STAT C206A / MATH C223A : Stein's method and applications — Fall 2007

Lecture 7

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1 Method of Exchangeable Pair

Suppose (W, W') is an exchangeable pair of random variables, *i.e.* $(W, W') \stackrel{d}{=} (W', W)$, and there is a constant $\lambda \in (0, 1)$ such that

$$\mathbf{E}(W' - W|W) = -\lambda W \text{ a.e.} \tag{1}$$

Also suppose $\mathbf{E}W^2 = 1$, then we have

$$\mathbf{Wass}(W, Z) \leq \sqrt{\frac{2}{\pi}\mathrm{Var}\left(\mathbf{E}(\frac{1}{2\lambda}(W' - W)^2|W)\right)} + \frac{1}{3\lambda}\mathbf{E}|W' - W|^3 \tag{2}$$

where $Z \sim \mathrm{N}(0, 1)$.

In general λ is a small positive number, usually of the oder $1/n$ and W' is obtained by applying a small perturbation to W.

Note that from the given conditions we have $\mathbf{E}W = \mathbf{E}W'$ and $\mathbf{E}W^2 = \mathbf{E}W'^2 = 1$. Using this information along with (1) we have

1. $\mathbf{E}W = 0$. Since $\mathbf{E}[-\lambda W] = \mathbf{E}[\mathbf{E}(W' - W|W)] = \mathbf{E}[W' - W] = 0$ and $\lambda \neq 0$.
2. $\mathbf{E}(W' - W)^2 = 2\lambda$. Since

$$\begin{aligned}\mathbf{E}(W' - W)^2 &= \mathbf{E}[W'^2 + W^2 - 2W'W] \\ &= \mathbf{E}[2W^2 - 2W'W] \\ &= \mathbf{E}[2W(W - W')] = \mathbf{E}[2W\mathbf{E}(W - W'|W)] = \mathbf{E}[2\lambda W^2] = 2\lambda.\end{aligned}$$

Now take any twice differentiable function f with $\|f\|_\infty \leq 1, \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}$ and $\|f''\|_\infty \leq 2$. Let

$$F(x) = \int_0^x f(y)dy.$$

Clearly F is a well defined thrice differentiable function. So using Taylor series expansion for F we have

$$\begin{aligned}0 &= \mathbf{E}[F(W') - F(W)] \\ &= \mathbf{E}\left[(W' - W)f(W) + \frac{1}{2}(W' - W)^2 f'(W) + \text{Remainder}\right]\end{aligned} \tag{3}$$

where $|\text{Remainder}| \le \frac{1}{6}|W-W'|^3\|f''\|_\infty \le \frac{1}{3}|W-W'|^3$. Now

$$\begin{aligned}
-\lambda\mathbf{E}[Wf(W)] &= \mathbf{E}[(W'-W)f(W)] \\
&= -\mathbf{E}\left[\frac{1}{2}(W'-W)^2 f'(W) + \text{Remainder}\right] \\
&= -\mathbf{E}\left[\frac{1}{2}\mathbf{E}[(W'-W)^2|W]\ f'(W)\right] + \mathbf{E}[\text{Remainder}].
\end{aligned}$$

Dividing both sides by λ we get

$$|\mathbf{E}f'(W) - \mathbf{E}Wf(W)| \le \left|\mathbf{E}\left[f'(W)\cdot\left(\mathbf{E}(\frac{1}{2\lambda}(W-W')^2|W) - 1\right)\right]\right| + \frac{1}{3\lambda}\mathbf{E}|W-W'|^3.$$

Since $\|f'\|_\infty \le \sqrt{\frac{2}{\pi}}$ and $\mathbf{E}(W-W')^2 = 2\lambda$ we have

$$\begin{aligned}
|\mathbf{E}f'(W) - \mathbf{E}Wf(W)| &\le \sqrt{\frac{2}{\pi}}\cdot\mathbf{E}\left|\mathbf{E}(\frac{1}{2\lambda}(W-W')^2|W) - 1\right| + \frac{1}{3\lambda}\mathbf{E}|W-W'|^3 \\
&\le \sqrt{\frac{2}{\pi}\text{Var}\left(\mathbf{E}(\frac{1}{2\lambda}(W'-W)^2|W)\right)} + \frac{1}{3\lambda}\mathbf{E}|W-W'|^3.
\end{aligned}$$

Remark 1 *If W, W' just have the same distribution (need not be exchangeable) then also the above result holds.*

Now let us apply this method to the simplest case of sums of independent random variables.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with mean 0 and variance 1. Define

$$W = \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i.$$

Let $X_1', X_2', \ldots, X_n'$ be an independent copy of $X_1, X_2, \ldots, X_n$. Choose an index I uniformly at random from $\{1, 2, \ldots, n\}$. Replace X_I by X_I'. Let

$$W' = \frac{1}{\sqrt{n}}\sum_{j\ne I} X_j + \frac{X_I'}{\sqrt{n}} = W + \frac{X_I' - X_I}{\sqrt{n}}.$$

Lemma 2 (W, W') *is an exchangeable pair.*

Proof: Exercise. □

Note that $W' - W = \frac{X_I' - X_I}{\sqrt{n}}$. Hence we have

$$\begin{aligned}\mathbf{E}[W' - W|W] &= \frac{1}{\sqrt{n}}\mathbf{E}[X_I' - X_I|W] \\ &= \frac{1}{\sqrt{n}} \cdot \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[X_i' - X_i|W] = -\frac{1}{n}\mathbf{E}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i|W] = -\frac{1}{n}W.\end{aligned}$$

Here condition (2) is satisfied with $\lambda = n^{-1}$. Now,

$$\frac{1}{3\lambda}\mathbf{E}|W' - W|^3 = \frac{n}{3n^{3/2}}\mathbf{E}|X_I' - X_I|^3 = \frac{1}{3n^{3/2}}\sum_{i=1}^{n}\mathbf{E}|X_i' - X_i|^3 \leq \frac{8}{3n^{3/2}}\sum_{i=1}^{n}\mathbf{E}|X_i|^3$$

and

$$\mathbf{E}\left[\frac{1}{2\lambda}(W' - W)^2|W\right] = \frac{n}{2n}\mathbf{E}((X_I' - X_I)^2|W) = \frac{1}{2n}\sum_{i=1}^{n}\mathbf{E}((X_i' - X_i)^2|W).$$

Note that,

$$\mathbf{E}((X_i' - X_i)^2|W) = \mathbf{E}(X_i'^2 - 2X_i'X_i + X_i^2|W) = 1 + \mathbf{E}(X_i^2|W).$$

Hence

$$\begin{aligned}\mathrm{Var}\left(\mathbf{E}(\frac{1}{2\lambda}(W' - W)^2|W)\right) &= \mathrm{Var}\left(\mathbf{E}(\frac{1}{2n}\sum_{i=1}^{n}X_i^2|W)\right) \\ &\leq \mathrm{Var}\left(\frac{1}{2n}\sum_{i=1}^{n}X_i^2\right) = \frac{1}{4n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i^2) \leq \frac{1}{4n^2}\sum_{i=1}^{n}\mathbf{E}X_i^4\end{aligned}$$

and we have,

$$\mathbf{Wass}(W, Z) \leq \sqrt{\frac{1}{2\pi n^2}\sum_{i=1}^{n}\mathbf{E}X_i^4} + \frac{8}{3n^{3/2}}\sum_{i=1}^{n}\mathbf{E}|X_i|^3.$$