining these two observations, we conclude that μ_t converges weakly as $t \to \infty$ to a probability measure μ_∞.

Now for any bounded, continuous function f

$$\lim_{t\to\infty} \frac{d}{dt}\mu_t f = \int_{\{x<y\}} \mu_\infty(dx)\mu_\infty(dy)\,[f(x)-f(y)]\Phi(x,y). \tag{5}$$

In order that $\lim_{t\to\infty}\mu_t f$ exists, the left-hand side of (5) must be 0. Taking f to be strictly increasing, we conclude that μ_∞ must be a point mass at some point in $\operatorname{supp}\rho$.

Another application of (4) shows that $t \to \mu_t(]-\infty, b]) = \mu_t([a,b])$ is nondecreasing for all $b > a$. Thus

$$\mu_\infty([a,b]) \geq \limsup_{t\to\infty} \mu_t([a,b]) \geq \rho([a,b]) > 0$$

for all $b > a$ and $\mu_\infty = \delta_a$.

□

PROPOSITION 8 *Suppose that μ is a solution of* $\mathtt{LV}(\rho,\Phi)$*, with E a metric space, $\mathcal{E}$ the Borel σ-field, $\rho \in \mathcal{P}$ compactly supported, and $\Phi(x,y) = \phi(y) - \phi(x)$, where ϕ is continuous on* $\operatorname{supp}\rho$ *and such that there exists a unique $z \in \operatorname{supp}\rho$ with the property $\phi(z) = \min\{\phi(x) : x \in \operatorname{supp}\rho\}$. Then μ_t converges weakly as $t \to \infty$ to the point mass δ_z.*

Proof. Write $\tilde{\mu}_t$ for the push–forward of μ_t by ϕ. Manipulations similar to those that established (4) give

$$\frac{d}{dt}\tilde{\mu}_t f = -\int_{\{x<y\}} \tilde{\mu}_t(dx)\tilde{\mu}_t(dy)\,[f(x)-f(y)][x-y].$$

A virtual reprise of the rest of the proof of Proposition 7 shows that $\tilde{\mu}_t$ converges weakly as $t \to \infty$ to $\delta_{\tilde{z}}$, where $\tilde{z} = \min\{\phi(x) : x \in \operatorname{supp}\rho\} = \phi(z)$.

The convergence of μ_t to δ_z is then clear.

□

Let $\leq$ denote the usual component–wise partial order on $\mathbb{R}^d$. That is, $(x_1,\ldots,x_d) \leq (y_1,\ldots,y_d)$ if $x_i \leq y_i$ for $1 \leq i \leq d$. Write $(x_1,\ldots,x_d) < (y_1,\ldots,y_d)$ if $x_i < y_i$ for $1 \leq i \leq d$. Note that $x < y$ implies that $x \leq y$ and $x \neq y$, but the converse is false for $d \neq 1$. For $x \in \mathbb{R}^d$ put $[x,\infty[= \{y \in \mathbb{R}^d : x \leq y\}$ and $]x,\infty[= \{y \in \mathbb{R}^d : x < y\}$. Define $]-\infty, x]$ and $]-\infty, x[$ similarly. The following result is immediate from Proposition 5 in the case $d = 1$.

PROPOSITION 9 *Suppose that $\mu \in \mathcal{M}$ is a solution of $\mathtt{LV}(\rho, \Phi)$ with $E = \mathbb{R}^d$, $\mathcal{E}$ the usual Borel σ-field, $\operatorname{supp} \rho$ bounded below in the partial order $\leq$, and $\Phi(x, y) = 1_{\{x \leq y\}} - 1_{\{y \leq x\}}$. Then μ_t converges weakly as $t \to \infty$ to a probability measure with support contained in the set $\{z \in \operatorname{supp} \rho :]-\infty, z[\cap \operatorname{supp} \rho = \emptyset\}$.*

Proof. We have for $z \in \mathbb{R}^d$ that

$$\begin{aligned} \frac{d}{dt}\mu_t([z, \infty[) &= (\mu_t \otimes \mu_t)(\{(x, y) : z \leq x, x \leq y\}) \\ &\quad - (\mu_t \otimes \mu_t)(\{(x, y) : z \leq x, y \leq x\}) \\ &= (\mu_t \otimes \mu_t)(\{(x, y) : z \leq x, x \leq y\}) \\ &\quad - (\mu_t \otimes \mu_t)(\{(x, y) : z \leq y, x \leq y\}) \\ &= -(\mu_t \otimes \mu_t)(\{(x, y) : z \not\leq x, z \leq y, x \leq y\}) \\ &\leq 0. \end{aligned} \tag{6}$$

