# Lecture 4

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In this lecture we are going to study the solution of the differential equation

$$f'(x) - xf(x) = g(x) - \mathbf{E}g(Z), \quad Z \sim N(0,1). \tag{1}$$

**Lemma 1** *Given function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{E}|g(Z)| < \infty$ where $Z \sim N(0,1)$,*

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2}(g(y) - \mathbf{E}g(Z))dy \tag{2}$$

*is an absolutely continuous solution of (1).*

*Moreover, any a.c. solution $\tilde{f}$ of (1) is of the form*

$$\tilde{f}(x) = f(x) + ce^{x^2/2}, \quad c \in \mathbb{R}.$$

*Finally, $f$ is the only solution that satisfies $\lim_{|x|\to\infty} f(x)e^{-x^2/2} = 0$.*

**Proof:** By the method of integrating factors, we have that if $f$ is a solution to (1), then

$$\frac{d}{dx}(e^{-x^2/2}f(x)) = e^{-x^2/2}(f'(x) - xf(x)) = e^{-x^2/2}(g(x) - \mathbf{E}g(Z)).$$

So, (2) is a reasonable candidate as a solution of (1). And it is easy to verify directly that (2) indeed satisfies (1).

If $\tilde{f}$ is any other solution of (1), then

$$\frac{d}{dx}\left(e^{-x^2/2}(f(x) - \tilde{f}(x))\right) = 0.$$

Hence, $\tilde{f}(x) = f(x) + ce^{x^2/2}$ for some $c \in \mathbb{R}$.

Clearly, from definition

$$\lim_{x\to-\infty} f(x)e^{-x^2/2} = 0 \quad \text{(by DCT)}.$$

Note that since $Z \sim N(0,1)$, we have

$$\int_{-\infty}^{\infty} e^{-y^2/2}(g(y) - \mathbf{E}g(Z))dy = 0.$$

So, $f$ can also be written as follows

$$f(x) = -e^{x^2/2} \int_x^\infty e^{-y^2/2}(g(y) - \mathbf{E}g(Z))dy. \tag{3}$$

Therefore, by DCT, $\lim_{x\to+\infty} f(x)e^{-x^2/2} = 0$.

□

**Remark 2** *If, instead of standard gaussian, $Z$ follows any other distribution then all of the statements of the above lemma still hold except* $\lim_{x\to+\infty} f(x)e^{-x^2/2} = 0$.

## 0.1 Another form of the solution

**Lemma 3** *Assume $g$ is Lipschitz. Then*

$$f(x) = -\int_0^1 \frac{1}{2\sqrt{t(1-t)}}\mathbf{E}\left[Zg(\sqrt{t}x + \sqrt{1-t}Z)\right]dt, \quad Z \sim N(0,1) \tag{4}$$

*is a solution of* (1). *In fact, it must be the same as* (2), *because* $\lim_{|x|\to\infty} f(x)e^{-x^2/2} = 0$.

**Proof:** Let $g$ is $C$-Lipschitz. Then[1] $|g'|_\infty \leq C$.

On differentiating $f$ and carrying the derivative inside the integral and expectation which can be justified using DCT, we have

$$f'(x) = -\int_0^1 \frac{1}{2\sqrt{1-t}}\mathbf{E}\left[Zg'(\sqrt{t}x + \sqrt{1-t}Z)\right]dt. \tag{5}$$

On the other hand, the Stein identity gives us

$$\mathbf{E}\left[Zg(\sqrt{t}x + \sqrt{1-t}Z)\right] = \sqrt{1-t}\mathbf{E}\left[g'(\sqrt{t}x + \sqrt{1-t}Z)\right].$$

Thus,

$$\begin{aligned} f'(x) - xf(x) &= \int_0^1 \mathbf{E}\left[\left(-\frac{Z}{2\sqrt{1-t}} + \frac{x}{2\sqrt{t}}\right)g'(\sqrt{t}x + \sqrt{1-t}Z)\right]dt \\ &= \int_0^1 \mathbf{E}\left[\frac{d}{dt}g'(\sqrt{t}x + \sqrt{1-t}Z)\right]dt \\ &= \mathbf{E}\left[\int_0^1 \frac{d}{dt}g'(\sqrt{t}x + \sqrt{1-t}Z)dt\right] = g(x) - \mathbf{E}g(Z). \end{aligned}$$

[1] Any Lipschitz function $g$ is absolutely continuous. Hence, it is (Lebesgue) almost surely differentiable. Define $g'$ to be derivative of $g$ at the points where it exists and 0 elsewhere.

□

Recall the notation $Ng := \mathbf{E}g(Z)$. Now we will prove that if $g : \mathbb{R} \to \mathbb{R}$ is bounded,

$$I.\ |f|_\infty \le \sqrt{\frac{\pi}{2}}\,|g - Ng|_\infty \quad \text{and} \quad II.\ \left|f'\right|_\infty \le 2\,|g - Ng|_\infty$$

and if $g$ is Lipschitz, but not necessarily bounded, then

$$III.\ |f|_\infty \le \left|g'\right|_\infty, \quad IV.\ \left|f'\right|_\infty \le \sqrt{\frac{2}{\pi}}\left|g'\right|_\infty, \quad \text{and} \quad V.\ \left|f''\right|_\infty \le 2\left|g'\right|_\infty.$$

This will prove the Lemma 1 of Lecture 3. The bounds (I), (II) and (V) were obtained by Stein.

