Stat 200A Fall 2007 Practice Problems

Instructor: Sourav Chatterjee

1. Suppose two independent events A and B have $P(A) = 0.5$ and $P(B) = 0.6$. Compute $P(A \cup B)$, $P(A \cap B \mid A \cup B)$, $P(B \mid A \cup B)$, and $P(A \cap B \mid A)$.

2. If you split a shuffled deck of 52 cards equally among four players, what is the chance that all four aces are given to the same player?

3. What is the expected number of students in class of size 50 who share their birthday with at least one other person in the class?

4. Suppose $X \sim Poisson(\lambda)$. Given a number b, compute $E((1+b)^X)$.

5. Suppose a point (X, Y) is picked uniformly from the rhombus with vertices $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$. What is the joint density function of (X, Y)? What is the marginal density function of X? What is its expected value and variance? What is the conditional density function of X given $Y = y$?

6. Consider an equilateral triangle whose sides each have length s. Let a point be uniformly chosen from one side of the triangle. Let X denote the distance of the point chosen from the opposite vertex. Find the distribution function and density of X.

7. If $X_1, \ldots, X_n \overset{i.i.d.}{\sim} Exp(1)$, compute the probability density, expectation, and variance of $\min_{1 \le i \le n} X_i$ and $\max_{1 \le i \le n} X_i$. Also compute their joint density function.

8. Suppose X and Y are independent $N(0, 1)$ random variables. Compute the probability density function of $X^2 + Y^2$.

9. Suppose $X \sim Unif[0, 2\pi]$. Compute the probability density function of $\cos X$. Compute its expected value and variance.

10. Suppose $X \sim Exp(1)$, and Y is another random variable such that the conditional distribution of Y given $X = x$ is $Exp(x)$. Using the formula
$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)},$$

compute the joint density of (X, Y). Compute the marginal density of Y and expected value of Y. Also compute the conditional density and conditional expectation of X given $Y = y$.

11. Suppose $X_1, X_2, \ldots \overset{i.i.d.}{\sim} N(\mu, 1)$. Suppose you want to test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. What is the standard test at level 5%? Show that as $n \to \infty$, the power function for your test converges to 1 at every $\mu \neq 0$.

12. Suppose $X_1, X_2, \ldots \overset{i.i.d}{\sim} Poisson(\lambda)$. How would you construct a crude confidence interval for λ? What is a more accurate interval using variance stabilizing transforms? If a preliminary estimate suggests that $\lambda \approx 1$, how large a sample should you collect to ensure that the variance stabilized confidence interval has length ≤ 0.1?

13. Suppose $X_1, X_2, \ldots \overset{i.i.d}{\sim} Poisson(\lambda)$. What is the Fisher information function for this model? What is the smallest possible variance of an unbiased estimator of λ based on $X_1, \ldots, X_n$?

14. Suppose $X_1, X_2, \ldots \overset{i.i.d}{\sim} Geo(p)$. What is the Fisher information function for this model? Compute the MLE $\hat{p}_n$ of p based on $X_1, \ldots, X_n$. Find out the limiting distribution of $\sqrt{n}(\hat{p}_n - p)$ as $n \to \infty$.

15. Suppose $X_1, \ldots, X_n$ are independent random variables, and for each i, X_i follows the exponential distribution with $E(X_i) = i\beta$, where β is an unknown parameter. Compute the MLE of β.