Thus $t \mapsto \mu_t([z, \infty[)$ is nonincreasing for each $z \in \mathbb{R}^d$ and, in particular, $\lim_{t\to\infty} \mu_t([z, \infty[)$ exists. This, coupled with the boundedness below of $\operatorname{supp} \mu_t = \operatorname{supp} \rho$ (recall part (iii) of Theorem 1), shows that $\mu_\infty = \lim_{t\to\infty} \mu_t$ exists.

Suppose that $w \in \mathbb{R}^d$ is such that $]-\infty, w[\cap \operatorname{supp} \rho \neq \emptyset$ and yet $w \in \operatorname{supp} \mu_\infty$. Then there exists $z \in \mathbb{R}^d$ such that $w \in]z, \infty[$ and $]-\infty, z[\cap \operatorname{supp} \rho \neq \emptyset$. By the above, we have

$$\begin{aligned} \inf_t \mu_t([z, \infty[) &= \lim_{t\to\infty} \mu_t([z, \infty[) \\ &\geq \lim \inf_{t\to\infty} \mu_t(]z, \infty[) \\ &\geq \mu_\infty(]z, \infty[) \\ &> 0. \end{aligned} \tag{7}$$

An argument similar to the above shows that $t \mapsto \mu_t(]-\infty, z[)$ is nondecreasing and so

$$\inf_t \mu_t(]-\infty, z[) \geq \rho(]-\infty, z[) > 0. \tag{8}$$

From (6) we have

$$\begin{aligned} \frac{d}{dt}\mu_t([z, \infty[) &= -(\mu_t \otimes \mu_t)(\{(x, y) : z \not\leq x, z \leq y, x \leq y\}) \\ &\leq -(\mu_t \otimes \mu_t)(\{(x, y) : x < z, z \leq y, x \leq y\}) \\ &= -\mu_t(]-\infty, z[)\mu_t([z, \infty[); \end{aligned} \tag{9}$$

and by (7) and (8) the rightmost quantity is bounded strictly below 0. This is clearly impossible, since $\mu_t([z, \infty[) \geq 0$ for all t.

□

4 SYSTEMS OF COALESCING MASSES.

Suppose now that E is a locally compact metric space. Write E^∂ for the one–point compactification of E with point at infinity ∂. Write $\mathcal{P}^\partial$ for the space of probability measures on E^∂ and let d be a metric on $\mathcal{P}^\partial$ that is complete and induces the topology of weak convergence. We will also use the notation d for the restriction of d to $\mathcal{P}$.

Fix $, \in b(\mathcal{E} \times \mathcal{E})$. Let $D = \{e_1, \dots, e_N\} \subset E$ be a finite subset of E. Write

$$\mathcal{S}_N = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \ : \ \sum_{k=1}^N x_k = 1, \ x_k \geq 0, \forall k\}.$$

For $k = 1, \dots N$ define $G_k : \mathcal{S}_N \times (\{1, \dots, N\})^2 \times [0, \|, \|] \to \mathbb{R}$ by

$$G_k(y, (i,j), u) = \begin{cases} y_j, & \text{if } i = k \neq j, \ 0 \leq u \leq \ , (e_i, e_j) y_i, \\ -y_j & \text{if } i \neq k = j, \ 0 \leq u \leq \ , (e_i, e_j) y_i, \\ 0, & \text{otherwise.} \end{cases}$$

Define over some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ an iid collection $N_{i,j}$, $(i,j) \in (\{1, \dots, N\})^2$, of Poisson random measures on $[0, \|, \|] \times \mathbb{R}_+$, with common intensity Lebesgue measure.

For $x \in \mathcal{S}_N$ it is straightforward to construct a unique càdlàg $\mathcal{S}_N$–valued process X such that

$$X_k(t) = x_k + \sum_{i,j} \int_{[0, \|\Gamma\|] \times [0,t]} G_k(X(s-), (i,j), u) \, N_{i,j}(du, ds).$$

Thus X is a jump–hold Markov process such that a jump from state

$$(y_1, \dots, y_{i-1}, y_i, y_{i+1}, \dots, y_{j-1}, y_j, y_{j+1}, \dots, y_N)$$

to state

$$(y_1, \dots, y_{i-1}, y_i + y_j, y_{i+1}, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_N)$$

occurs at rate $, (e_i, e_j) y_i$.