**Proof of bound (III) :** Applying Stein's identity on (4), we have

$$f(x) = -\int_0^1 \frac{1}{2\sqrt{t}}\mathbf{E}\left[g'(\sqrt{t}x + \sqrt{1-t}Z)\right]dt.$$

Hence,

$$|f|_\infty \le |g'|_\infty \int_0^1 \frac{1}{2\sqrt{t}} = |g'|_\infty.$$

□

**Proof of bound (IV) :** From (5), it follows that

$$|f|_\infty \le (\mathbf{E}|Z|)\,|g'|_\infty \int_0^1 \frac{1}{2\sqrt{1-t}} = \sqrt{\frac{2}{\pi}}|g'|_\infty.$$

□

**Exercise 4** *Get the bound (V) from the representation* (4).

**Proof of bound (I) :** Take $f$ as in (2). Suppose $x > 0$. Using the representation in (3), we have

$$|f(x)| \le |g - Ng|_\infty \left(e^{x^2/2}\int_x^\infty e^{-y^2/2}dy\right).$$

Now, $\frac{d}{dx}e^{x^2/2}\int_x^\infty e^{-y^2/2}dy = -1 + xe^{x^2/2}\int_x^\infty e^{-y^2/2}dy \le 0 \quad \forall x > 0$. The last step follows from Mill's ratio inequality which says that $\int_x^\infty e^{-y^2/2}dy \le \frac{1}{x}e^{-x^2/2}$ ( for a quick proof, note that LHS $\le \int_x^\infty \frac{y}{x}e^{-y^2/2}dy =$ RHS ).

So, $e^{x^2/2}\int_x^\infty e^{-y^2/2}dy$ is maximized at $x = 0$ on $[0, \infty)$ where its value is $\sqrt{\frac{\pi}{2}}$. Hence,

$$|f(x)| \le \sqrt{\frac{\pi}{2}}|g - Ng|_\infty \quad \forall x > 0.$$

For $x < 0$, use the form (2) and proceed in the similar manner. $\square$

**Proof of bound (II) :** Again, we will only consider $x > 0$ case. The other case will be similar.

Note that

$$f'(x) = g(x) - Ng + xf(x) = g(x) - Ng - xe^{x^2/2} \int_x^\infty e^{-y^2/2}(g(y) - Ng)dy.$$

Therefore,

$$\begin{aligned} |f'(x)| &\leq |g - Ng|_\infty \left(1 + xe^{x^2/2} \int_x^\infty e^{-y^2/2} dy\right) \\ &\leq 2|g - Ng|_\infty \quad \text{(By Mill's ratio inequality)}. \end{aligned}$$

$\square$

**Proof of bound (V) :** On differentiating (1) and rearranging

$$\begin{aligned} f''(x) &= g'(x) + f(x) + xf'(x) \\ &= g'(x) + f(x) + x(g(x) - Ng + xf(x)) \\ &= g'(x) + x(g(x) - Ng) + (1 + x^2)f(x). \end{aligned} \tag{6}$$

We can write $g(x) - Ng$ in terms of $g'$ as follows,

$$\begin{aligned} g(x) - Ng &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2/2}(g(x) - g(y))dy \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^x \int_y^x g'(z)e^{-y^2/2} dz dy - \int_x^\infty \int_x^y g'(z)e^{-y^2/2} dz dy\right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^x g'(z) \int_{-\infty}^z e^{-y^2/2} dy dz - \int_x^\infty g'(z) \int_z^\infty e^{-y^2/2} dy dz\right] \\ &= \int_{-\infty}^x g'(z)\Phi(z)dz - \int_x^\infty g'(z)\overline{\Phi}(z)dz \end{aligned}$$

where $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ is the distribution function for standard normal and $\overline{\Phi}(z) = 1 - \Phi(z)$.

Similarly,

$$
\begin{aligned}
f(x) &= e^{x^2/2}\int_{-\infty}^{x} e^{-y^2/2}(g(y)-\mathbf{E}g(Z))dy \\
&= e^{x^2/2}\int_{-\infty}^{x} e^{-y^2/2}\left(\int_{-\infty}^{y} g'(z)\Phi(z)dz - \int_{y}^{\infty} g'(z)\overline{\Phi}(z)\right)dzdy \\
&= e^{x^2/2}\left(\int_{-\infty}^{x} g'(z)\Phi(z)\int_{z}^{x} e^{-y^2/2}dydz - \int_{-\infty}^{\infty} g'(z)\overline{\Phi}(z)\int_{-\infty}^{z\wedge x} e^{-y^2/2}dydz\right) \\
&= \sqrt{2\pi}e^{x^2/2}\left(\int_{-\infty}^{x} g'(z)\Phi(z)(\overline{\Phi}(z)-\overline{\Phi}(x))dz\right. \\
&\qquad \left. - \int_{-\infty}^{x} g'(z)\overline{\Phi}(z)\Phi(z)dz - \int_{x}^{\infty} g'(z)\overline{\Phi}(z)\Phi(x)dz\right) \\
&= -\sqrt{2\pi}e^{x^2/2}\left[\overline{\Phi}(x)\int_{-\infty}^{x} g'(z)\Phi(z)dz + \Phi(x)\int_{x}^{\infty} g'(z)\overline{\Phi}(z)dz\right]
\end{aligned}
$$

Substituting the above expressions for $g - Ng$ and $f$ in (6), we get

$$
\begin{aligned}
f''(x) = g'(x) &+ \left(x - \sqrt{2\pi}(1+x^2)e^{x^2/2}\overline{\Phi}(x)\right)\int_{-\infty}^{x} g'(z)\Phi(z)dz \\
&+ \left(-x - \sqrt{2\pi}(1+x^2)e^{x^2/2}\Phi(x)\right)\int_{x}^{\infty} g'(z)\overline{\Phi}(z)dz.
\end{aligned}
$$

*To be continued in the next lecture.*