Put $\zeta = \sum_{k=1}^N x_k \delta_{e_k} \in \mathcal{P}$ and define a $\mathcal{P}$–valued process Z by $Z_t f = \sum_{k=1}^N X_k(t) \delta_{e_k}$. We can also think of ζ as an element of $\mathcal{P}^\partial$ and Z as a $\mathcal{P}^\partial$–valued process. Observe that Z has the evolution dynamics described in Section 1. That is, the jump

$$\sum_k y_k \delta_{e_k} \longrightarrow \sum_{k \notin \{i,j\}} y_k \delta_{e_k} + (y_i + y_j)\delta_{e_i} + 0.\delta_{e_j}$$

occurs at rate , $(e_i, e_j)y_i$

Note that Z has the semimartingale decomposition

$$Z_t f = \zeta f + \int_0^t Z_s(f.Z_s \hat{,}\,)\, ds + M_t^f, \tag{10}$$

where M_t^f is a martingale with

$$\langle M^f \rangle_t = \int_0^t \sum_{i,j} (f(e_i) - f(e_j))^2 \, X_j(s)^2, \, (e_i, e_j) X_i(s)\, ds, \tag{11}$$

and we recall the notation $\hat{,}\,(x,y) = \,,\,(x,y) - \,,\,(y,x)$.

LEMMA 10 *In the above notation,*

$$\mathbb{P}[\sum_{k=1}^N X_k(t)^2] \le e^{2\|\Gamma\|t} \sum_{k=1}^N x_k^2 = e^{2\|\Gamma\|t} (\zeta \otimes \zeta)(\{(v,w) \,:\, v = w\}).$$

Proof. From Itô's formula for stochastic integrals against uncompensated Poisson random measures (cf. Theorem II.5.1 of [10]) we have

$$\begin{aligned} \mathbb{P}[\sum_k X_k^2(t)] &= \sum_k x_k^2 + \int_0^t \mathbb{P}[2 \sum_{i,j} X_i(s)^2 X_j(s), \, (e_i, e_j)]\, ds \\ &\le \sum_{k=1}^N x_k^2 + 2\|, \| \int_0^t \mathbb{P}[\sum_k X_k(s)^2]\, ds, \end{aligned}$$

and the result follows from Gronwall's lemma.

□

For $n \in \mathbb{N}$ let Z^n be a process constructed on $(\Omega, \mathcal{F}, \mathbb{P})$ via the same recipe that was used for constructing Z, with the defining ingredients D and ζ replaced, respectively, by some finite subset $D^n = \{e_1^n, \ldots, e_{N(n)}^n\}$ and probability measure ζ^n with support D^n.

THEOREM 11 *Suppose that* $\lim_{n\to\infty} \zeta^n = \rho$ *in the weak topology, where* $\rho \in \mathcal{P}$ *is a diffuse probability measure such that the product measure* $\rho \otimes \rho$ *assigns zero mass to the set of discontinuities of* $\hat{,}$ *. Then the sequence* $\{Z^n\}_{n=1}^\infty$ *converges in probability (with respect to the Skorohod topology induced by the metric d) to the unique deterministic function* $\mu \in \mathcal{M}$ *that solves* $\mathtt{LV}(\rho, \hat{,}\,)$*.*

Proof. By assumption, $\lim_{n\to\infty}(\zeta^n \otimes \zeta^n)(\{(v,w) : v = w\}) = 0$.

Fix $f \in C(\mathcal{P}^\partial)$ Let $M^{f,n}$ be the analogue of M^f for Z^n. For each $n \in \mathbb{N}$ let τ_n be a stopping-time for the natural filtration of Z^n such that for all n we have $0 \le \tau_n \le T$ for some constant $T \ge 0$, and let $0 \le \delta_n \le 1$. Applying (10), (11) and Lemma 10 gives

$$\begin{aligned}\mathbb{P}\left[|Z^n_{\tau_n+\delta_n} f - Z^n_{\tau_n} f|\right] &\le \|f\| \|\hat{,}\| \delta_n \left\{\mathbb{P}\left[\langle M^{f,n}_{\tau_n+\delta_n}\rangle - \langle M^{f,n}_{\tau_n}\rangle\right]\right\}^{1/2} \\ &\le \|f\| \|\hat{,}\| \delta_n + \left\{2\|f\|^2 \|,\| e^{2\|\Gamma\|(T+1)} (\zeta^n \otimes \zeta^n)(\{(v,w) : v = w\})\delta_n\right\}^{1/2}.\end{aligned}$$

Let $P^{f,n}$ denote the law of the càdlàg $\mathbb{R}$–valued processes $(Z^n_t f)_{t\ge 0}$. It follows from Aldous's criterion for tightness [1] that the sequence $(P^{f,n})_{n\in\mathbb{N}}$ is tight in the space of probability measures on $D(\mathbb{R}_+,\mathbb{R})$, the Skorohod space of $\mathbb{R}$–valued càdlàg paths.

Write P^n for the the law of the càdlàg (with respect to the metric d) $\mathcal{P}^\partial$–valued processes Z^n. The sequence $(P^n)_{n\in\mathbb{N}}$ is tight in the space of probability measures on the Skorohod space $D(\mathbb{R}_+, (\mathcal{P}^\partial, d))$ (cf. Theorem 3.7.1 of [6]).

Again from (11) and Lemma 10, we see that

$$\lim_{n\to\infty} \mathbb{P}[\sup_{0\le s\le t} \{M^{f,n}_s\}^2] = 0. \tag{12}$$

Let $\tilde{Z}$ be a process that has as its law a particular subsequential limit of the sequence $(P^n)_{n\in\mathbb{N}}$ (in the topology of weak convergence of laws on $D(\mathbb{R}_+, (\mathcal{P}^\partial, d))$). It follows from (12) that almost surely

$$\tilde{Z}_t f \le \rho f + \|\hat{,}\| \int_0^t \tilde{Z}_s f \, ds$$

for all $f \in pC(E^\partial)$ and all $t \in \mathbb{R}_+$. Hence, by Gronwall's lemma and a monotone class argument, almost surely $\tilde{Z}_t$ is absolutely continuous with respect to ρ for all $t \in \mathbb{R}_+$ and

$$\frac{d\tilde{Z}_t}{d\rho} \le e^{\|\hat{\Gamma}\|t}. \tag{13}$$

The inequality (13) has two consequences. Firstly, almost surely $\tilde{Z}_t$ is supported on E for all $t \in \mathbb{R}_+$. Hence the law of $\tilde{Z}$ is actually a subsequential limit of the sequence $(P^n)_{n\in\mathbb{N}}$ in the topology of weak convergence of laws on the Skorohod space $D(\mathbb{R}_+, (\mathcal{P}, d))$. Secondly, almost surely $\tilde{Z}_t \otimes \tilde{Z}_t$ assigns zero mass to the set of discontinuities of $\hat{,}$ for all $t \in \mathbb{R}_+$. Thus, again using (12), almost surely

$$\tilde{Z}_t f = \rho f + \int_0^t \tilde{Z}_s(f.\tilde{Z}_s \hat{,})\, ds$$

for all $f \in C(E^\partial)$ and all $t \in \mathbb{R}_+$. It follows from a monotone class argument and parts (i) of Theorem 1 that $\tilde{Z} = \mu$ almost surely, and the result follows from the familiar fact that convergence in law of a sequence of random variables to a constant implies convergence in probability.

□

Suppose now that we "tag" one of the atoms of $Z_0^n = \zeta^n$ and let W^n be the $D^n \subset E$–valued process that keeps track of the label of this piece of mass as it is moved around by coalescent events. That is, $W_0^n = w^n \in D^n$, W^n has jump–hold paths that can only jump when Z^n jumps, and W^n jumps from e_j^n to e_i^n at time t if and only if Z^n jumps at time t by moving mass at e_j^n to e_i^n. It is straightforward to check that

$$f(W_t^n) - f(w^n) - \int_0^t \left[\int_E Z_s^n(dx),\, (x, W_s^n)(f(x) - f(W_s^n)) \right] ds$$

is a martingale for all $f \in bC(E)$. We omit the standard argument using Theorem 11 that establishes the following.

COROLLARY 12 *Suppose that the assumptions of Theorem 11 hold and* $\lim_{n\to\infty} w^n = w \in E$. *Then the law of* W^n *converges as* $n \to \infty$ *in the topology of weak convergence of probability measures on* $D(\mathbb{R}_+, E)$ *to the law of a time–inhomogeneous, jump–hold Markov process* W *on* E *that jumps at rate* $\mu_t(dx),\, (x, y)$ *from state* y *to state* x *at time* t. *That is,* W *is the unique in law process such that*

$$f(W_t) - f(w) - \int_0^t \left[\int_E \mu_s(dx),\, (x, W_s)(f(x) - f(W_s)) \right] ds$$

is a martingale for all $f \in bC(E)$.

5 PROOF OF THEOREM 1(vi).

We begin with a particular case of Theorem 1(vi). Let λ denote Lebesgue measure on $[0,1[$ and suppose that $H \in bp(\mathcal{B}([0,1[) \times \mathcal{B}([0,1[))$ is product–simple with corresponding partition of the form $[a_0, a_1[, [a_1, a_2[, \ldots [a_{m-1}, a_m[$ for $0 = a_0 < a_1 < \cdots < a_m = 1$. Note that $\lambda \otimes \lambda$ assigns zero mass to the discontinuities of H. Apply Theorem 11 with $D^n = \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, ζ^n given by $\zeta^n(\{\frac{k}{n}\}) = \frac{1}{n}$ for $1 \le k \le n$, and $, = H$, to see that the equation $\mathtt{LV}(\lambda, \hat{H})$ has a solution.

Now consider the general case of Theorem 1(vi). Recall that any skew–symmetric $\Phi \in b(\mathcal{E} \times \mathcal{E})$ can be written as $\Phi = \hat{\Psi}$ with $\Psi = \frac{1}{2}(\Phi + \|\Phi\|)$. As the nonnegative product–simple functions are $L^1(\rho \otimes \rho)$-dense in $b(\mathcal{E} \times \mathcal{E})$, we see by Theorem 1(iv) that it suffices to consider the case when $\Phi = \hat{\Psi}$ with $\Psi = \sum_{i,j} c_{ij} 1_{A_i \times A_j}$ a product–simple function and $c_{ij} \geq 0$.

Determine $0 = a_0 < a_1 < \cdots < a_m = 1$ by the requirements $\lambda([a_{k-1}, a_k[) = a_k - a_{k-1} = \rho(A_k)$ and define $H \in bp(\mathcal{B}([0,1[) \times \mathcal{B}([0,1[))$ by $H = \sum_{i,j} c_{ij} 1_{[a_{i-1},a_i[\times[a_{j-1},a_j[}$. We know by the above that $\mathtt{LV}(\lambda, \hat{H})$ has a solution η.

By one direction of Theorem 1(v), η is of the form $\eta_t(dz) = (\sum_k h_k(t) 1_{[a_{k-1},a_k[}(z))\lambda(dz)$ where the functions $h_1, \ldots, h_m$ solve the system

$$h_i(t) = h_i(0) + \int_0^t h_i(s) \sum_{j=1}^m b_{ij} h_j(s)\, ds, \quad 1 \leq i \leq m,$$

with $h_i(0) = \lambda([a_{i-1}, a_i[) = \rho(A_i)$ and $b_{ij} = c_{ij}\lambda([a_{i-1}, a_i[)\lambda([a_{j-1}, a_j[) = c_{ij}\rho(A_i)\rho(A_j)$. Theorem 1(v) in the other direction gives that $\mu_t(dx) = (\sum_k h_k(t) 1_{A_k}(x))\rho(dx)$ solves $\mathtt{LV}(\rho, \Phi)$.

□

6 A SERIES EXPANSION.

Suppose that μ solves $\mathtt{LV}(\rho, \Phi)$. Observe that if $f_1, \ldots, f_n \in b\mathcal{E}$, then

$$\begin{aligned}
\frac{d}{dt} \prod_{i=1}^n \mu_t f_i &= \sum_{i=1}^n \Big[\prod_{j \neq i} \mu_t f_j\Big] \mu_t(f_i . \mu_t \Phi) \\
&= \int \mu_t^{\otimes(n+1)}(dx) \Big[\prod_{i=1}^n f_i(x_i)\Big] \Big[\sum_{j=1}^n \Phi(x_j, x_{n+1})\Big].
\end{aligned}$$

Hence, if $F \in b(\mathcal{E}^n)$, then

$$\begin{aligned}
\frac{d}{dt}\mu_t^{\otimes n} F &= \int \mu_t^{\otimes(n+1)}(dx)(F \otimes 1)(x) \Big[\sum_{j=1}^n \Phi(x_j, x_{n+1})\Big] \\
&= \sum_{j=1}^n \|\Phi\| \Big\{ \int \mu_t^{\otimes(n+1)}(dx)(F \otimes 1)(x) \Psi(x_j, x_{n+1}) - \mu_t^{\otimes n} F \Big\},
\end{aligned} \tag{14}$$

where $\Psi = \|\Phi\|^{-1}\Phi + 1 \geq 0$.

Let $\mathcal{A} = \bigcup_{n=1}^{\infty} b\mathcal{E}^n$. Write $\#F = n$ if $F \in \mathcal{A}$ belongs to $b\mathcal{E}^n$. Define a pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{P}$ and $\mathcal{A}$ by setting $\langle \xi, F \rangle = \xi^{\otimes(\#F)} F$. Define a jump–hold Markov process Y with state-space $\mathcal{A}$ and laws $(\mathbb{Q}^F)_{F \in \mathcal{A}}$ by declaring that if Y is in state $F \in b\mathcal{E}^n$, then it jumps at rate $n\|\Phi\|$, and when it jumps it chooses with equal probability one of the states $(x_1, \ldots, x_{n+1}) \mapsto (F \otimes 1)(x)\Psi(x_j, x_{n+1}) \in b\mathcal{E}^{n+1}$, $1 \leq j \leq n$. The identity (14) shows that the generator of Y and the generator of the flow μ (thought of as a deterministic Markov process) are in duality with respect to the pairing $\langle \cdot, \cdot \rangle$ in the sense described in, say, Section 4.4 of [7].

From Corollary 4.4.13 of [7], we see that

$$\mu_t f = \mathbb{Q}^f[\langle \rho, Y_t \rangle], \; f \in b\mathcal{E}.$$

It is clear from the definition of Y that the process $\#Y$ is a pure birth process for which jumps from state n to state $n+1$ are made at rate $n\|\Phi\|$. Thus $\mathbb{Q}^f\{\#Y_t = n\} = (1 - e^{-\|\Phi\| t})^{n-1} e^{-\|\Phi\| t}$, $n \in \mathbb{N}$, and

$$\mu_t f = \sum_{n=1}^{\infty} (1 - e^{-\|\Phi\| t})^{n-1} e^{-\|\Phi\| t} \times \frac{1}{(n-1)!} \sum_{i(n,2)=1}^{1} \cdots \sum_{i(n,n)=1}^{n-1} \int \rho^{\otimes n}(dx) f(x_1) \prod_{j=2}^{n} \Psi(x_{i(n,j)}, x_j).$$

Therefore, if we define finite measures ν_n, $n \in \mathbb{N}$, by

$$\nu_n f = \frac{1}{(n-1)!} \sum_{i(n,2)=1}^{1} \cdots \sum_{i(n,n)=1}^{n-1} \int \rho^{\otimes n}(dx) f(x_1) \prod_{j=2}^{n} \Psi(x_{i(n,j)}, x_j),$$

then

$$\mu_t = \sum_{n=1}^{\infty} (1 - e^{-\|\Phi\| t})^{n-1} e^{-\|\Phi\| t} \nu_n$$

Equating terms, we see that $\nu_n 1 = 1$ for all n and so $\nu_n \in \mathcal{P}$ for all n.

Note that this expansion is different to the the one obtained by simply iterating $\mathtt{LV}(\rho, \Phi)$. The latter would lead to a Taylor–type expansion of the form

$$\mu_t = \sum_{n=1}^{\infty} \frac{t^n}{n!} \xi_n,$$

where the ξ_n are just signed measures. In principle, it should be possible to derive weak convergence results of the form $\lim_{t\to\infty} \mu_t = \zeta$ by establishing that $\lim_{n\to\infty} \nu_n = \zeta$, but we don't have an example (other than the trivial one of Proposition 5 where it is possible to evaluate ν_n explicitly).

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